

— CT-2023 —

AN $(\infty, 2)$ -CATEGORICAL PASTING THEOREM

Martina Rovelli

joint with { Philip Hackney
Viktoriya Ozornova
Emily Riehl }

Pasting theorem: Why?

Proof



Pasting theorem: Why?

Example: Uniqueness of adjoints

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Prop: If $\begin{cases} f \dashv u \\ f' \dashv u \end{cases}$ in a 2-category \mathcal{D} then $f \cong f'$.

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Define $f \xrightarrow{\alpha} f' :=$



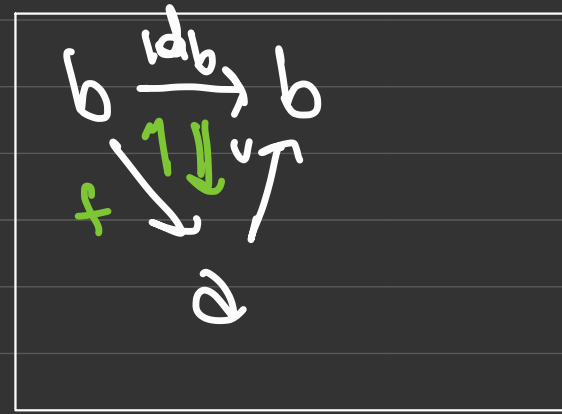
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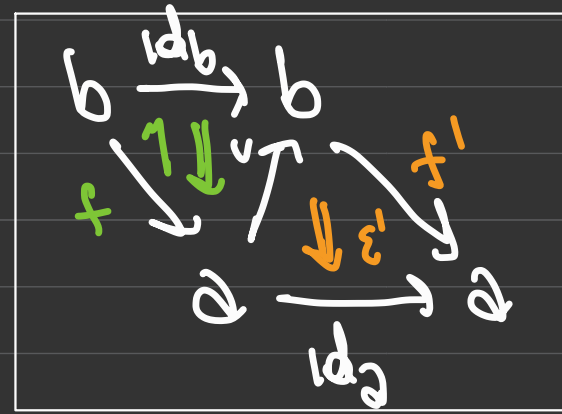
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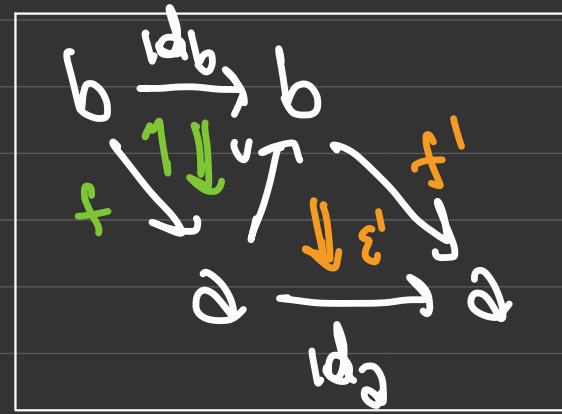
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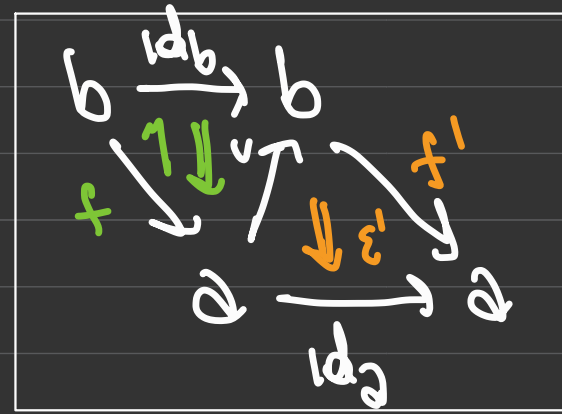
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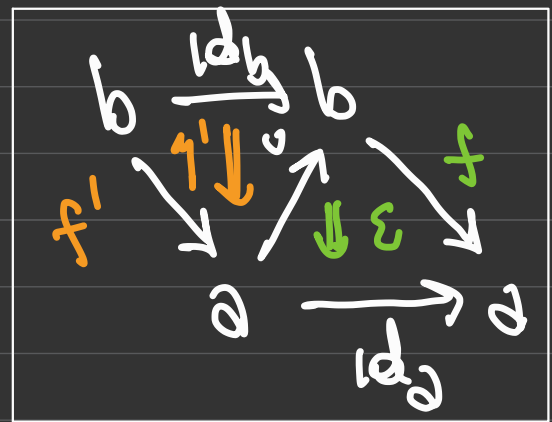
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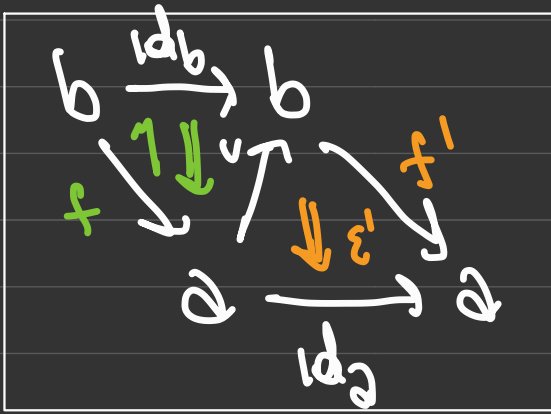
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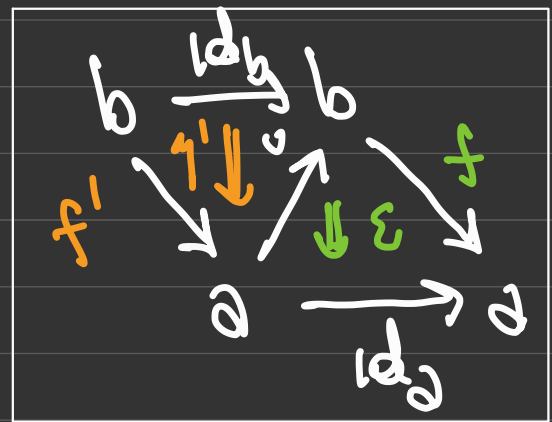
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$$u \circ \alpha \circ \beta =$$



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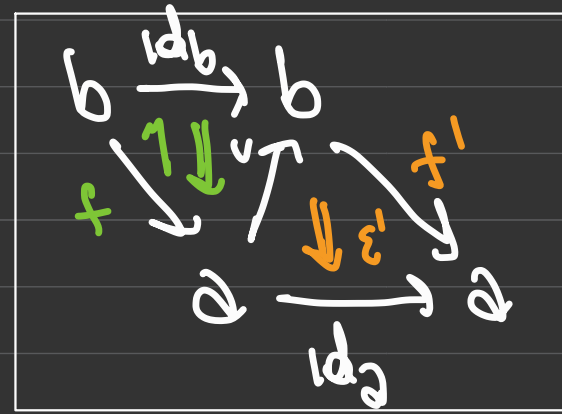
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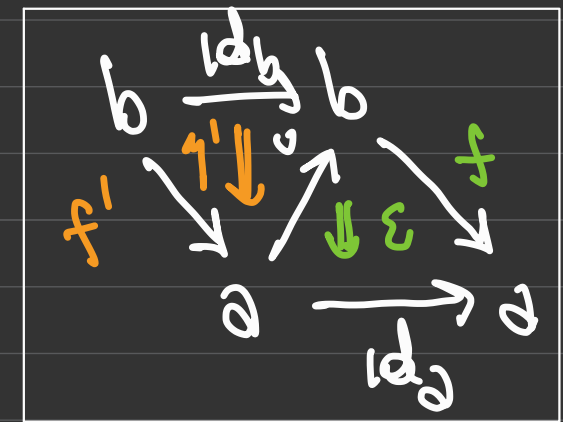
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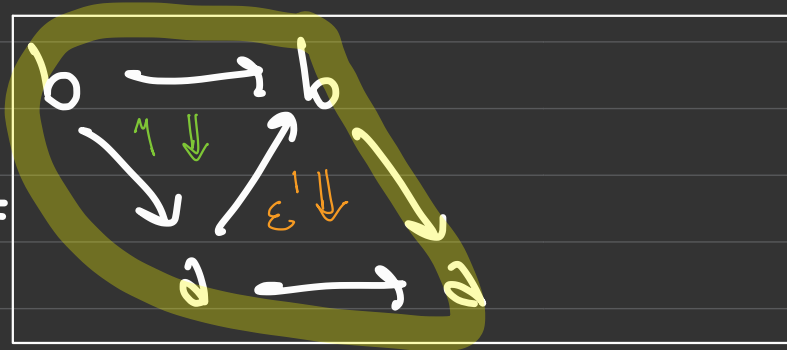
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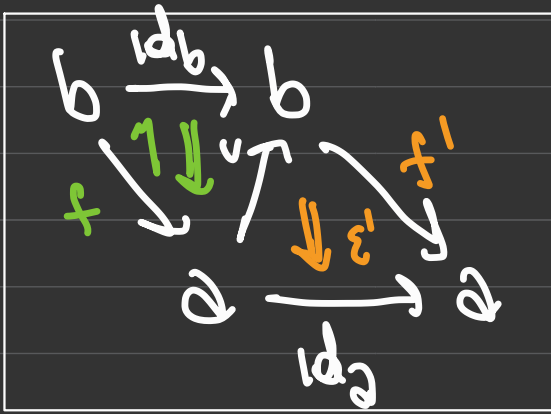
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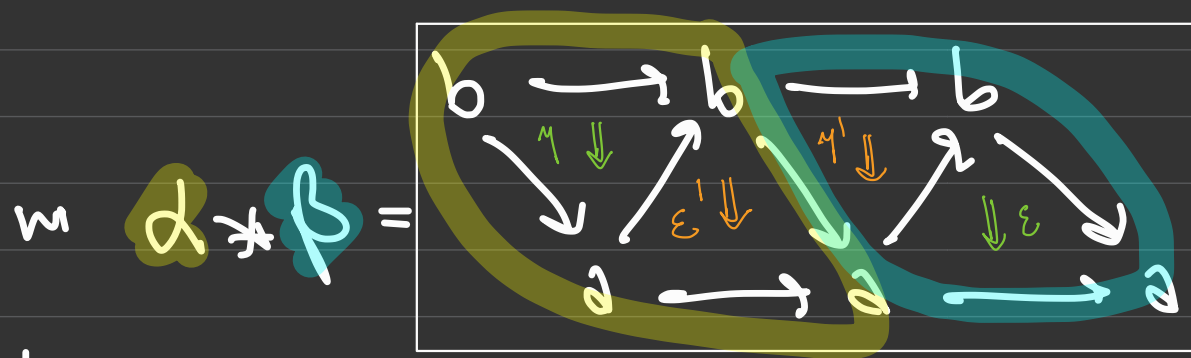
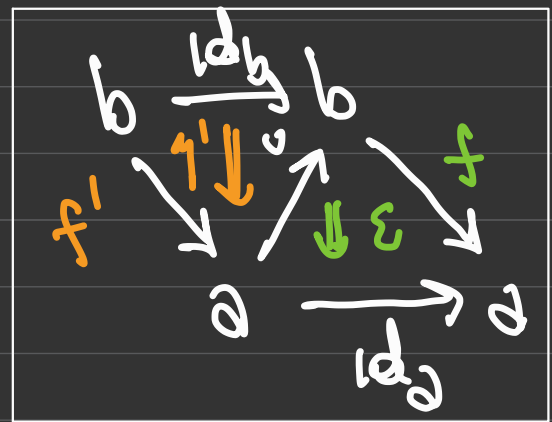
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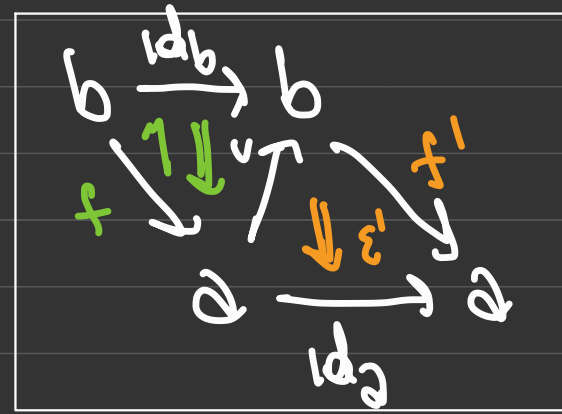
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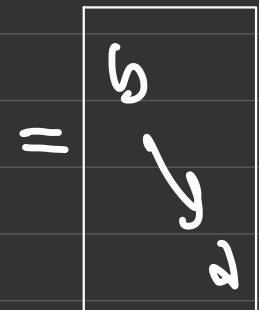
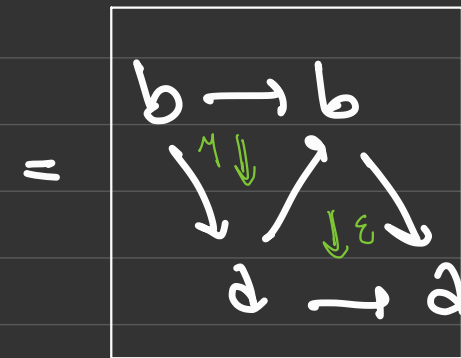
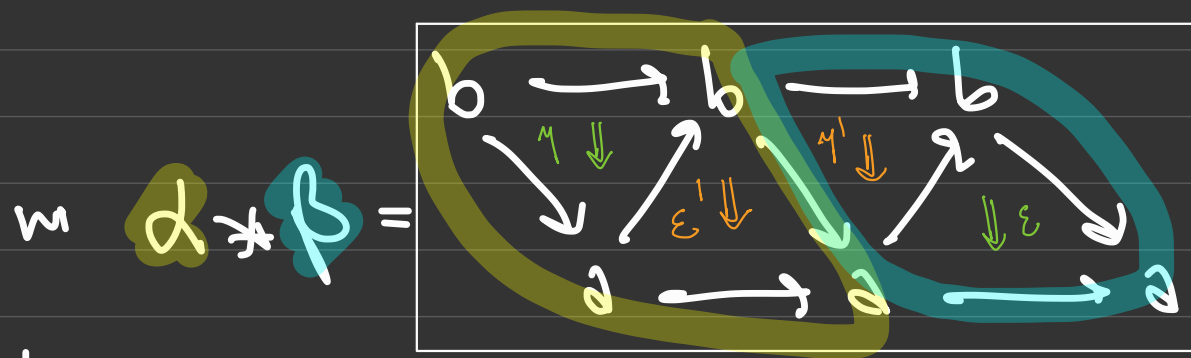
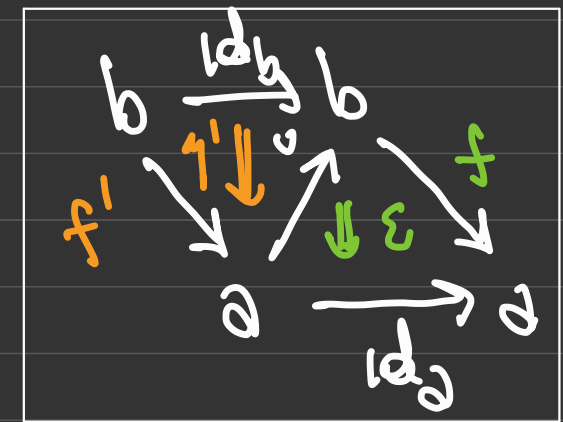
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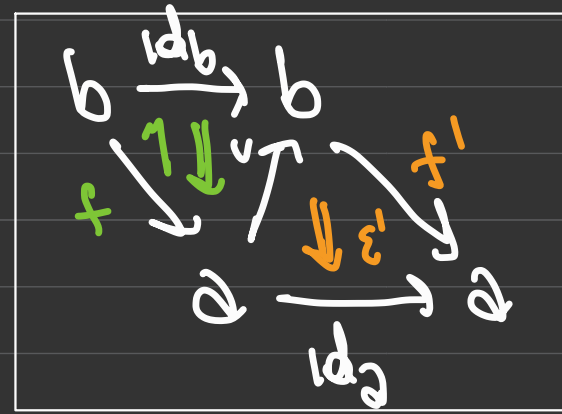
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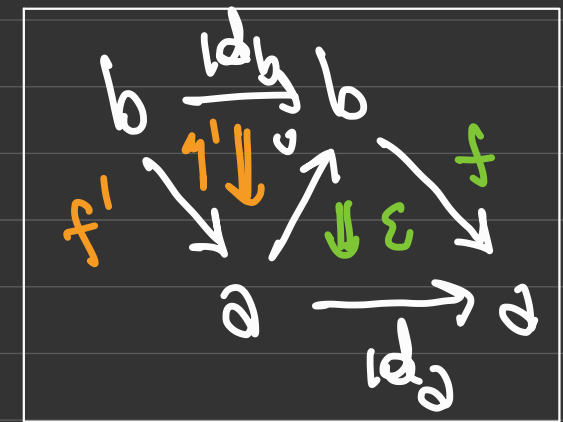
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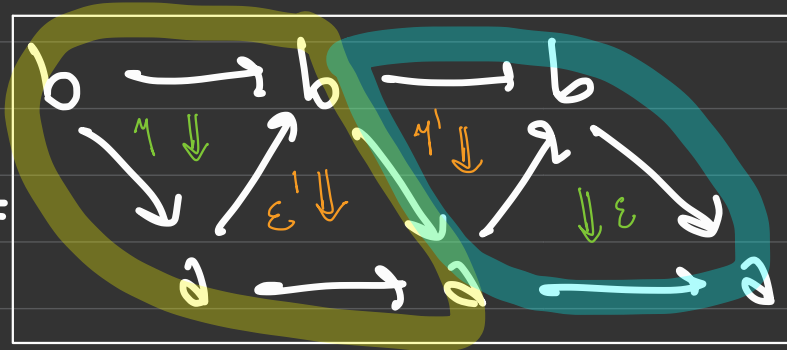
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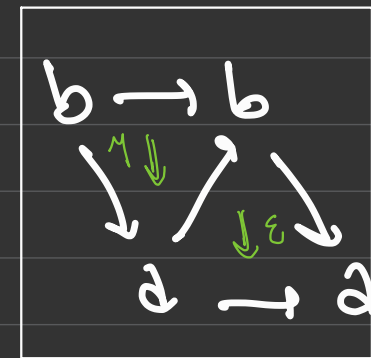
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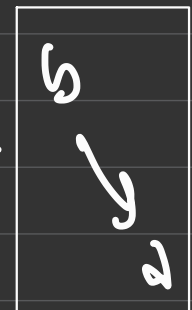
$$\alpha \circ \beta =$$



$$=$$



$$=$$



$$= id_f$$



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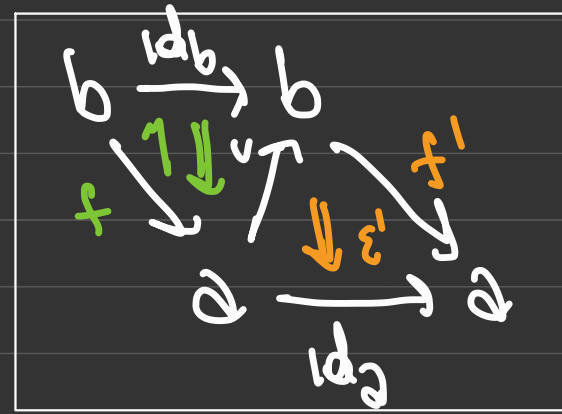
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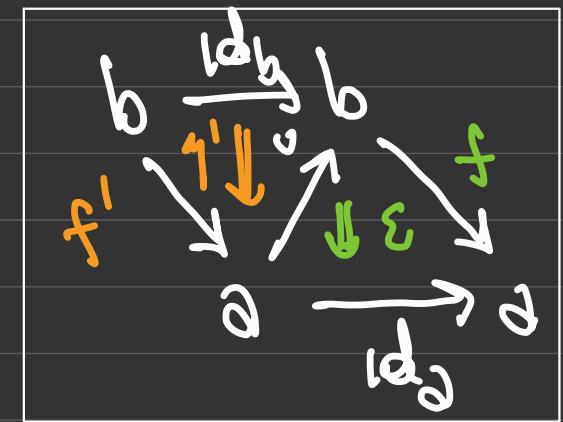
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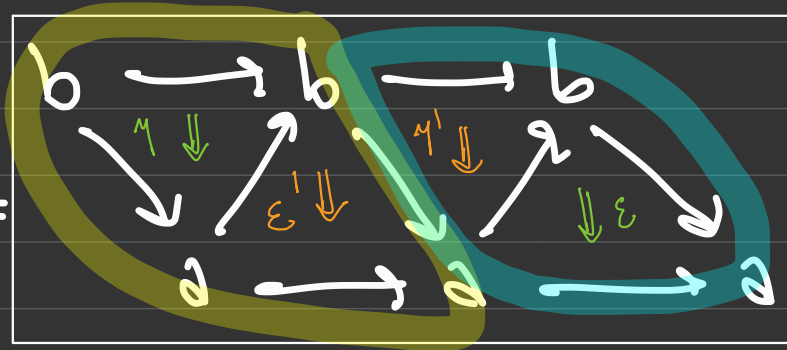
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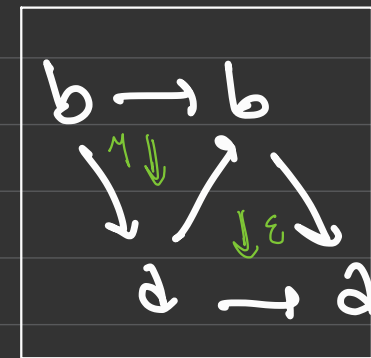
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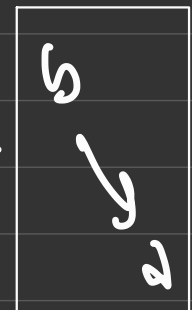
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$$\Rightarrow f \cong f' \quad \square$$

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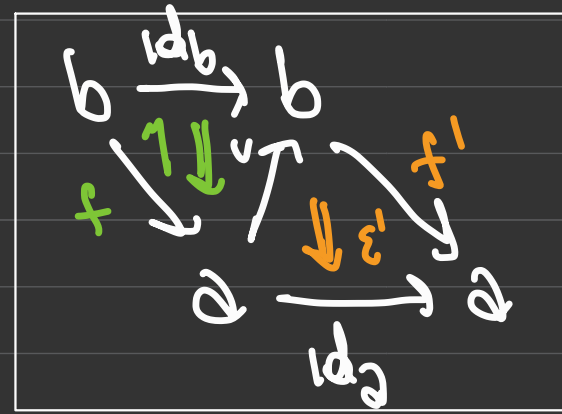
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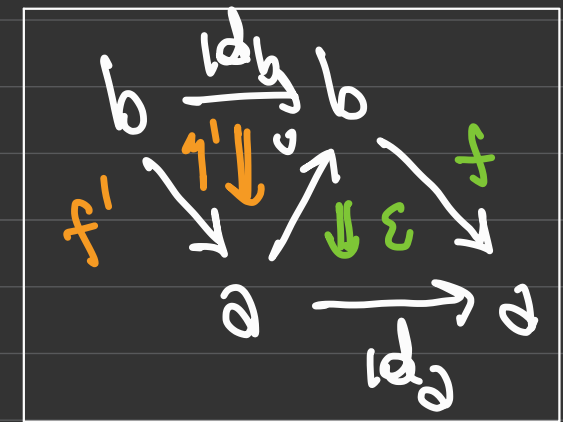
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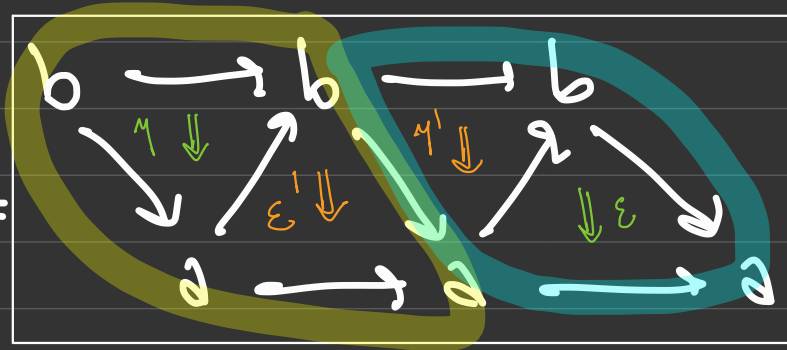
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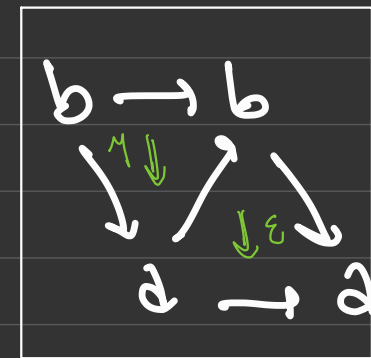
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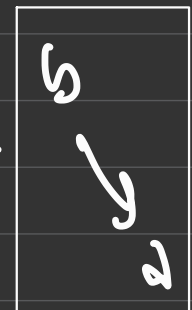
$$\alpha * \beta =$$



$$=$$



$$=$$



$$= id_f$$

\therefore Pasting theorem validates pasting calculus!

$$\therefore f \cong f' \quad \square$$

Pasting schemes

Pasting scheme =

Pasting schemes

Pasting scheme =

nice graph

Pasting schemes

finite connected nice graph

Pasting scheme =

Pasting schemes

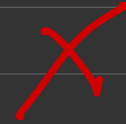
finite connected nice graph

Pasting scheme = nicely embedded in the plane

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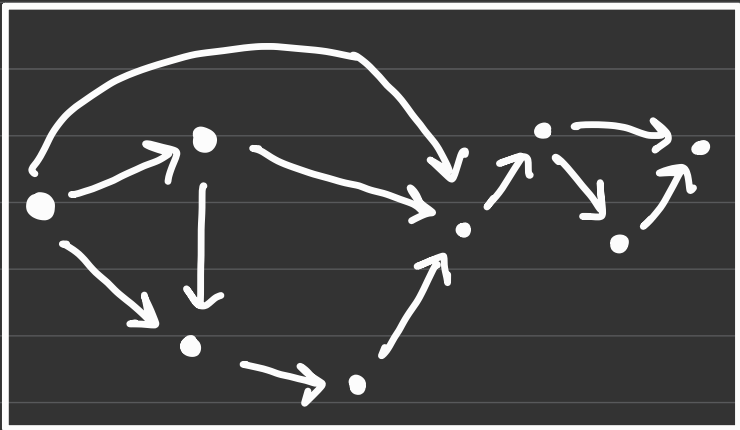
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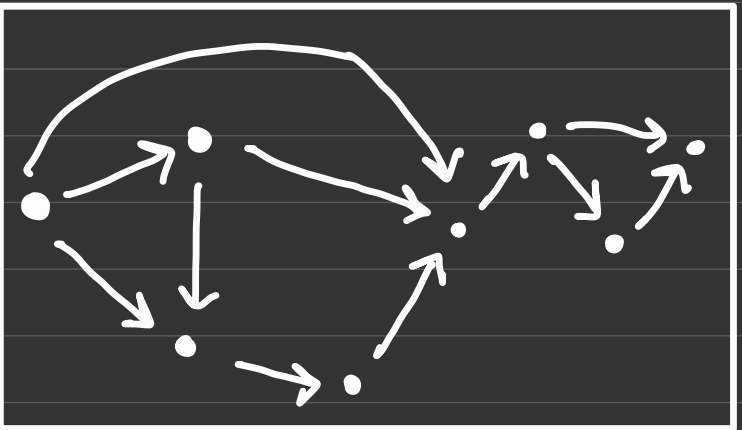
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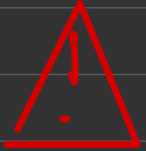
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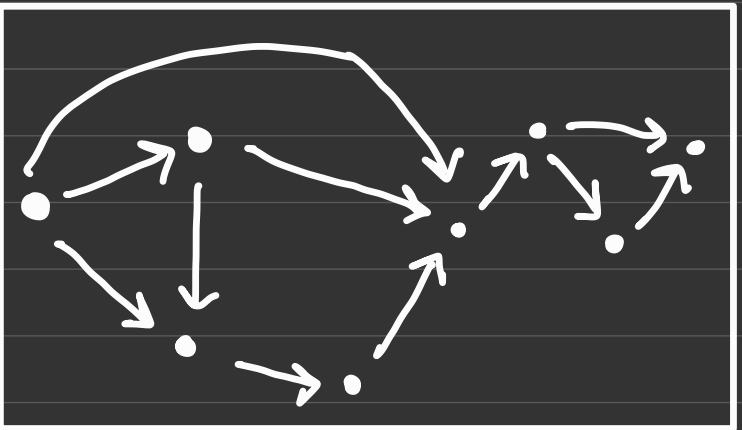
- Each region bounded by parallel paths



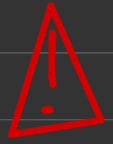
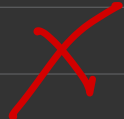
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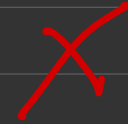
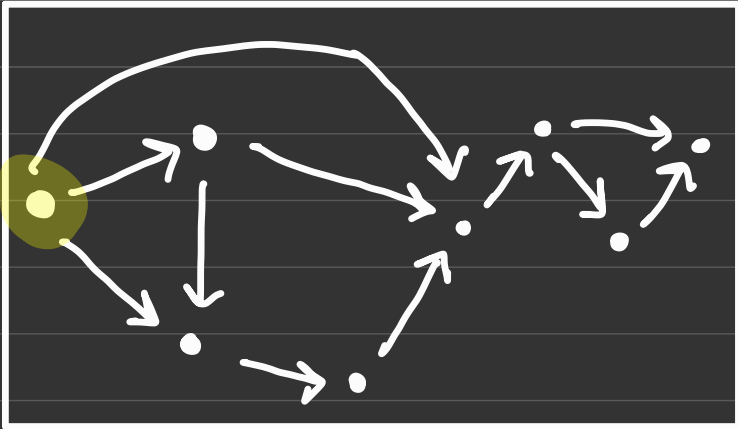
- Each region bounded by parallel paths
- Global source & sink



Pasting schemes

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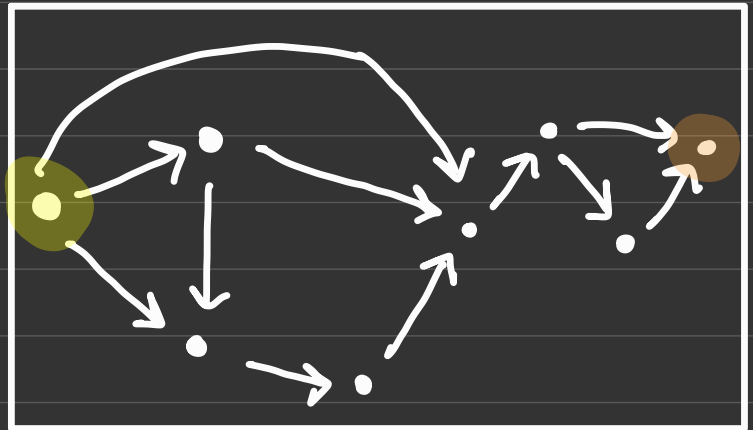
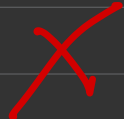
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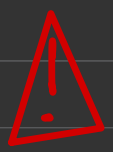
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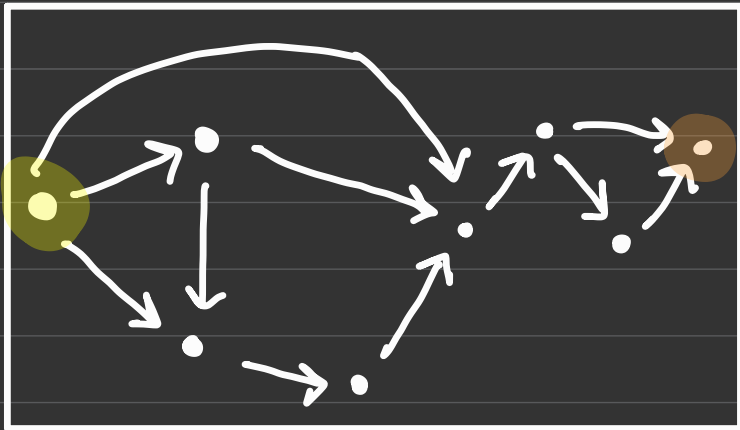
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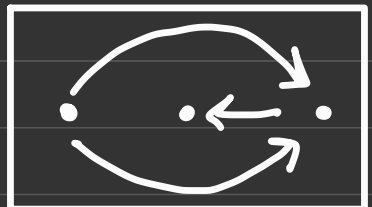
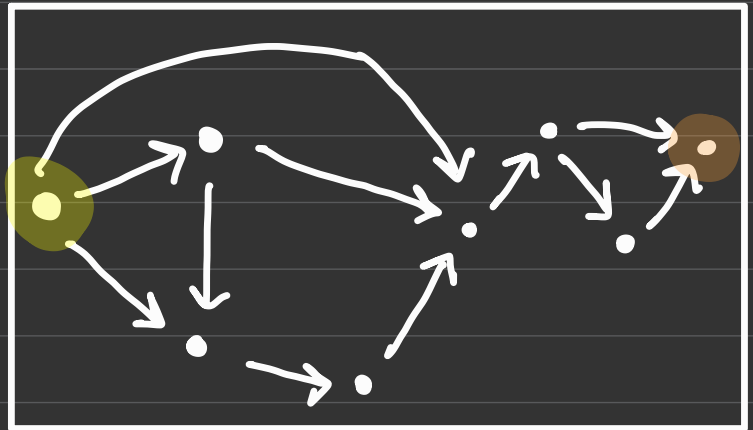
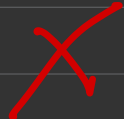
- Each region bounded by parallel paths
- Global **source** & **sink** on the outside



Pasting schemes

finite connected nice graph

Pasting scheme = nicely embedded in the plane



- internal sink



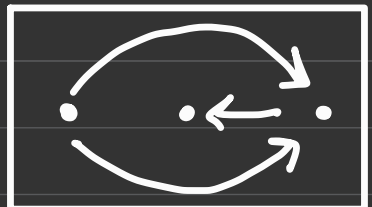
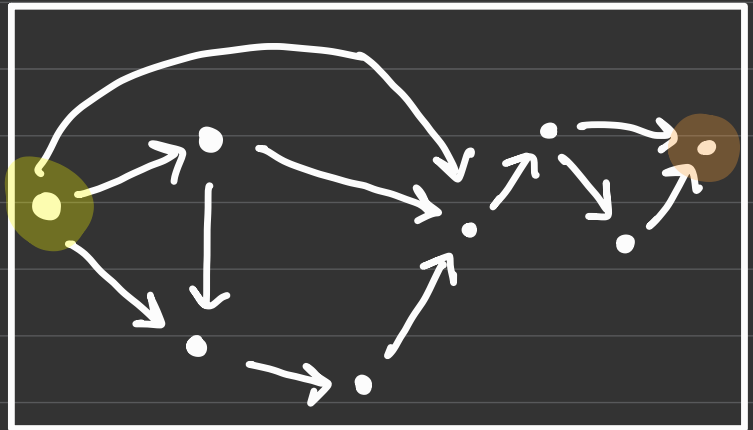
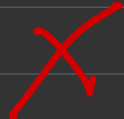
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Pasting schemes

finite connected nice graph

Pasting scheme = nicely embedded in the plane

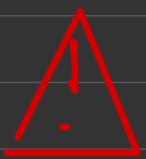


- internal sink



- loop

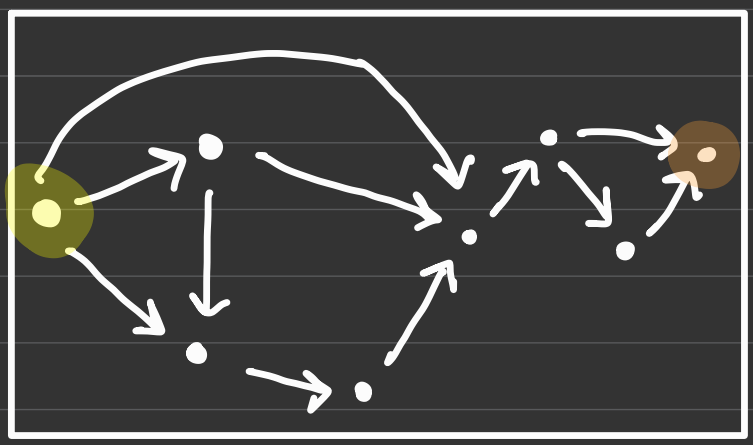
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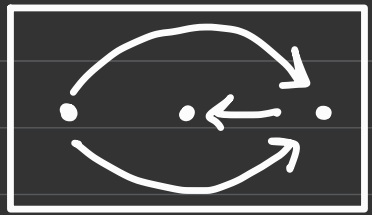
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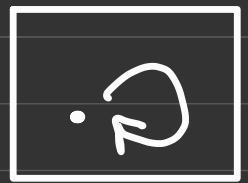
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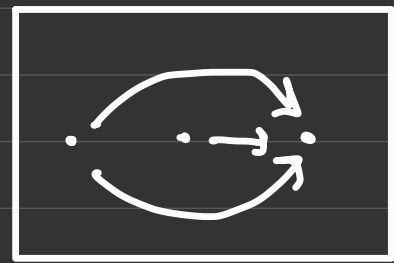
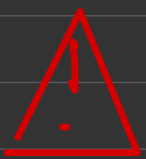
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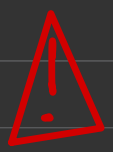
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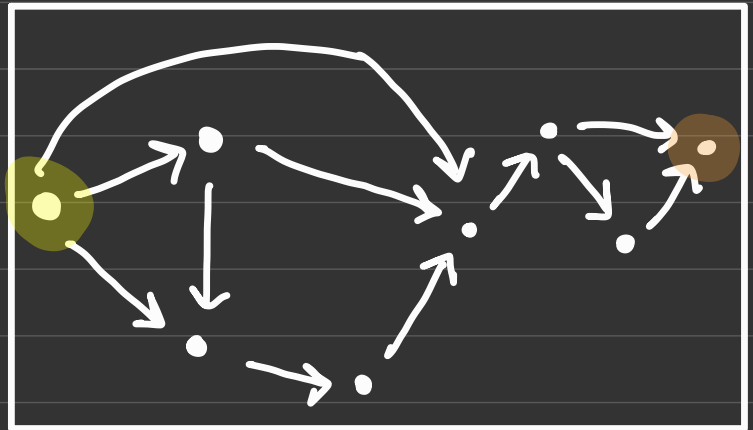
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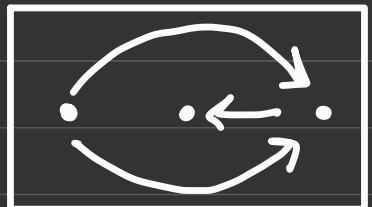
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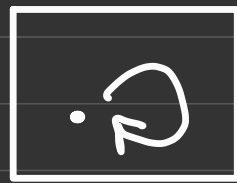
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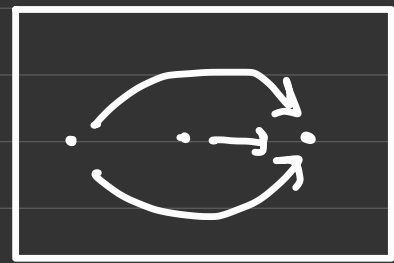
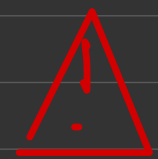
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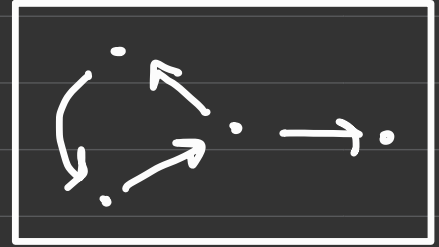
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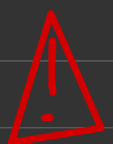
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- no source

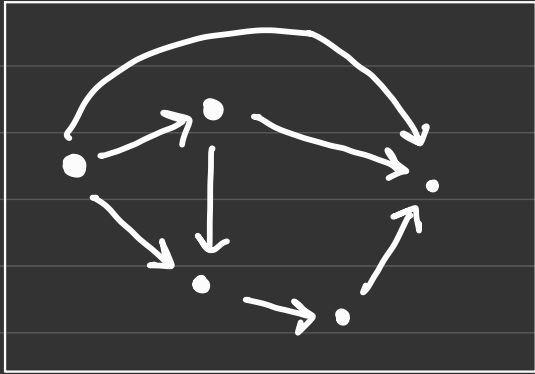


- no unique source



Pasting diagrams in a 2-category \mathcal{D}

\mathcal{D}
"

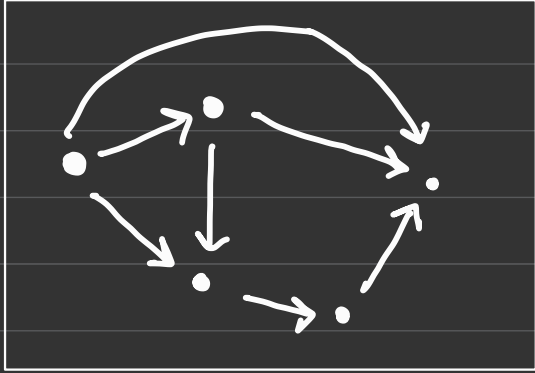


pasting scheme

\mapsto {

Pasting diagrams in a 2-category \mathcal{D}

P
" "



pasting scheme

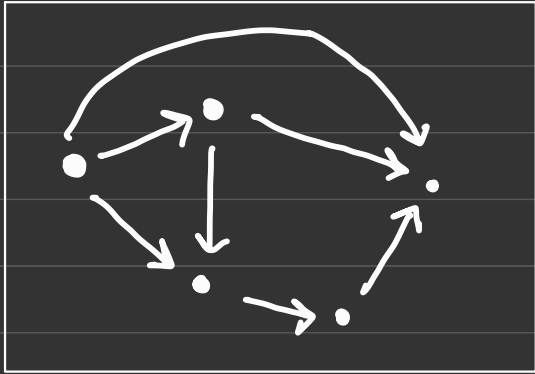
$\tilde{A}(P) \in 2\text{Cat}$

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\mapsto {

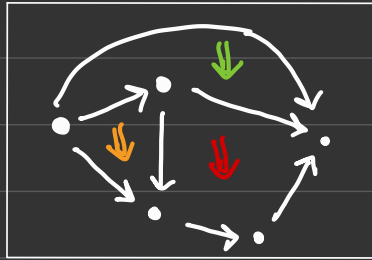
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pasting scheme

$\tilde{A}(\mathcal{P})$
" "

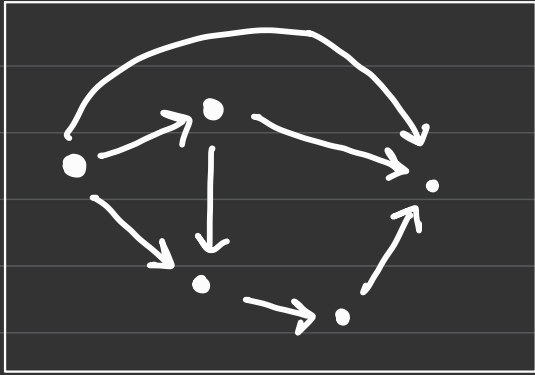


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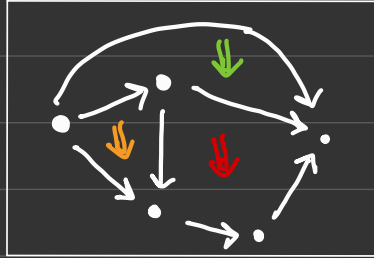
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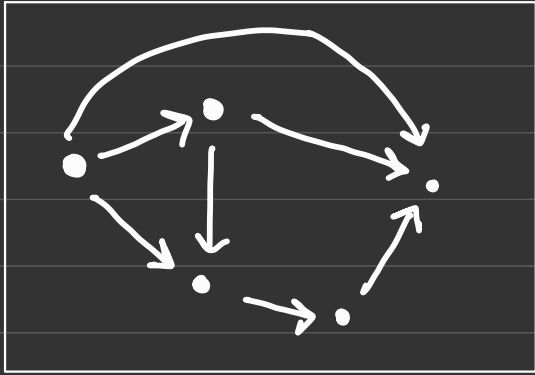
free 2-category
[Johnson-Yau]

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\mapsto {

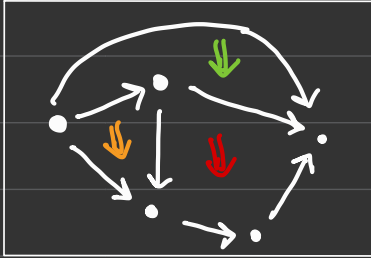
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pasting scheme

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free 2-category
[Johnson-Yau]

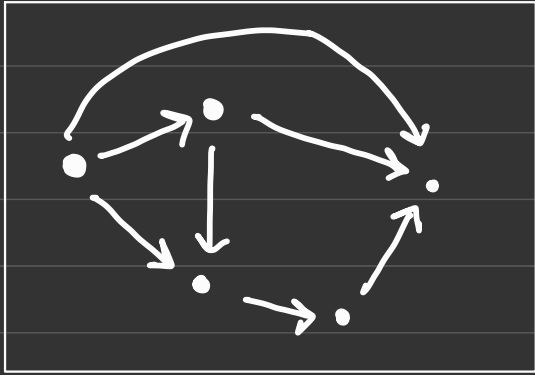
$A(\mathcal{P})$ $\in 2\text{Cat}$

2-category encoding
elementary pieces of \mathcal{P}

\mapsto {

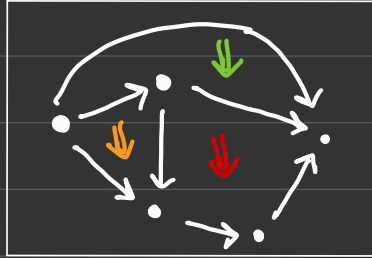
Pasting diagrams in a 2-category \mathcal{D}

\mathcal{P}
" "



pasting scheme

$\tilde{A}(\mathcal{P})$
" " $\in 2\text{Cat}$



free 2-category
[Johnson-Yau]

$A(\mathcal{P})$
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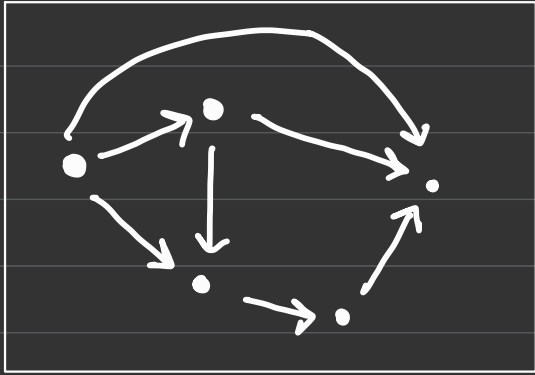


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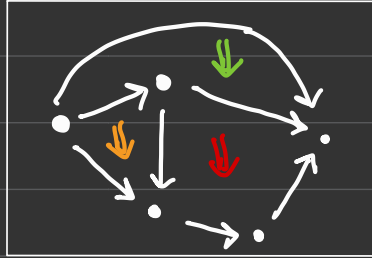
Pasting diagrams in a 2-category \mathcal{D}

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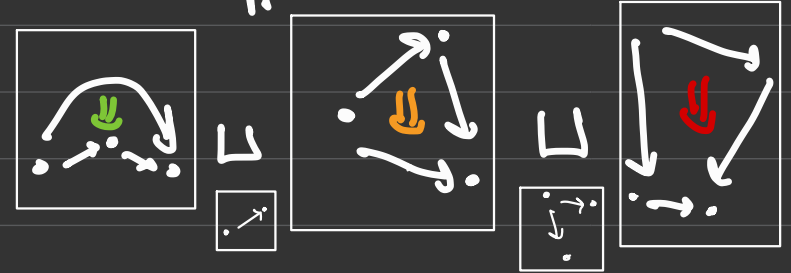
pasting scheme

$\tilde{A}(\mathcal{P}) \in 2\text{Cat}$
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free 2-category
[Johnson-Yau]

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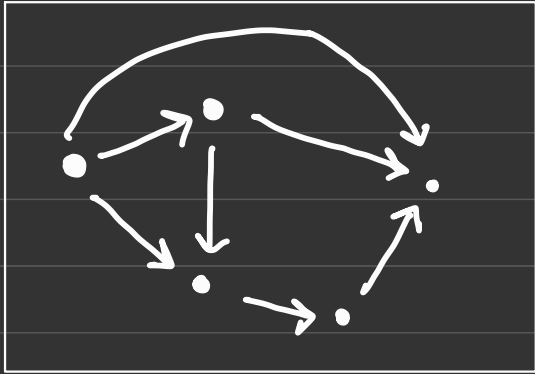


2-category encoding
elementary pieces of \mathcal{P}

\mapsto {

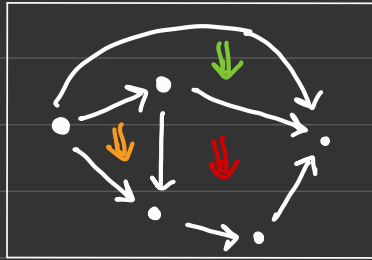
Pasting diagrams in a 2-category \mathcal{D}

P
" "



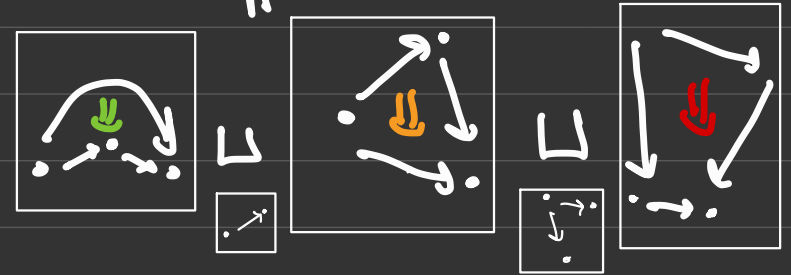
pasting scheme

$\tilde{A}(P)$ $\in 2\text{Cat}$
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free 2-category
[Johnson-Yau]

$A(P)$ $\in 2\text{Cat}$
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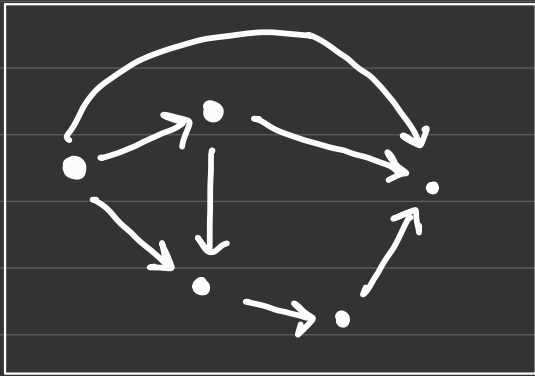


2-category encoding
elementary pieces of P

$$\mapsto \begin{cases} k : A(P) \rightarrow \mathcal{D} \\ \tilde{k} : \tilde{A}(P) \rightarrow \mathcal{D} \end{cases}$$

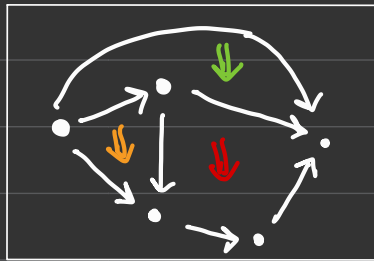
Pasting diagrams in a 2-category \mathcal{D}

P
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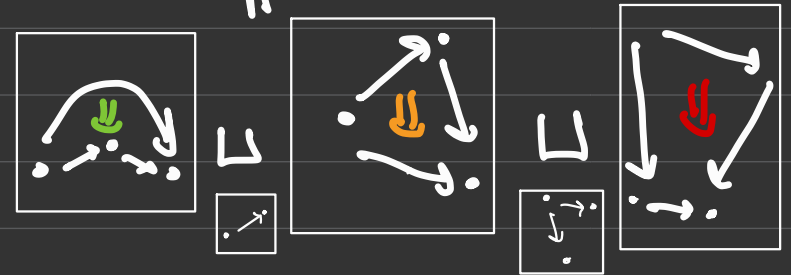
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free 2-category
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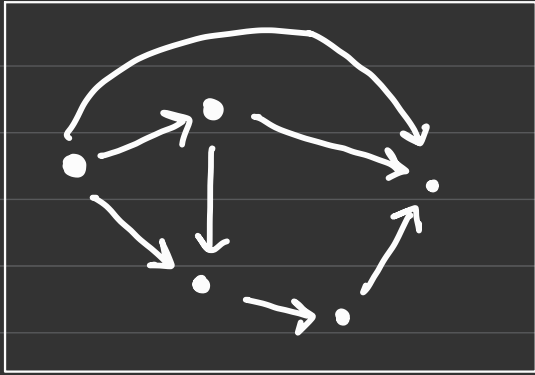


2-category encoding
elementary pieces of P

$$\mapsto \begin{cases} k : A(P) \rightarrow \mathcal{D} \\ \tilde{k} : \tilde{A}(P) \rightarrow \mathcal{D} \end{cases} \quad \text{turn pasting diagram of shape } P \text{ in } \mathcal{D}$$

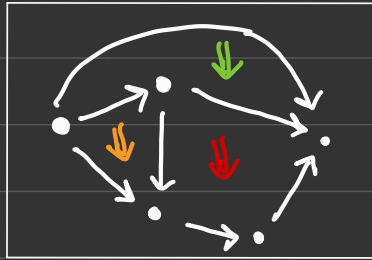
Pasting diagrams in a 2-category \mathcal{D}

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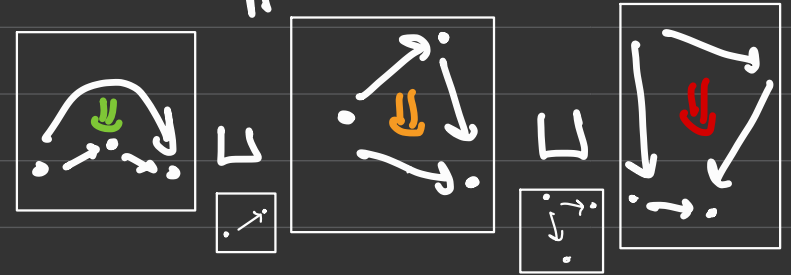
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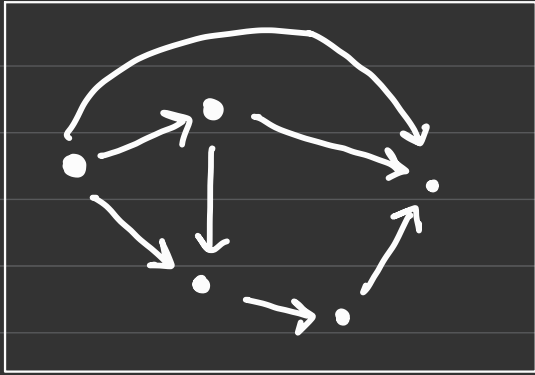


2-category encoding
elementary pieces of P

$$\mapsto \begin{cases} k : A(P) \rightarrow \mathcal{D} \leftarrow \text{pasting diagram of shape } P \text{ in } \mathcal{D} \\ \tilde{k} : \tilde{A}(P) \rightarrow \mathcal{D} \leftarrow \text{pasting diagram of shape } P \text{ in } \mathcal{D} \end{cases}$$

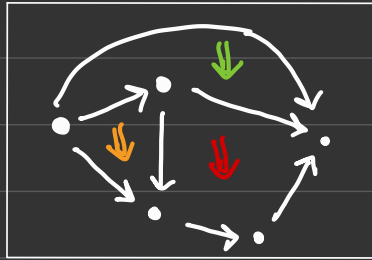
Pasting diagrams in a 2-category \mathcal{D}

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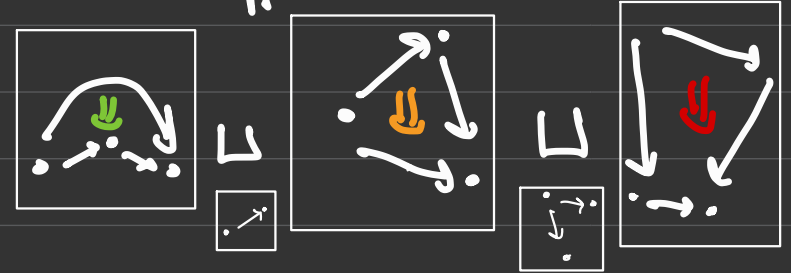
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free 2-category
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2-category encoding
elementary pieces of P

$\mapsto \begin{cases} K : A(P) \rightarrow \mathcal{D} \leftarrow \text{pasting diagram of shape } P \text{ in } \mathcal{D} \\ \tilde{K} : \tilde{A}(P) \rightarrow \mathcal{D} \leftarrow \text{pasting diagram of shape } P \text{ in } \mathcal{D} \\ \text{with specified composite(s)} \end{cases}$

Pasting Theorem for 2-categories

Power ~1990

Thm:

Idea



Pasting Theorem for 2-categories Power ~1990

Thm: A pasting diagram in a 2-category has a unique total composite

Idea



Pasting Theorem for 2-categories Power ~1990

Thm: A pasting diagram k in a 2-category \mathcal{D} has a unique total composite

Idea



Pasting Theorem for 2-categories Power ~1990

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Idea $A(P) \rightarrow \tilde{A}(P)$



Pasting Theorem for 2-categories Power ~1990

Thm: A pasting diagram k in a 2-category \mathcal{D} has a unique total composite

Idea $A(P) \rightarrow \tilde{A}(P)$

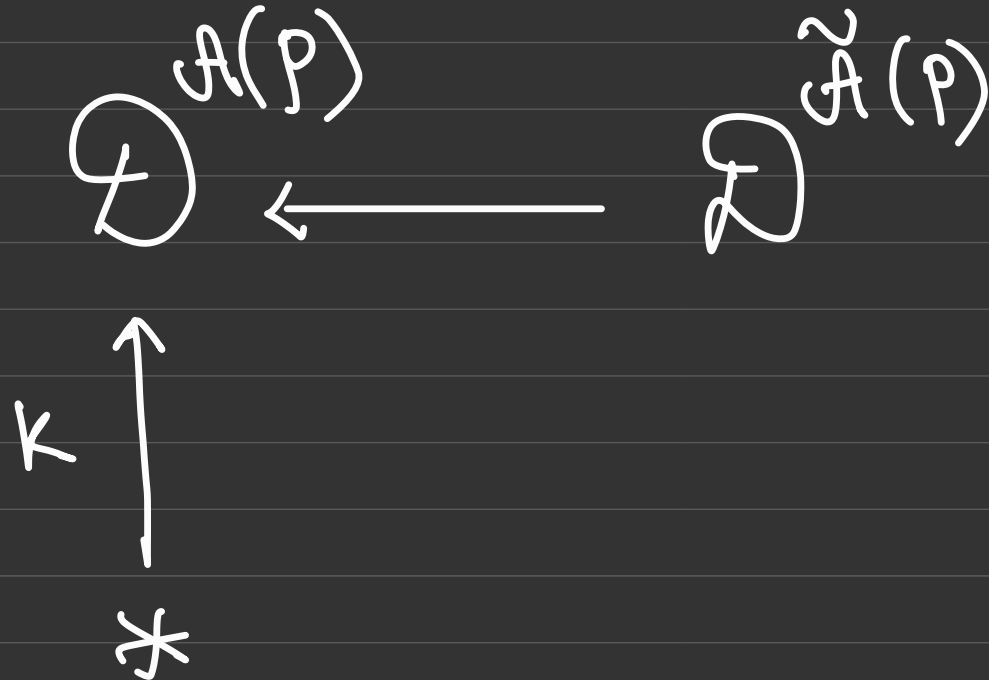
$$\mathcal{D}^{A(P)} \longleftarrow \mathcal{D}^{\tilde{A}(P)}$$



Pasting Theorem for 2-categories Power ~1990

Thm: A pasting diagram k in a 2-category \mathcal{D} has a unique total composite

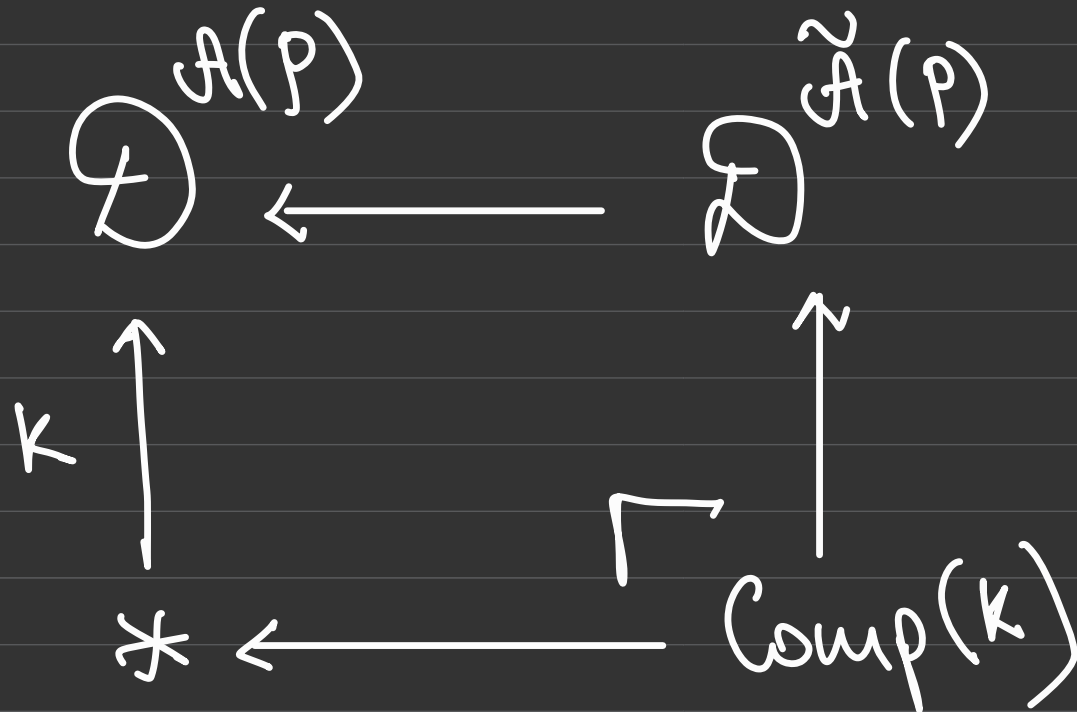
Idea $A(P) \rightarrow \tilde{A}(P)$



Pasting Theorem for 2-categories Power ~1990

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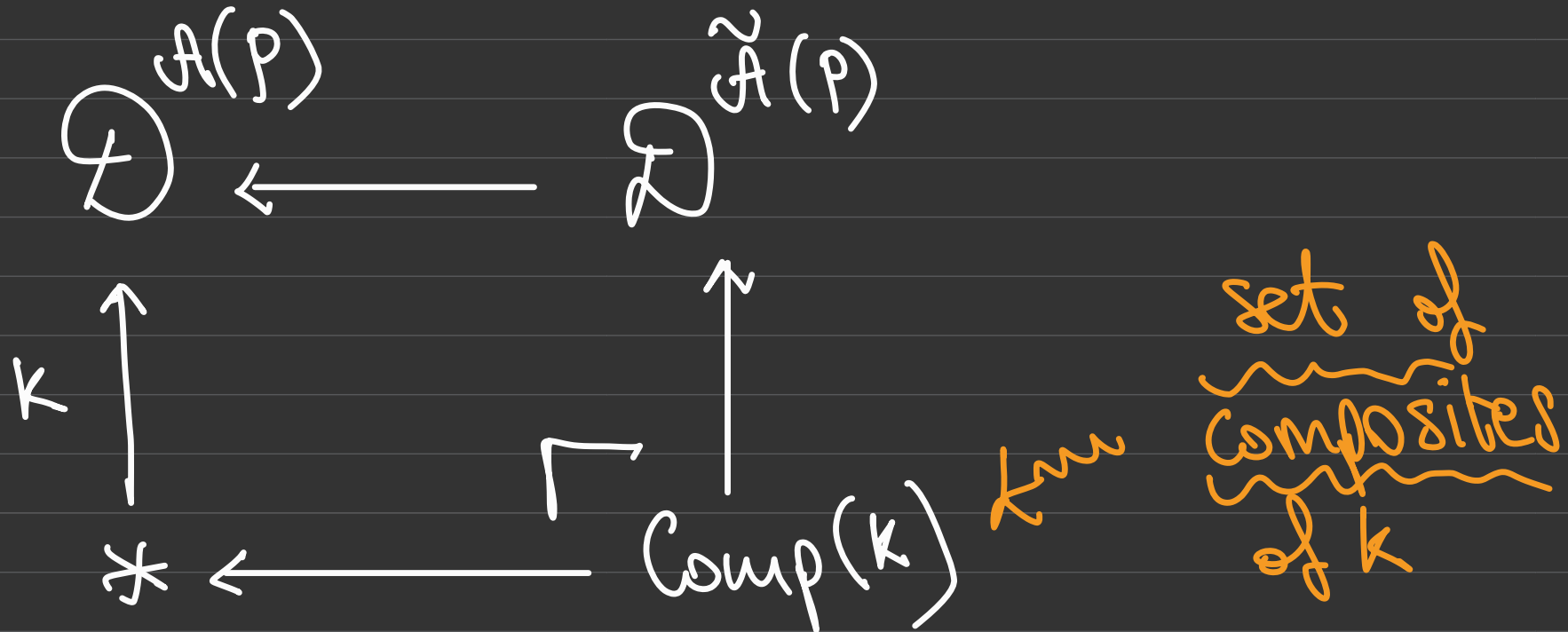
Idea $A(P) \rightarrow \tilde{A}(P)$



Pasting Theorem for 2-categories Power ~1990

Thm: A pasting diagram k in a 2-category \mathcal{D} has a unique total composite

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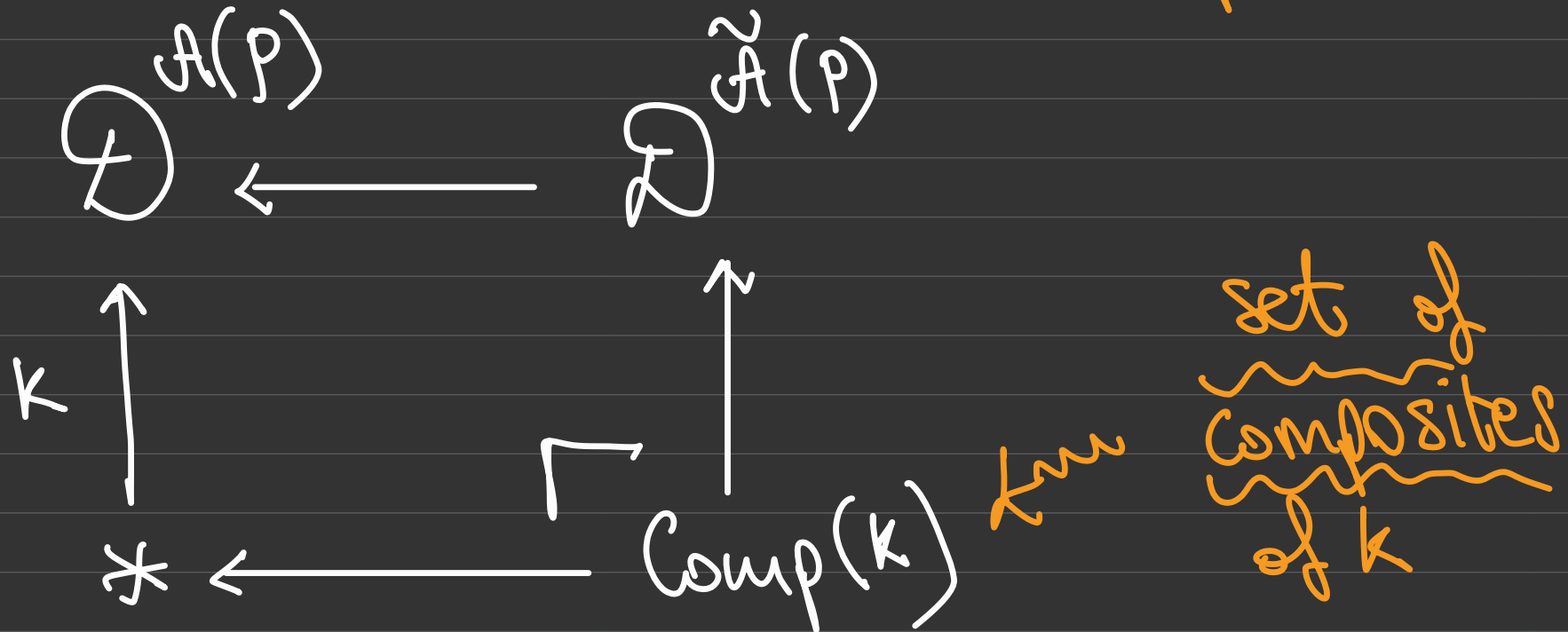


Pasting Theorem for 2-categories

Power ~1990

Thm: A pasting diagram k in a 2-category \mathcal{D} has a unique total composite

Idea $A(P) \xrightarrow{\cong} \tilde{A}(P)$ \leftarrow Problem encoded into shapes!

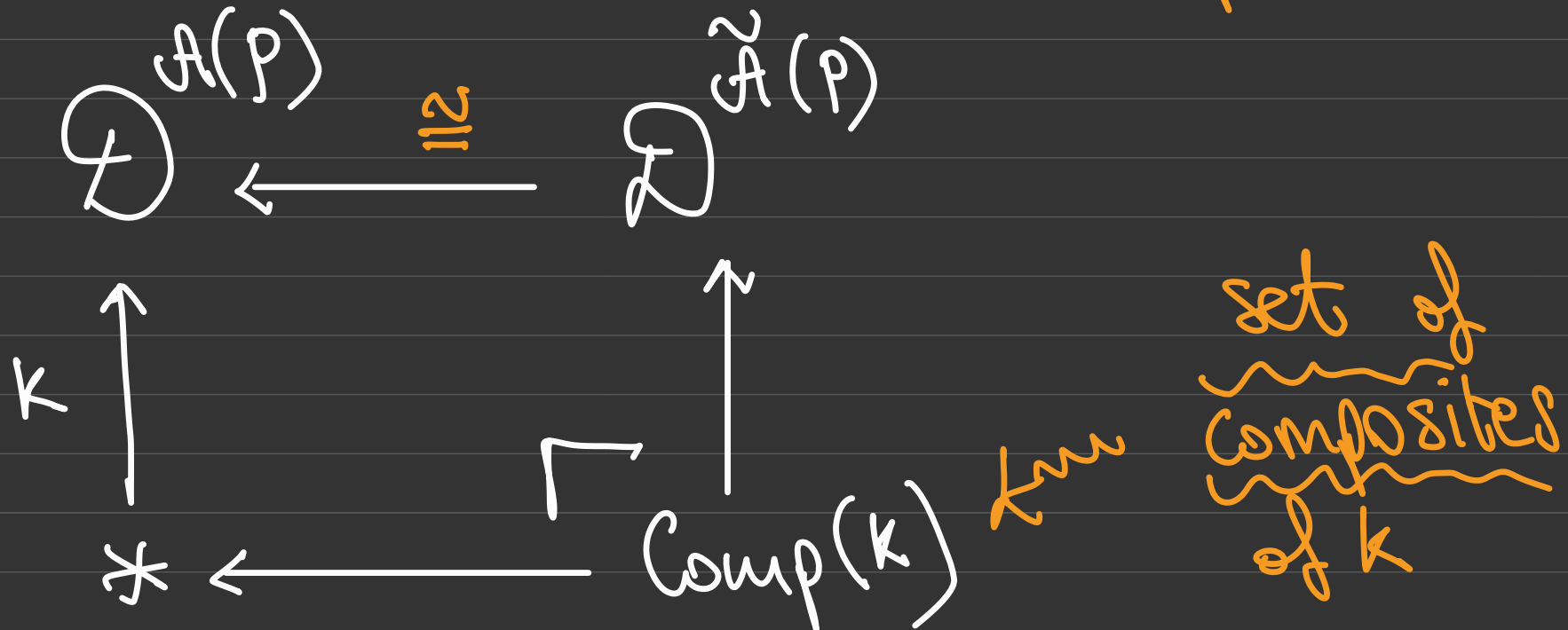


Pasting Theorem for 2-categories

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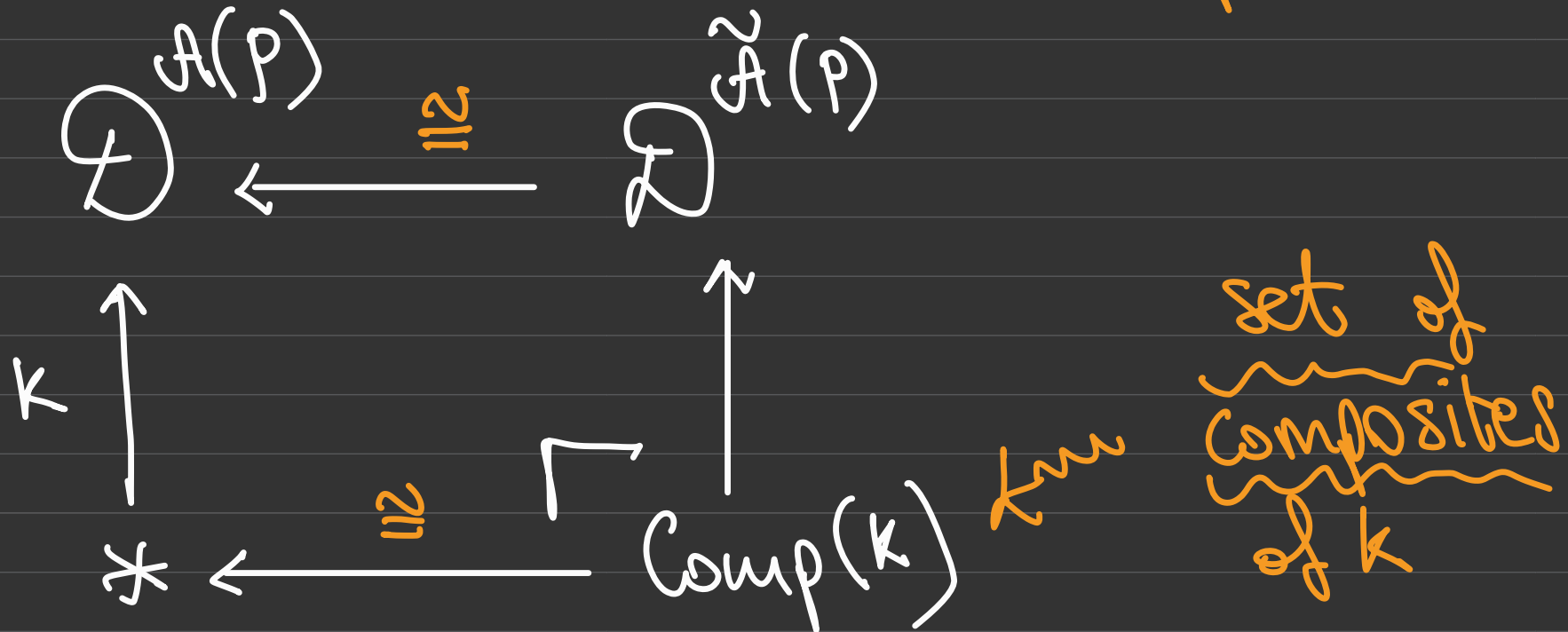


Pasting Theorem for 2-categories

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$(\infty, 2)$ -categories

$(\infty, 2)$ -categories

2-category

$(\infty, 2)$ -categories

2-category = category enriched over categories

$(\infty, 2)$ -categories

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Consists of " -

-

-

$(\infty, 2)$ -categories

2-category = category enriched over categories

Consists of ∞ - objects

-

-

$(\infty, 2)$ -categories

2-category = category enriched over categories

Consists of ∞ - objects A, B, \dots

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$(\infty, 2)$ -categories

2-category = category enriched over categories

Consists of ∞ - objects A, B, \dots

- hom categories

-

$(\infty, 2)$ -categories

2-category \mathcal{D} = category enriched over categories

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-

$(\infty, 2)$ -categories

2-category \mathcal{D} = category enriched over categories

Consists of ∞ - objects A, B, \dots
- hom categories $\mathcal{D}(A, B)$

-

$(\infty, 2)$ -categories

2-category \mathcal{D} = category enriched over categories

Consists of ∞ - objects A, B, \dots
- hom categories $\mathcal{D}(A, B)$
- composition functors

$(\infty, 2)$ -categories

2-category \mathcal{D} = category enriched over categories

Consists of ∞ - objects A, B, \dots

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$$\mathcal{D}(A, B)$$

- composition functors

$$\mathcal{D}(A, B) \times \mathcal{D}(B, C)$$

$(\infty, 2)$ -categories

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$$\mathcal{D}(A, B) \times \mathcal{D}(B, C) \rightarrow \mathcal{D}(A, C)$$

Nerve construction

$(\infty, 2)$ -categories

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$$\mathcal{D}(A, B)$$

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Nerve construction $N_{\mathcal{D}}: 2\text{Cat} \rightarrow (\infty, 2)\text{Cat}$

$(\infty, 2)$ -categories

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Nerve construction $N_{\mathcal{D}}: 2\text{Cat} \rightarrow (\infty, 2)\text{Cat} \quad \mathcal{D} \mapsto$

$(\infty, 2)$ -categories

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Consists of ∞ - objects A, B, \dots

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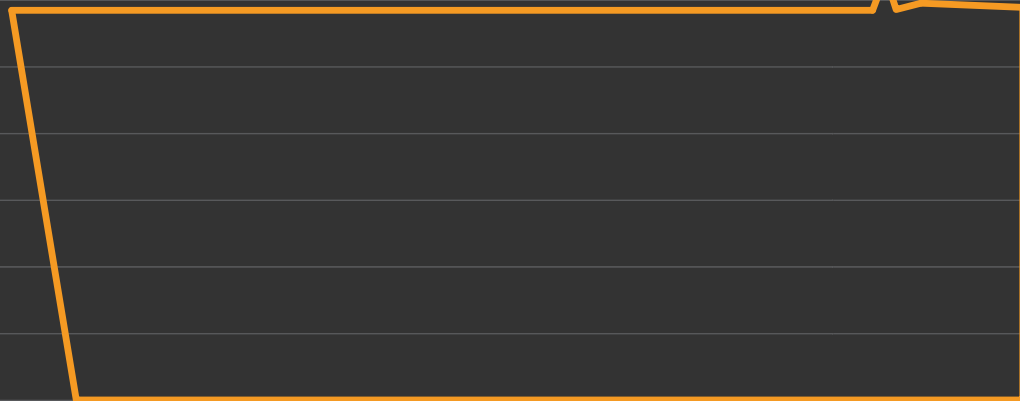
- composition ∞ -functors

$$\mathcal{D}(A, B)$$

$$\mathcal{D}(A, B) \times \mathcal{D}(B, C) \rightarrow \mathcal{D}(A, C)$$

Nerve construction

$$N_{\sharp}: 2\text{Cat} \rightarrow (\infty, 2)\text{Cat} \quad \mathcal{D} \mapsto N_{\sharp} \mathcal{D}$$



$(\infty, 2)$ -categories

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$$\mathcal{D}(A, B) \times \mathcal{D}(B, C) \rightarrow \mathcal{D}(A, C)$$

Nerve construction

$$N_*: 2\text{Cat} \rightarrow (\infty, 2)\text{Cat} \quad \mathcal{D} \mapsto N_* \mathcal{D}$$

$$\text{Ob}(N_* \mathcal{D}) =$$

$$(N_* \mathcal{D})(A, B) =$$

$(\infty, 2)$ -categories

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Warning: The nerve

-

-

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Warning: The nerve

- preserves limits

-

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$(\infty, 2)$ -categories

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Nerve construction

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Warning: The nerve

- preserves limits

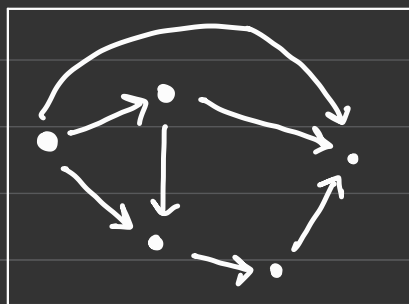
- does not preserve colimits

$$\text{Ob}(N_* \mathcal{D}) = \text{Ob}(\mathcal{D})$$

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Pasting diagrams in an $(\infty, 2)$ -category \mathcal{D}

\mathcal{P}



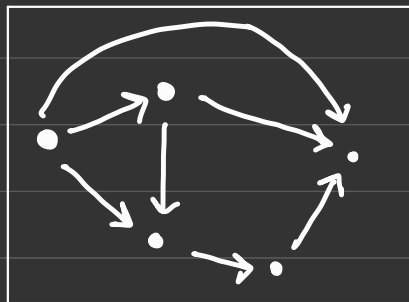
pasting
scheme

$\{$

Pasting diagrams in an $(\infty, 2)$ -category \mathcal{D}

\mathcal{P}

$\tilde{A}^\infty(\mathcal{P})$



pasting
scheme

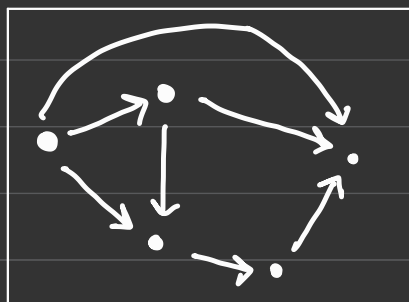
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Pasting diagrams in an $(\infty, 2)$ -category \mathcal{D}

\mathcal{P}

$\tilde{A}^\infty(\mathcal{P})$

$A^\infty(\mathcal{P})$



pasting
scheme

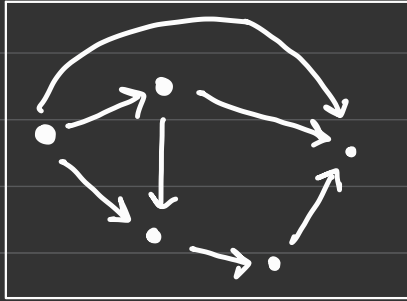
$\rightsquigarrow \left\{ \right.$

Pasting diagrams in an $(\infty, 2)$ -category \mathcal{D}

\mathcal{P}

$\tilde{A}^\infty(\mathcal{P})$

$A^\infty(\mathcal{P})$



pasting
scheme

\ll glue before
nerve \gg

\ll glue after
nerve \gg

$\rightsquigarrow \left\{ \right.$

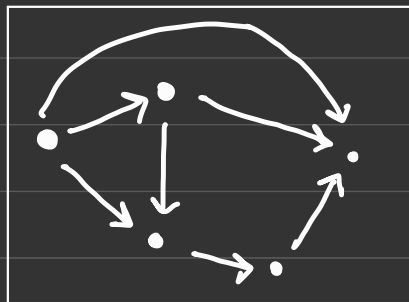
Pasting diagrams in an $(\infty, 2)$ -category \mathcal{D}

\mathcal{P}

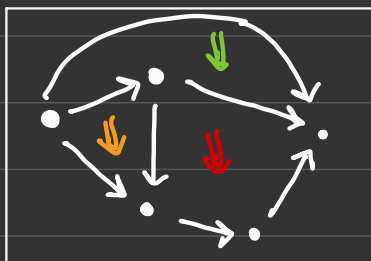
$\tilde{A}^\infty(\mathcal{P}) :=$

$\tilde{A}(\mathcal{P})$
 \equiv

$A^\infty(\mathcal{P})$



pasting
scheme



\ll glue before
nerve \gg

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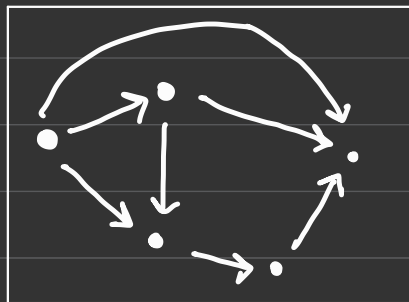
$\rightsquigarrow \left\{ \right.$

Pasting diagrams in an $(\infty, 2)$ -category \mathcal{D}

\mathcal{P}

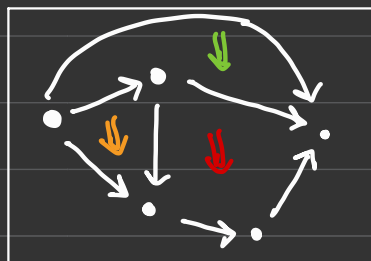
$$\tilde{A}^\infty(\mathcal{P}) := N_*(\tilde{A}(\mathcal{P}))$$

$$A^\infty(\mathcal{P})$$



pasting
scheme

N_*



\ll glue before
nerve \gg

\ll glue after
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\rightsquigarrow {

Pasting diagrams in an $(\infty, 2)$ -category \mathcal{D}

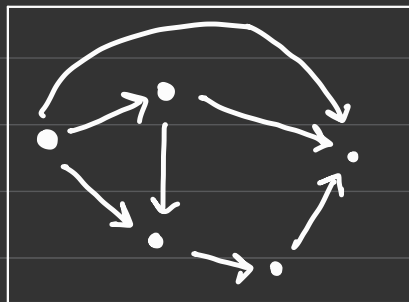
\mathcal{P}

$$\tilde{A}^\infty(\mathcal{P}) := N_* (\tilde{A}(\mathcal{P}))$$

=

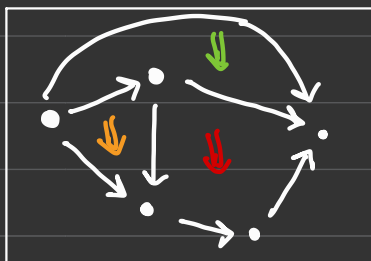
$$A^\infty(\mathcal{P})$$

=



pasting
scheme

N_*



« glue before
nerve »

N_*

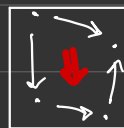


\sqcup



\sqcup

N_*



« glue after
nerve »

↪ {

Pasting diagrams in an $(\infty, 2)$ -category \mathcal{D}

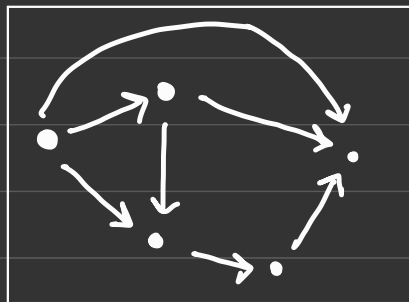
\mathcal{P}

$$\tilde{A}^\infty(\mathcal{P}) := N_*(\tilde{A}(\mathcal{P}))$$

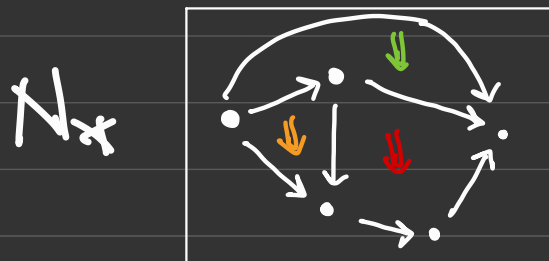
=

$$A^\infty(\mathcal{P})$$

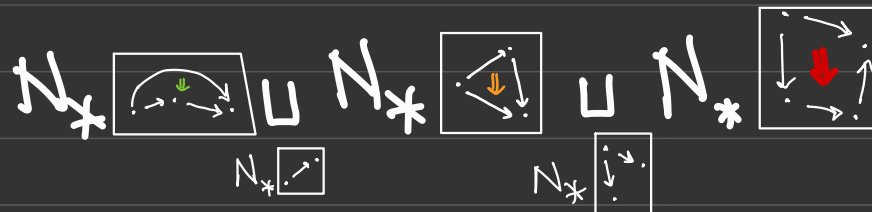
=



pasting
scheme



« glue before
nerve »



« glue after
nerve »

↪ {

Pasting diagrams in an $(\infty, 2)$ -category \mathcal{D}

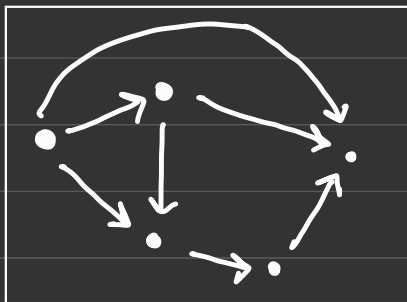
\mathcal{P}

$$\tilde{A}^\infty(\mathcal{P}) := N_*(\tilde{A}(\mathcal{P}))$$

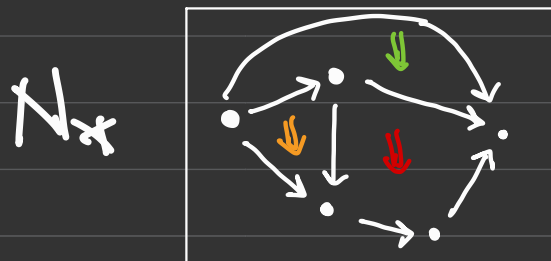
=

$$A^\infty(\mathcal{P})$$

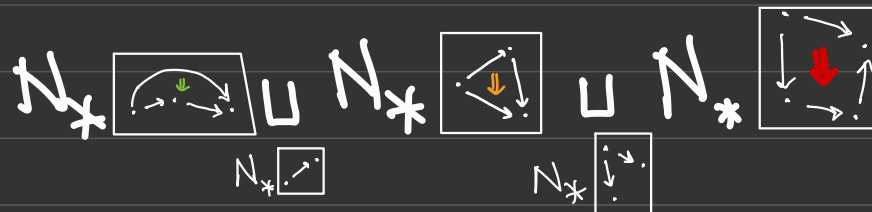
=



pasting
scheme



« glue before
nerve »



« glue after
nerve »

$$\rightsquigarrow \begin{cases} k : A^\infty(\mathcal{P}) \rightarrow \mathcal{D} \\ \tilde{k} : \tilde{A}^\infty(\mathcal{P}) \rightarrow \mathcal{D} \end{cases}$$

Pasting diagrams in an $(\infty, 2)$ -category \mathcal{D}

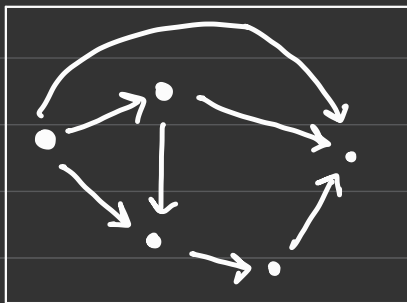
P

$$\tilde{A}^\infty(P) := N_* (\tilde{A}(P))$$

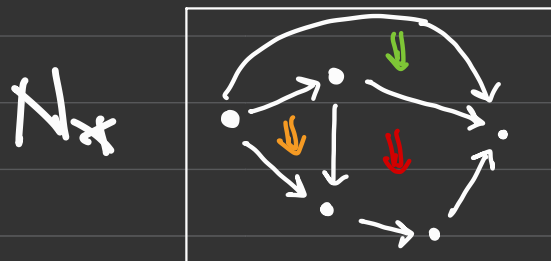
=

$$A^\infty(P)$$

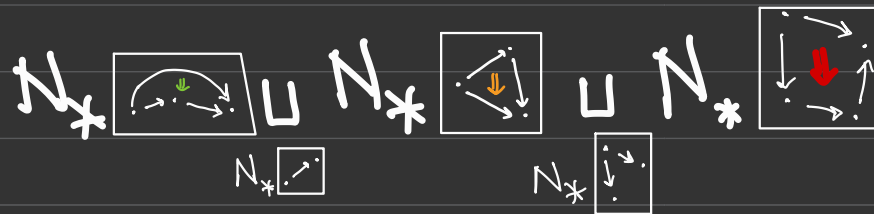
=



pasting
scheme



« glue before
nerve »



« glue after
nerve »

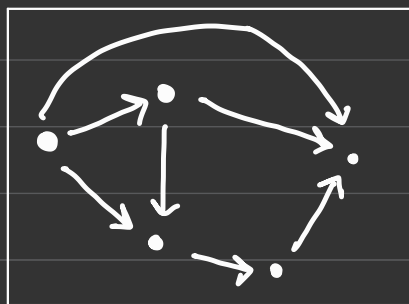
$$\rightsquigarrow \begin{cases} K : A^\infty(P) \rightarrow \mathcal{D} \\ \tilde{K} : \tilde{A}^\infty(P) \rightarrow \mathcal{D} \end{cases} \rightsquigarrow \text{pasting diagram of shape } P \text{ in } \mathcal{D}$$

Pasting diagrams in an $(\infty, 2)$ -category \mathcal{D}

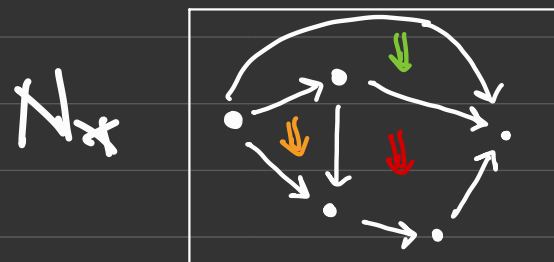
\mathcal{P}

$$\tilde{A}^\infty(\mathcal{P}) := N_*(\tilde{A}(\mathcal{P}))$$

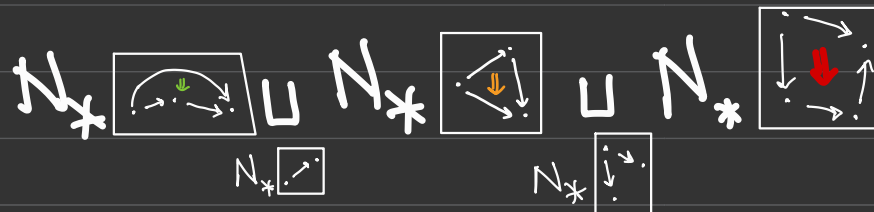
$$A^\infty(\mathcal{P})$$



pasting
scheme



« glue before
nerve »



« glue after
nerve »

$\begin{cases} K : A^\infty(\mathcal{P}) \rightarrow \mathcal{D} \leftarrow \text{pasting diagram of shape } \mathcal{P} \text{ in } \mathcal{D} \\ \tilde{K} : \tilde{A}^\infty(\mathcal{P}) \rightarrow \mathcal{D} \leftarrow \text{pasting diagram of shape } \mathcal{P} \text{ in } \mathcal{D} \\ \text{with specified composite(s)} \end{cases}$

Pasting Theorem for $(\infty, 2)$ -categories

Hackney-Ozornova
Riehl-R.

~ 2021

Thm:

Idea



Pasting Theorem for $(\infty, 2)$ -categories

Hackney-Ozornova ~ 2021
Riehl-R.
Columbus ~ 2017

Thm:

Idea



Pasting Theorem for $(\infty, 2)$ -categories

Hackney-Ozornova ~ 2021
Riehl-R.
Columbus ~ 2017

Thm: A pasting diagram in an $(\infty, 2)$ -category
has a contractible space of composites

Idea



Pasting Theorem for $(\infty, 2)$ -categories

Hackney-Ozornova ~ 2021
Riehl-R.
Columbus ~ 2017

Thm: A pasting diagram k in an $(\infty, 2)$ -category \mathcal{C} has a contractible space of composites

Idea



Pasting Theorem for $(\infty, 2)$ -categories

Hackney-Ozornova ~ 2021
Riehl-R.
Columbus ~ 2017

Thm: A pasting diagram k in an $(\infty, 2)$ -category \mathcal{C} has a contractible space of composites

Idea $A^\infty(P) \hookrightarrow \tilde{A}^\infty(P)$



Pasting Theorem for $(\infty, 2)$ -categories

Hackney-Ozornova ~ 2021
Riehl-R.
Columbus ~ 2017

Thm: A pasting diagram k in an $(\infty, 2)$ -category \mathcal{D} has a contractible space of composites

Idea $A^\infty(P) \hookrightarrow \tilde{A}^\infty(P)$

$$\mathcal{D}^{A^\infty(P)} \longleftarrow \mathcal{D}^{\tilde{A}^\infty(P)}$$

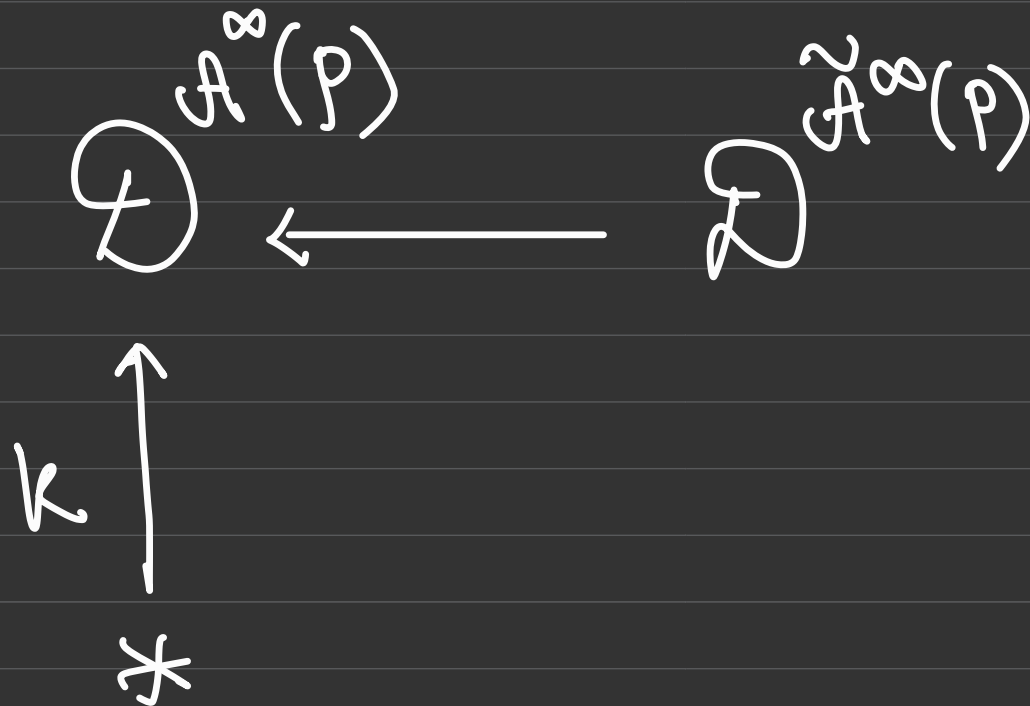


Pasting Theorem for $(\infty, 2)$ -categories

Hackney-Ozornova ~ 2021
Riehl-R.
Columbus ~ 2017

Thm: A pasting diagram k in an $(\infty, 2)$ -category \mathcal{D} has a contractible space of composites

Idea $A^\infty(P) \hookrightarrow \tilde{A}^\infty(P)$

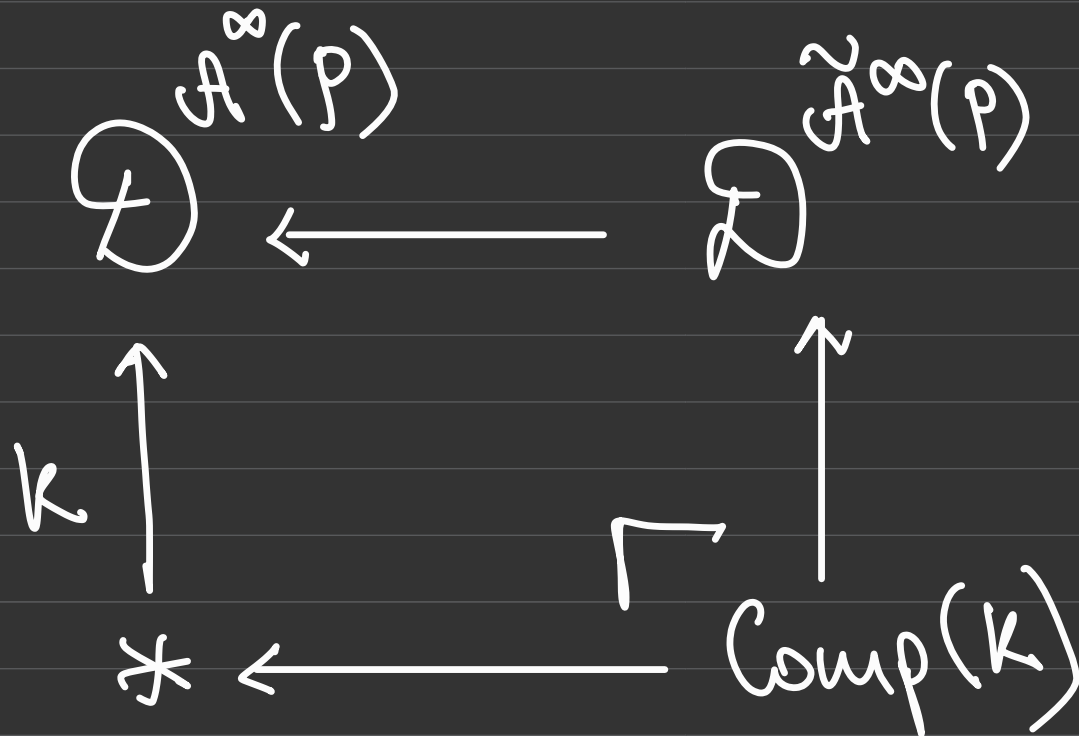


Pasting Theorem for $(\infty, 2)$ -categories

Hackney-Ozornova ~ 2021
Riehl-R.
Columbus ~ 2017

Thm: A pasting diagram k in an $(\infty, 2)$ -category \mathcal{D} has a contractible space of composites

Idea $A^\infty(P) \hookrightarrow \tilde{A}^\infty(P)$

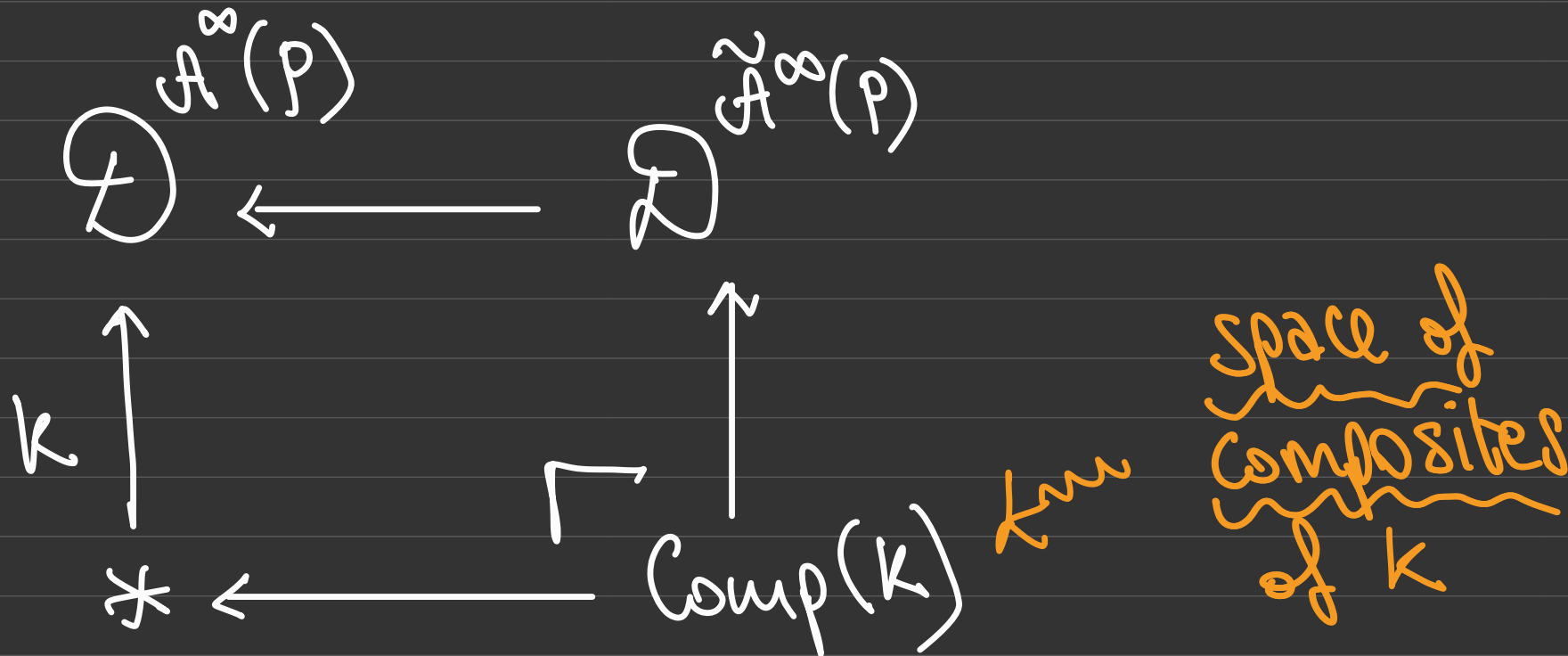


Pasting Theorem for $(\infty, 2)$ -categories

Hackney-Ozornova ~ 2021
Riehl-R.
Columbus ~ 2017

Thm: A pasting diagram k in an $(\infty, 2)$ -category \mathcal{D} has a contractible space of composites

Idea $A^\infty(P) \hookrightarrow \tilde{A}^\infty(P)$

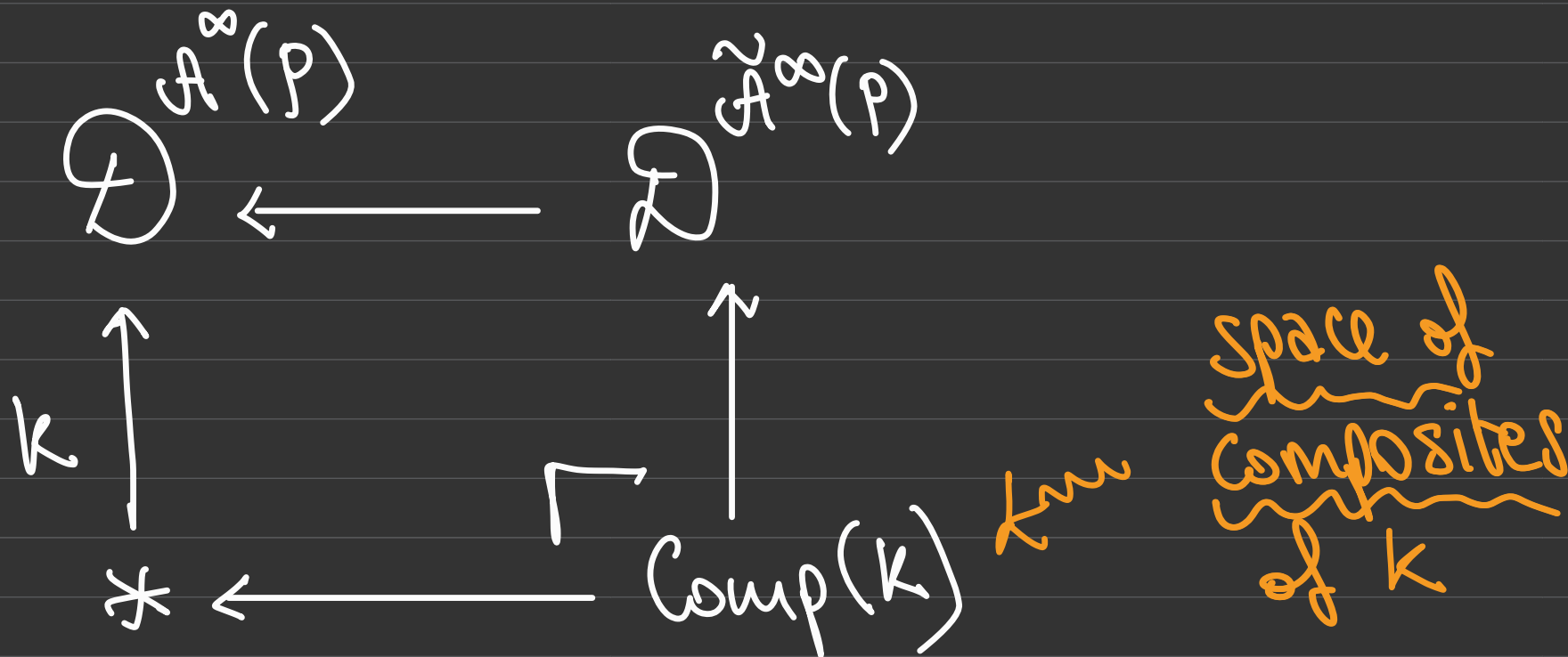


Pasting Theorem for $(\infty, 2)$ -categories

Hackney-Ozornova ~ 2021
Riehl-R.
Columbus ~ 2017

Thm: A pasting diagram k in an $(\infty, 2)$ -category \mathcal{D} has a contractible space of composites

Idea $A^\infty(P) \xrightarrow{\sim} \tilde{A}^\infty(P)$ \hookrightarrow Main ingredient



Pasting Theorem for $(\infty, 2)$ -categories

Hackney-Ozornova ~ 2021
Riehl-R.
Columbus ~ 2017

Thm: A pasting diagram k in an $(\infty, 2)$ -category \mathcal{D} has a contractible space of composites

Idea $A^\infty(P) \xrightarrow{\sim} \tilde{A}^\infty(P)$

\leadsto Main ingredient

$$\begin{array}{ccc} A^\infty(P) & \xleftarrow{\sim} & \tilde{A}^\infty(P) \\ \uparrow k & & \uparrow \\ * & \xleftarrow{\quad} & \text{Comp}(k) \end{array}$$

space of
composites
of k



Pasting Theorem for $(\infty, 2)$ -categories

Hackney-Ozornova ~ 2021
Riehl-R.
Columbus ~ 2017

Thm: A pasting diagram k in an $(\infty, 2)$ -category \mathcal{D} has a contractible space of composites

Idea $A^\infty(P) \xrightarrow{\sim} \tilde{A}^\infty(P)$

\leadsto Main ingredient

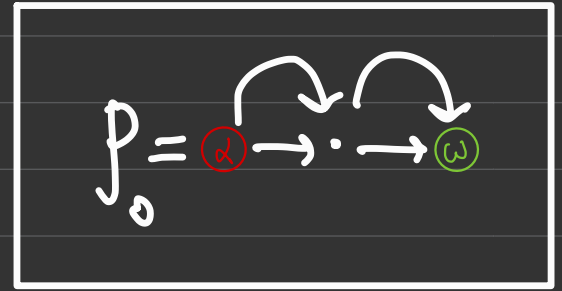
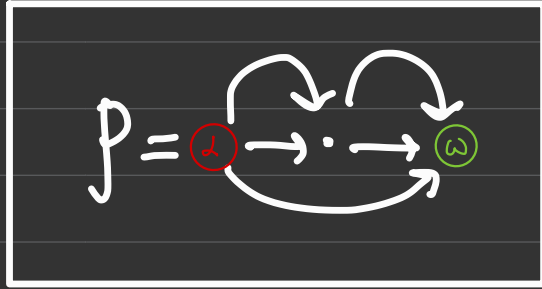
$$\begin{array}{ccc} A^\infty(P) & & \tilde{A}^\infty(P) \\ \uparrow k & \xleftarrow{\sim} & \uparrow \\ * & \xleftarrow{\sim} & \text{Comp}(k) \end{array}$$

space of
composites
of k



The main ingredient Thm $\tilde{A}^\infty(P) \xrightarrow{\sim} \tilde{\tilde{A}}^\infty(P)$ in $(\infty, 1)\mathbf{Cat}$

Idea Induction on
regions of P

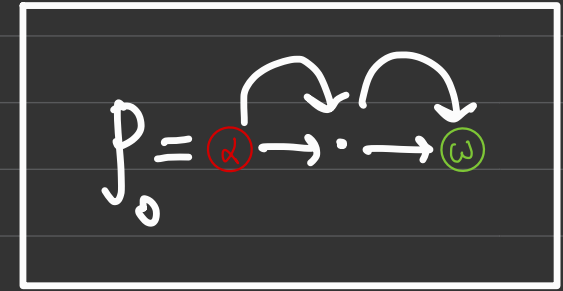
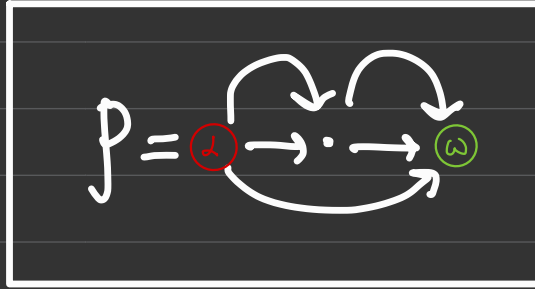


$$\begin{array}{ccccc}
 \tilde{A}^\infty P & \hookrightarrow & \tilde{A}^\infty P \sqcup \tilde{\tilde{A}}^\infty P_0 =: Q & \longrightarrow & \tilde{\tilde{A}}^\infty P \\
 \uparrow & & \downarrow \scriptstyle \tilde{A}^\infty P_0 & & \\
 \tilde{A}^\infty P_0 & \longrightarrow & \tilde{\tilde{A}}^\infty P_0 & &
 \end{array}$$



The main ingredient Thm $\tilde{A}^\infty(P) \xrightarrow{\sim} \tilde{\tilde{A}}^\infty(P)$ in $(\infty, 1)\text{Cat}$

Idea Induction on
regions of P

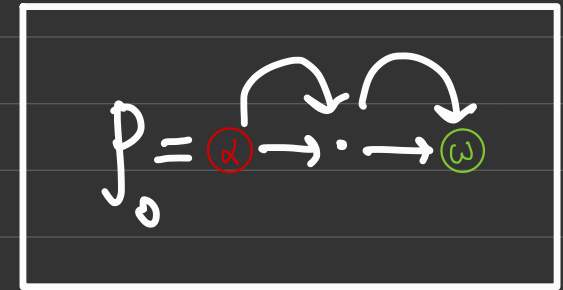
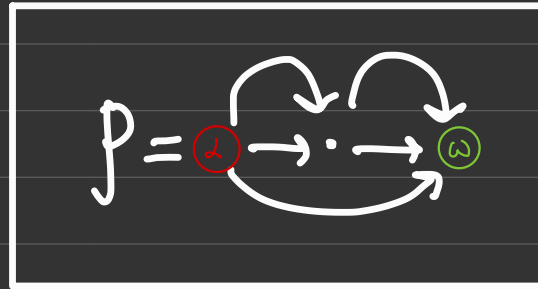


$$\begin{array}{ccccc}
 \tilde{A}^\infty P & \xrightarrow{\sim} & \tilde{A}^\infty P \sqcup \tilde{\tilde{A}}^\infty P_0 =: Q & \longrightarrow & \tilde{\tilde{A}}^\infty P \\
 \uparrow & & \downarrow \scriptstyle \tilde{A}^\infty P_0 & & \\
 & \nearrow & \uparrow & & \\
 \tilde{A}^\infty P_0 & \xrightarrow{\sim} & \tilde{\tilde{A}}^\infty P_0 & & \\
 & \nwarrow & \uparrow & & \\
 & & \text{induction hp} & &
 \end{array}$$



The main ingredient Thm $\tilde{A}(P) \xrightarrow{\cong} \tilde{A}^\infty(P)$ in $(\infty, 1)\text{Cat}$

Idea Induction on
regions of P



$$\begin{array}{ccccc}
 \tilde{A}^\infty P & \xrightarrow{\cong} & \tilde{A}^\infty P \sqcup \tilde{A}^\infty P_0 =: Q & \xrightarrow{\cong} & \tilde{A}^\infty P \\
 \uparrow & & \uparrow \scriptstyle \tilde{A}^\infty P_0 & & \\
 & \nearrow & & \nearrow & \\
 \tilde{A}^\infty P_0 & \xrightarrow{\cong} & \tilde{A}^\infty P_0 & & \\
 \uparrow & & \uparrow & & \\
 N\tilde{A}^\infty P_0(\alpha, \omega) \sqcup N[1]_{N[0]} & \xrightarrow{\cong} & N[\tilde{A}^\infty P_0(\alpha, \omega) \sqcup [1]_{[0]}] & &
 \end{array}$$

Intermediate maps and objects in the diagram:

- $Q(\alpha, \omega) \xrightarrow{\cong} \tilde{A}^\infty P(\alpha, \omega)$
- $N\tilde{A}^\infty P_0(\alpha, \omega) \sqcup N[1]_{N[0]} \xrightarrow{\cong} N[\tilde{A}^\infty P_0(\alpha, \omega) \sqcup [1]_{[0]}]$
- $\tilde{A}^\infty P_0 \xrightarrow{\cong} \tilde{A}^\infty P_0$ (identity)
- $\tilde{A}^\infty P \xrightarrow{\cong} \tilde{A}^\infty P$ (identity)

induction hp

Thm [HOLL] Nerve is pushout of Dwyer maps □

— THANK YOU! —