Limits in (∞, n)-Categories

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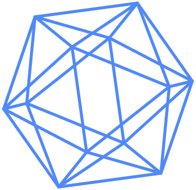


07.07.2023

Upshot

We define **weighted limits** in non-strict models of (∞, n)-**categories** via **double** $(\infty, n-1)$-**categories of cones**, such that

1. It is computationally feasible using fibrational methods.
2. It compares correctly to limits in strict models.

As part of this we introduced nerves and straightening for (∞, n)-categories, which can have many further applications.

Theorem (Upshot of the upshot)

There is a correct, computationally meaningful and comparable to strict definition of limits in (∞, n)-categories that we introduce and study using (∞, n)-homotopical nerves and (∞, n)-straightening!

Limits are everywhere

Even binary coproducts are amazing:

1. Disjoint union of topological spaces.
2. Wedge sum of pointed sets.
3. Free product of groups.
4. Direct product of abelian groups.
5. Tensor product of commutative rings.
6. Union of two open subsets of a topological space.

Sometimes categories aren't structured enough ...

Example

The category of categories does not suffice. We need natural transformations for adjunctions, equivalences, ...

Example

Loop spaces of topological spaces have a natural composition, which is only unital and associative up to homotopy.

... enter higher categories

The solution: Various forms of higher categories, which provide further structure to our categories.

Solution

Categories form a 2-**category**, with (categories, functors, natural transformations). Adjunctions can be defined via 2-morphisms.

Solution

Spaces form a **weak** 1-**category** or $(\infty, 1)$-**category** given by (spaces, continuous functions, homotopies). Loop spaces are a group object there.

$\Rightarrow$ Combining these two gives us weak *n*-categories, or (∞, n)-**categories**.

But do we still need limits?

Yes!

Geometry	Logic
$\Downarrow$	$\Downarrow$
6 – Functor Formalism, Correspondences	Directed Type Theory, Higher n – Topos Theory
$\Downarrow$	$\Downarrow$
Gaitsgory–Rozenblyum, Scholze, Mann	Riehl–Shulman, Buchholtz–Weinberger, Cisinski–Nguyen–Walde
⋮	⋮

"Higher" Limits: Examples

"Pullback" of categories:

$$\begin{array}{ccc} F \downarrow G & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow G \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{D} \end{array}$$

$\Longrightarrow$ Partially lax pullback

"Pullback" of spaces:

$$\begin{array}{ccc} \Omega_x X & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ * & \xrightarrow[(x,x)]{} & X \times X \end{array}$$

$\Longrightarrow$ Homotopy pullback

$\Longrightarrow$ Both are **weighted limits**.

Weights

Weights for a given diagram I are functors $I \to \mathcal{V}$ and provide use with an "instruction manual" how to adjust the universal property, meaning they give us a *universal property of universal properties*.

Enrichment	**Theory**	**Weighted Limits**
$\mathcal{S}\mathrm{et}$	Categories	Limits
$\mathcal{C}\mathrm{at}$	2-Categories	partially lax limits
$\mathcal{K}\mathrm{an}$	$(\infty, 1)$-Categories	Homotopy limits
$\mathcal{QC}\mathrm{at}$	$(\infty, 2)$-Categories	partially lax homotopy limits
$\mathcal{C}\mathrm{at}_{(\infty,n-1)}$	(∞, n)-Categories	partially lax n-homotopy limits

All's well that ends well?

So, are we done? Nope!

$\mathcal{C}\mathrm{at}_{(\infty,n)}$

Strict:
Enriched categories
Relative categories

⋮

⇑

Limits for all n defined only here

Non-strict:
$n-$ Quasi-categories
$n-$ fold complete Segal spaces
complete Segal Θ_n- spaces
$n-$ complicial sets

⋮

⇑

Limits defined for n=1

Joyal's Insight

Non-strict models of $(\infty, 1)$-categories, such as quasi-categories, only satisfy some lifting conditions, making any notion of functor impossible to use.

Exercise

A functor $F : I \to \mathcal{C}$ has a limit L, if there is an isomorphism of categories

$$\begin{array}{ccc} \mathcal{C}_{/L} & \xrightarrow{\cong} & \mathcal{C}_{/F} \\ & \searrow \quad \swarrow & \\ & \mathcal{C} & \end{array} \quad \Leftarrow \text{(Category of cones)}$$

Easy math – deep insight!

Limits of $(\infty, 1)$-Categories via fibrations

We can use this method to define limits in all non-strict models of $(\infty, 1)$-categories by defining ∞-categorical cones.

Model	**Work**
Quasi-categories	Joyal, Lurie, Rovelli, ...
(complete) Segal spaces	R.
Segal categories, 1-complicial sets, ...	Riehl–Verity

In a way that compares appropriately to the strict case (Riehl–Verity, Rovelli).

Now all's well that ends well?

This suggests the following approach:

1. Define limits in strict n-categories via slices.
2. Identify the appropriate analogous (∞, n)-construction.
3. Voila! Definition of (∞, n)-limits.

clingman & Moser's Insight

Theorem (clingman–Moser[a])

This approach fails in every possible way imaginable for 2-categories.

Theorem (clingman–Moser[b], Grandis[c])

This approach does work for 2-categories if we generalize our slices to double categories.

[a] 2-limits and 2-terminal objects are too different. Applied Categorical Structures. 30 (2022), pp. 1283–1304.

[b] Bi-initial objects and bi-representations are not so different. Cahiers de Topologie et Géométrie Différentielle Catégoriques. Volume LXIII-3 (2022), pp. 259-330.

[c] Higher dimensional categories. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020. From double to multiple categories.

Now finally (hopefully) all's well that ends well?

This suggests the following updated approach:

1. Define limits in strict n-categories via double $(n-1)$-categorical slices.
2. Identify appropriate analogous (∞, n)-construction as a double $(\infty, n-1)$-category.
3. Voila! Definition of (∞, n)-limits.

Our insight!

Theorem (Moser – R – Rovelli)

This works!
We have a notion of limits for non-strict models of (∞, n)-categories via double $(\infty, n-1)$-categories that coincides with the limit for strict models.

Here double $(\infty, n-1)$-categories are an appropriately chosen generalization of double categories to weak n-categories.

Stop! Other work

For the specific case of $n = 2$, there is a different approach by Gagna–Harpaz–Lanari, which keeps the slices and changes the universality.[1]

[1] Gagna, Andrea and Harpaz, Yonatan and Lanari, Edoardo. Fibrations and lax limits of $(\infty, 2)$-categories. arXiv:2012.04537.

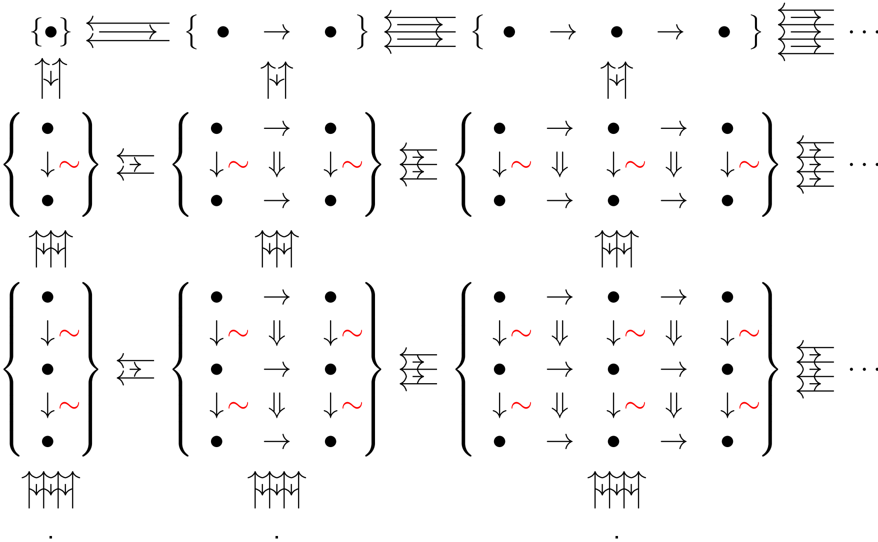
From the world of ideals into our world

How can we realize such a claim? First we need a *theory of fibrations of* (∞, n)*-categories*.

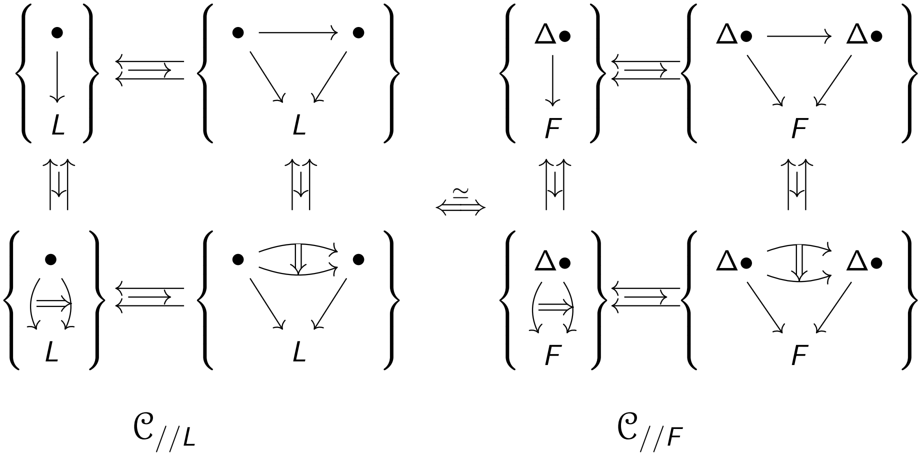
1. A definition of such fibration over a given (∞, n)-category $\mathcal{C}$.
2. A way to construct such fibrations $\mathcal{C}_{//F} \to \mathcal{C}$ out of a functor $F : \mathcal{D} \to \mathcal{C}$.
3. A representability result, giving us meaningful criteria when $\mathcal{C}_{//F}$ is representable i.e. a *Yoneda lemma*.

Turns out I introduced such fibrations with these properties. The paper is aptly called *Yoneda Lemma for* $\mathcal{D}$*-simplicial spaces* (arXiv:2108.06168).

Illustration of a double $(\infty, 1)$-category($(\infty, 2)$-Category)



L is the limit of $F : \mathcal{D} \to \mathcal{C}$



To limits and beyond

Fibrations enable us to define things, but not compare. For that we need way more math:

1. We need a homotopy coherent nerve of (∞, n)-categories, which gives us control over general diagrams.[2]
2. The ability to "unstraighten" general diagrams out of the categorification.[3]

So, our pursuit of limits has led to a deep study of (∞, n)-categories that is of independent interest.

[2] A homotopy coherent nerve for (∞, n)-categories, arXiv:2208.02745
[3] Soon!

Upshot Again

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Talk to us!

If you have any use for limits in (∞, n)-categories, talk to us!

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- Website: https://guests.mpim-bonn.mpg.de/rasekh/