Effective descent $\mathcal{V}$-functors

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CT2023 – Louvain-la-Neuve

July 7, 2023

Project

Can we provide criteria for a $\mathcal{V}$-functor to be an effective descent morphism in $\mathcal{V}$-Cat?

Goal: Provide an answer when $\mathcal{V}$ with finite limits for the cartesian monoidal structure.

Strategy

- Problem is solved for effective descent morphisms in $\mathsf{Cat}(\mathcal{C})$, when $\mathcal{C}$ has finite limits (Le Creurer, 1999);
- $\mathcal{V}$-$\mathsf{Cat} \to \mathsf{Cat}(\mathcal{V})$ reflects effective descent morphisms for suitable $\mathcal{V}$ (Lucatelli Nunes, 2018);
- If $\mathcal{V}$ has finite limits, the free coproduct completion $\mathsf{Fam}(\mathcal{V})$ is suitable (Carboni, Lack, Walters, 1993; Gray, 1966; Borceux, Janelidze 2001);
- Confirm whether $\mathcal{V}$-$\mathsf{Cat} \to \mathsf{Fam}(\mathcal{V})$-$\mathsf{Cat}$ reflects effective descent morphisms.

Bénabou-Roubaud, Beck

$\mathcal{C}$ category w/ finite limits, $p \colon E \to B$ morphism.

$$\mathcal{C}/B \underset{p^*}{\overset{p_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{C}/E$$

Theorem (Bénabou-Roubaud, 1970; Beck)

$$\mathsf{Desc}(p) \simeq T^p\text{-}\mathsf{Alg}$$

Effective descent morphisms

Eilenberg-Moore factorization of p^*:

$$\begin{array}{ccc} \mathcal{C}/B & \xrightarrow{\ \mathcal{K}^p\ } & \mathsf{Desc}(p) \\ & {\scriptstyle p^*}\searrow \quad \swarrow & \\ & \mathcal{C}/E & \end{array}$$

- p *effective descent morphism* $\iff$ $\mathcal{K}^p$ equivalence,
- p *descent morphism* $\iff$ $\mathcal{K}^p$ fully faithful,
- p *almost descent morphism* $\iff$ $\mathcal{K}^p$ faithful.

Effective descent internal functors

$\mathcal{V}$ category w/ finite limits.

Theorem (Le Creurer, 1999)

An internal $\mathcal{V}$-functor $F \colon \mathscr{C} \to \mathscr{D}$ such that

$$\begin{aligned} F_1 \colon \mathscr{C}_1 \to \mathscr{D}_1 &\quad \text{is an effective descent morphism,} \\ F_2 \colon \mathscr{C}_2 \to \mathscr{D}_2 &\quad \text{is a descent morphism,} \\ F_3 \colon \mathscr{C}_3 \to \mathscr{D}_3 &\quad \text{is an almost descent morphism} \end{aligned}$$

is an effective descent morphism in $\mathsf{Cat}(\mathcal{V})$.

$\mathcal{V}$ *extensive* if coproducts are *disjoint* + *universal*.

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & \ulcorner & \downarrow \\ B & \longrightarrow & A+B \end{array} \qquad \begin{array}{ccc} Y_j & \longrightarrow & Y \\ \downarrow & \ulcorner & \downarrow{\scriptstyle f} \\ X_j & \xrightarrow[\iota_j]{} & \sum_{j\in J} X_j \end{array}$$

Examples

Extensive categories

- Any Grothendieck topos (Set),
- Cat,
- Top.

Non-extensive categories

- Any meet semilattice,
- FinSet,
- CHaus,
- Stn.

Effective descent enriched functors

$\mathcal{V}$ extensive category w/ finite limits.

Theorem (Lucatelli Nunes, 2018)

If $- \cdot 1 \colon \mathsf{Set} \to \mathcal{V}$ is fully faithful, then

$$\mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Cat}(\mathcal{V})$$

reflects effective descent morphisms ($\mathcal{V}$ cartesian monoidal).

Free coproduct completion

Free coproduct completion of $\mathcal{V}$: $\mathsf{Fam}(\mathcal{V})$.

Objects: families $(X_j)_{j \in J}$.

Morphisms $(X_j)_{j \in J} \to (Y_k)_{k \in K}$: function $f\colon J \to K$, family $\phi = (\phi_j\colon X_j \to Y_{fj})_{j \in J}$.

Free coproduct completion

Lemma (Carboni, Lack, Walters, 1993)

$\mathsf{Fam}(\mathcal{V})$ extensive category.

Lemma (Gray, 1966)

$\mathcal{V}$ finite limits $\implies$ $\mathsf{Fam}(\mathcal{V})$ finite limits.

Lemma (Borceux, Janelidze, 2001)

$- \cdot 1 \colon \mathsf{Set} \to \mathsf{Fam}(\mathcal{V})$ fully faithful.

Corollary

$\mathsf{Fam}(\mathcal{V})\text{-}\mathsf{Cat} \to \mathsf{Cat}(\mathsf{Fam}(\mathcal{V}))$ reflects effective descent morphisms.

Effective descent morphisms in bilimits

Lemma (Lucatelli Nunes, 2018)

In a pseudopullback of categories

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle H} \\ \mathcal{C} & \xrightarrow[K]{} & \mathcal{D} \end{array}$$

a morphism f in $\mathcal{A}$ is effective for descent if

$$\begin{aligned} Ff \quad & \text{is an effective descent morphism,} \\ Gf \quad & \text{is an effective descent morphism,} \\ KGf \cong HFf \quad & \text{is a descent morphism.} \end{aligned}$$

2-cartesian unit

Lemma (Weber, 2007)

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\eta} & \mathsf{Fam}(\mathcal{V}) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \mathsf{Fam}(F)} \\ \mathcal{W} & \xrightarrow[\eta]{} & \mathsf{Fam}(\mathcal{W}) \end{array}$$

is a 2-pullback.

Enrichment 2-functor

Lemma (Fujii, Lack 2022)

The 2-functor $\mathsf{CartCat} \to \mathsf{CAT}$ given by $\mathcal{V} \mapsto \mathcal{V}\text{-}\mathsf{Cat}$ preserves pseudopullbacks.

Corollary

We have a pseudopullback

$$\begin{array}{ccc} \mathcal{V}\text{-}\mathsf{Cat} & \xrightarrow{\eta_!} & \mathsf{Fam}(\mathcal{V})\text{-}\mathsf{Cat} \\ \downarrow & & \downarrow \\ \mathsf{Set} & \longrightarrow & \mathsf{Set}\text{-}\mathsf{Cat} \end{array}$$

Lemma (P., 2023)

$\eta_! \colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Fam}(\mathcal{V})\text{-}\mathsf{Cat}$ reflects effective descent morphisms.

Main result

Theorem (P., 2023)

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a $\mathcal{V}$-functor. If

$$F\colon (\mathcal{C}(x_0, x_1))_{Fx_i = y_i} \to \mathcal{D}(y_0, y_1)$$

is an effective descent morphism,

$$F \times F\colon (\mathcal{C}(x_1, x_2) \times \mathcal{C}(x_0, x_1))_{Fx_i = y_i} \to \mathcal{D}(y_1, y_2) \times \mathcal{D}(y_0, y_1)$$

is a descent morphism, and

$$F \times F \times F\colon (\mathcal{C}(x_2, x_3) \times \mathcal{C}(x_1, x_2) \times \mathcal{C}(x_0, x_1))_{Fx_i = y_i} \to \mathcal{D}(y_2, y_3) \times \mathcal{D}(y_1, y_2) \times \mathcal{D}(y_0, y_1)$$

is an almost descent morphism, then F is effective for descent.

Applications

- Connected Grothendieck toposes $\to$ all toposes.
- $\mathcal{V}$ frame $\to$ recovers Clementino and Hofmann's result.
- Provides a description of effective descent CHaus-functors and Stn-functors.

Problems

- Is there a good description of effective descent morphisms of $\mathsf{Fam}(\mathcal{V})$?
- Is there a similar approach for (symmetric) monoidal categories $\mathcal{V}$?

Merci!