Towards a Substitution Calculus for Dinatural Transformations

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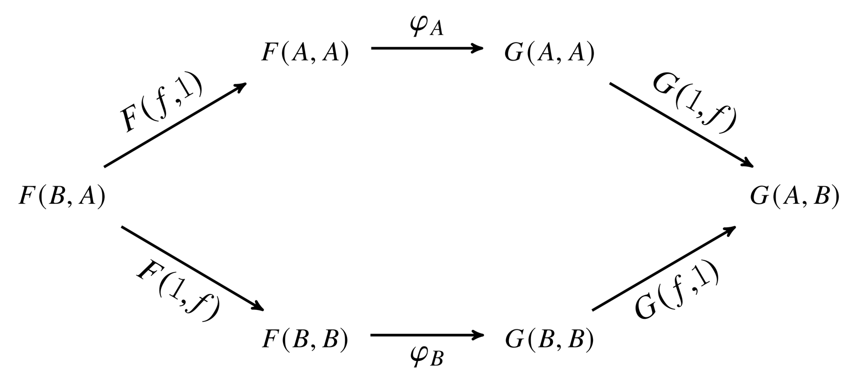
Article:

References:

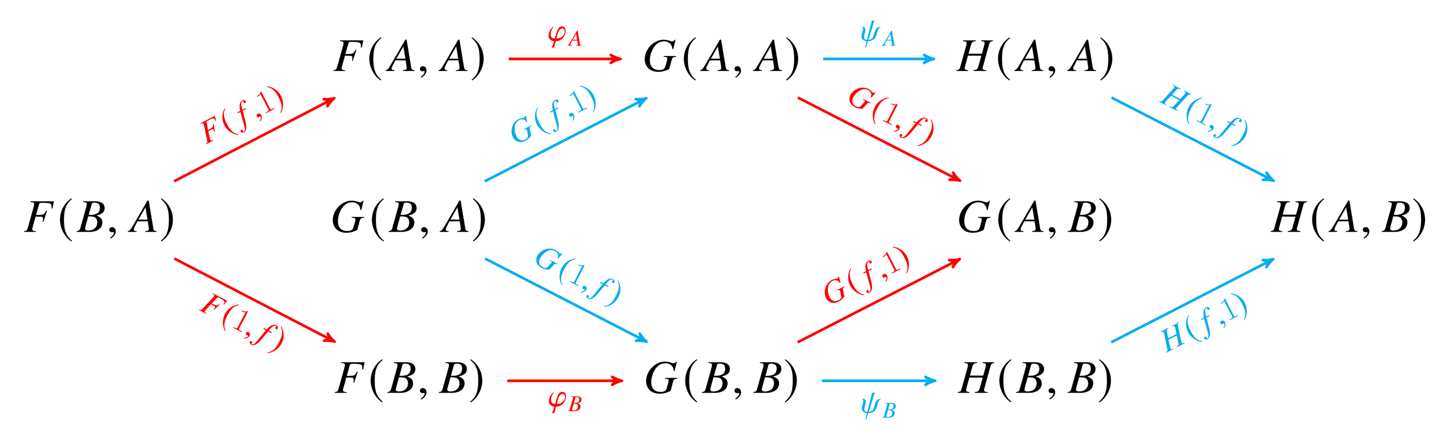
Dinatural transformations

Dinatural transformations [4] generalise natural ones for mixed-variance functors.

Let $F, G\colon \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{D}$ be functors. A family $\varphi = (\varphi_A\colon F(A,A) \to G(A,A))_{A\in\mathbb{C}}$ is a *dinatural transformation* $\varphi\colon F \to G$ iff for all $f\colon A \to B$ in $\mathbb{C}$ the following commutes:



Dinatural transformations arise in many different contexts: they define the notion of *end* [13, 18] of certain functors, which in appropriate enriched contexts allows to compute Kan extensions [5, 9]; they correspond to proofs in intuitionistic and multiplicative linear logic [2, 7, 15]; they provide a semantics for the notion of *parametric polymorphism* in the second-order λ-calculus [1, 3, 6, 17]; they characterise fixed point operators on many categories of domains [16]; and very recently they were proved to provide an operational semantics [8] as well as to model subtyping and bounded quantification in a game semantics [11, 12] for higher-order languages.
Yet, they suffer from a troublesome shortcoming: they do not compose.

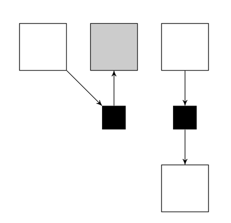


There is no way, in general, to infer the commutativity of the outer hexagon from that of the two inner ones. If either φ or ψ is natural, or if the middle diamond above is a pushout or a pullback, then $\psi \circ \varphi$ is dinatural. However, these are far from being satisfactory solutions for the compositionality problem, for either they are too restrictive (as in the first case), or they speak of properties enjoyed not by φ and ψ themselves, but rather by other structures, namely one of the functors involved.

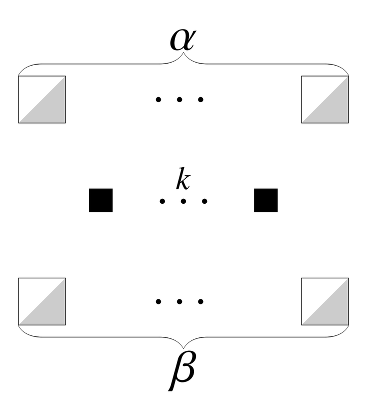
Vertical composition

Let $\mathbb{C}$ be a cartesian closed category, and consider the evaluation morphism $\mathrm{eval}_{A,B}\colon A \times (A \Rightarrow B) \to B$. We see it as a transformation (i.e. a family of morphisms) $\mathrm{eval}\colon F \to \mathrm{id}_{\mathbb{C}}$ dinatural in A and natural in B, where $\begin{array}{ccc} \mathbb{C}\times\mathbb{C}^{op}\times\mathbb{C} & \xrightarrow{F} & \mathbb{C} \\ (X,Y,Z) & \longmapsto & X \times (Y \Rightarrow Z) \end{array}$.

We can associate to eval a *graph* that reflects its signature:



The 3 upper boxes correspond to the arguments of F, the 1 lower box to the argument of $\mathrm{id}_{\mathbb{C}}$, the 2 middle black squares to the variables of eval. The edges tell us which arguments of F and $\mathrm{id}_{\mathbb{C}}$ we have to equate to which variable of the transformation eval when we write down its general component eval_{AB}. The direction of the edges and the colour shade of the upper/lower boxes keep track of the mixed variance of F and $\mathrm{id}_{\mathbb{C}}$.
In general, we consider arbitrary categories $\mathbb{B}$ and $\mathbb{C}$, functors $F\colon \mathbb{B}^{\alpha} \to \mathbb{C}$ and $G\colon \mathbb{B}^{\beta} \to \mathbb{C}$, with $\alpha, \beta \in \mathrm{List}\{+,-\}$, and transformations $\varphi = (\varphi_{A_1,\dots,A_k})\colon F \to G$ in k variables *together with a graph* $\Gamma(\varphi)$ that reflects its signature:



In the above, the i-th upper/lower box is linked with the j-th middle black square if the i-th argument of F/G is A_j when we write down the general component $\varphi_{A_1,\dots,A_k}$ of φ.

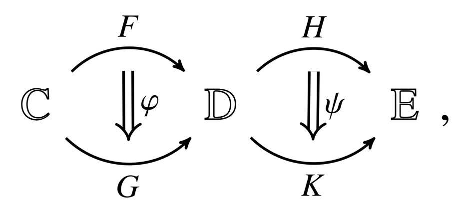
Theorem (Vertical Compositionality), cf.[14]

Let $\varphi\colon F \to G$ and $\psi\colon G \to H$ be transformations in many variables as above, dinatural in all their variables. If the composite graph $\Gamma(\psi) \circ \Gamma(\varphi)$, obtained by glueing together $\Gamma(\varphi)$ and $\Gamma(\psi)$ along the G-boxes, is *acyclic*, then $\psi \circ \varphi$ is dinatural in all its variables.

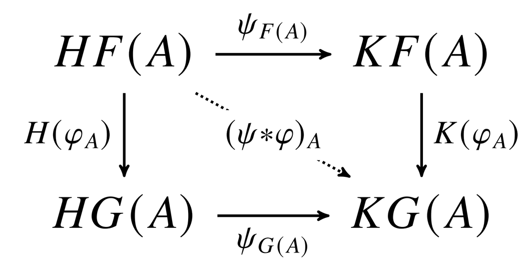
Notice: if either φ or ψ is natural, then their composite graph is always acyclic. If instead they are both dinatural and of the form $\varphi_A\colon F(A,A) \to G(A,A)$ and $\psi_A\colon G(A,A) \to H(A,A)$, then $\Gamma(\psi) \circ \Gamma(\varphi)$ is *never* acyclic! We need to consider many-variable functors and transformations to unlock compositionality.

Horizontal composition

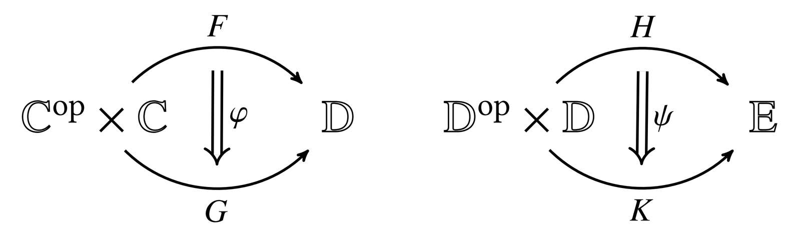
Natural transformations compose also horizontally. In the following situation:



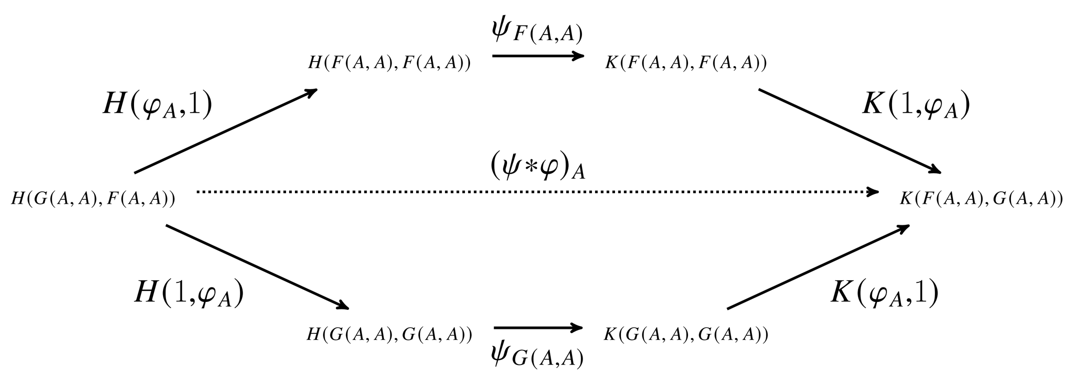
their horizontal composition $\psi * \varphi\colon HF \to KG$ is the natural transformation whose A-th component, $A \in \mathbb{C}$, is either leg of



$*$ is associative, unitary, and satisfies an interchange law with vertical composition.
Let now φ, ψ be dinatural transformations.



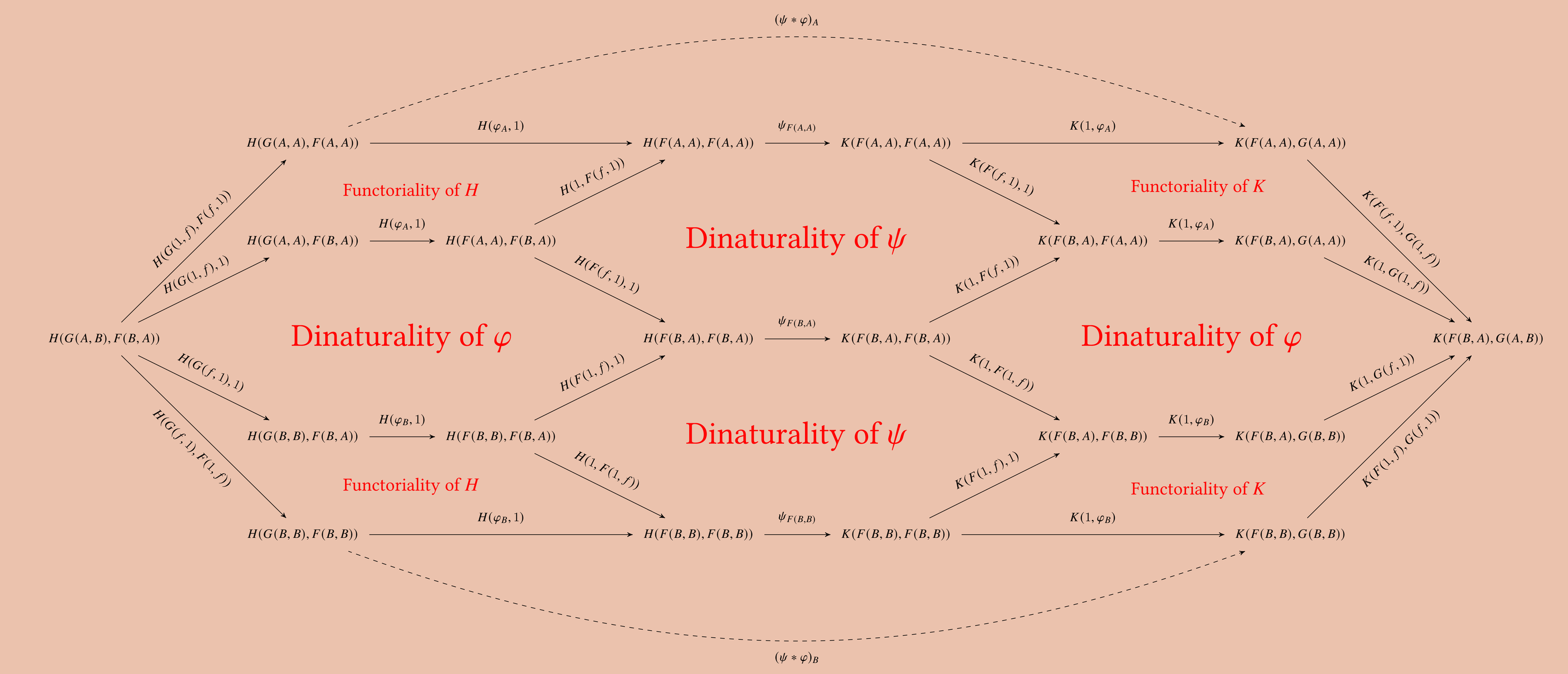
$\psi * \varphi\colon H(G^{op}, F) \to K(F^{op}, G)$ is the family of morphisms whose A-th component, $A \in \mathbb{C}$, is either leg of



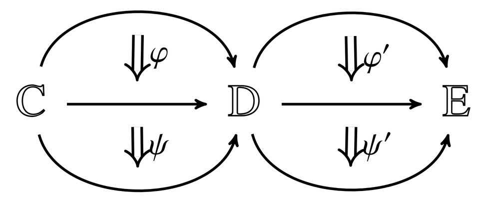
Theorem (Horizontal compositionality)

If φ and ψ are dinatural, so is $\psi * \varphi$.

Proof. The outer hexagon below is the dinaturality condition of $\psi * \varphi$:

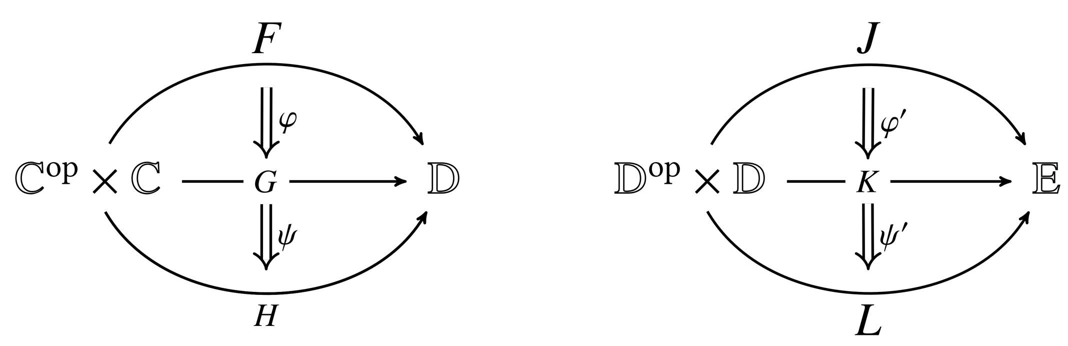


$*$ is associative and unitary. How about an interchange law with vertical composition? Recall in the natural case:



$(\psi' \circ \varphi') * (\psi \circ \varphi) = (\psi' * \psi) \circ (\varphi' * \varphi)$

Let now $\varphi, \psi, \varphi', \psi'$ be dinatural:



such that $\psi \circ \varphi$ and $\psi' \circ \varphi'$ are dinatural. Then

$$\varphi' * \varphi\colon J(G^{op}, F) \to K(F^{op}, G) \qquad \psi' * \psi\colon K(H^{op}, G) \to L(G^{op}, H)$$

They are not composable at all, not even as transformations!

Towards a substitution calculus

The usual calculus of natural transformations stems from the cartesian closedness of $\mathbb{C}\mathrm{at}$, as it is embodied in the functor $M\colon [\mathbb{B},\mathbb{C}] \times [\mathbb{A},\mathbb{B}] \to [\mathbb{A},\mathbb{C}]$. We aim to develop a calculus of many-variable functors and dinatural transformations centred around the notion of *substitution* as originally envisioned by Kelly [10]. Substitution generalises ordinary composition of functors and horizontal composition of (di)natural transformations for the many-variable case. We shall define a generalised functor category $\{\mathbb{B},\mathbb{C}\}$ over an appropriate category $\mathbb{G}$ of graphs, and prove that $\{\mathbb{B},-\}$ has a left adjoint $- \circ \mathbb{B}$, making $\mathbb{C}\mathrm{at}/_{\mathbb{G}}$ monoidal closed. The analogue of the functor M above, now of the form $\{\mathbb{B},\mathbb{C}\} \circ \{\mathbb{A},\mathbb{B}\} \to \{\mathbb{A},\mathbb{C}\}$, will yield the desired calculus.