UNIVERSAL PROPERTIES IN PROBABILITY THEORY

PAOLO PERRONE   University of Oxford

Based on joint work with

NOÉ ENSARGUET   ANTONIO LORENZIN

TOBIAS FRITZ   SEAN MOSS

TOMÁŠ GONDA   DARIO STEIN   & AL.

UNIVERSAL PROPERTIES

(The best thing about category theory!)

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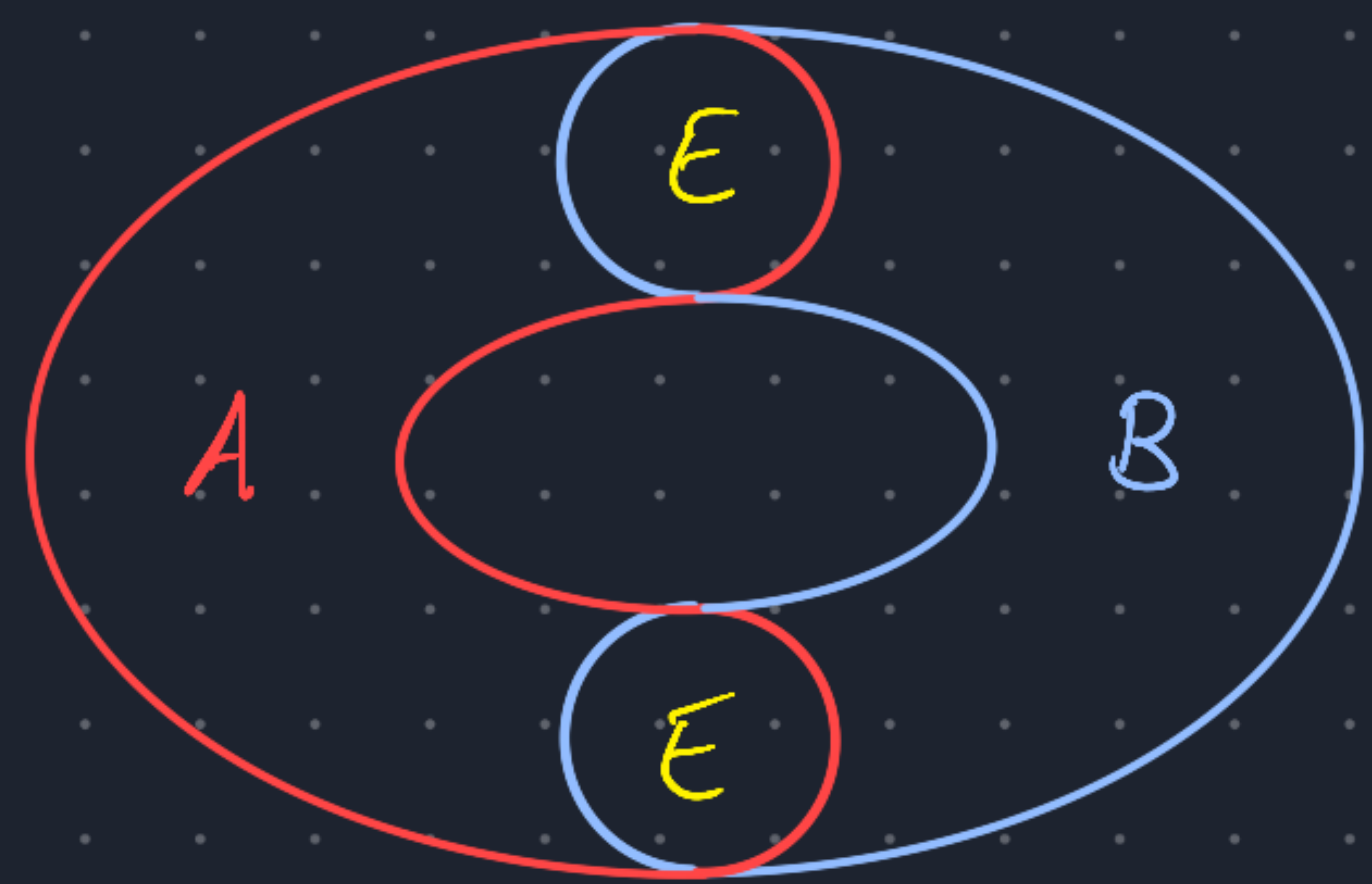

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\begin{tikzcd}
E \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] & A +_E B \arrow[ul, phantom, "\ulcorner", very near start]
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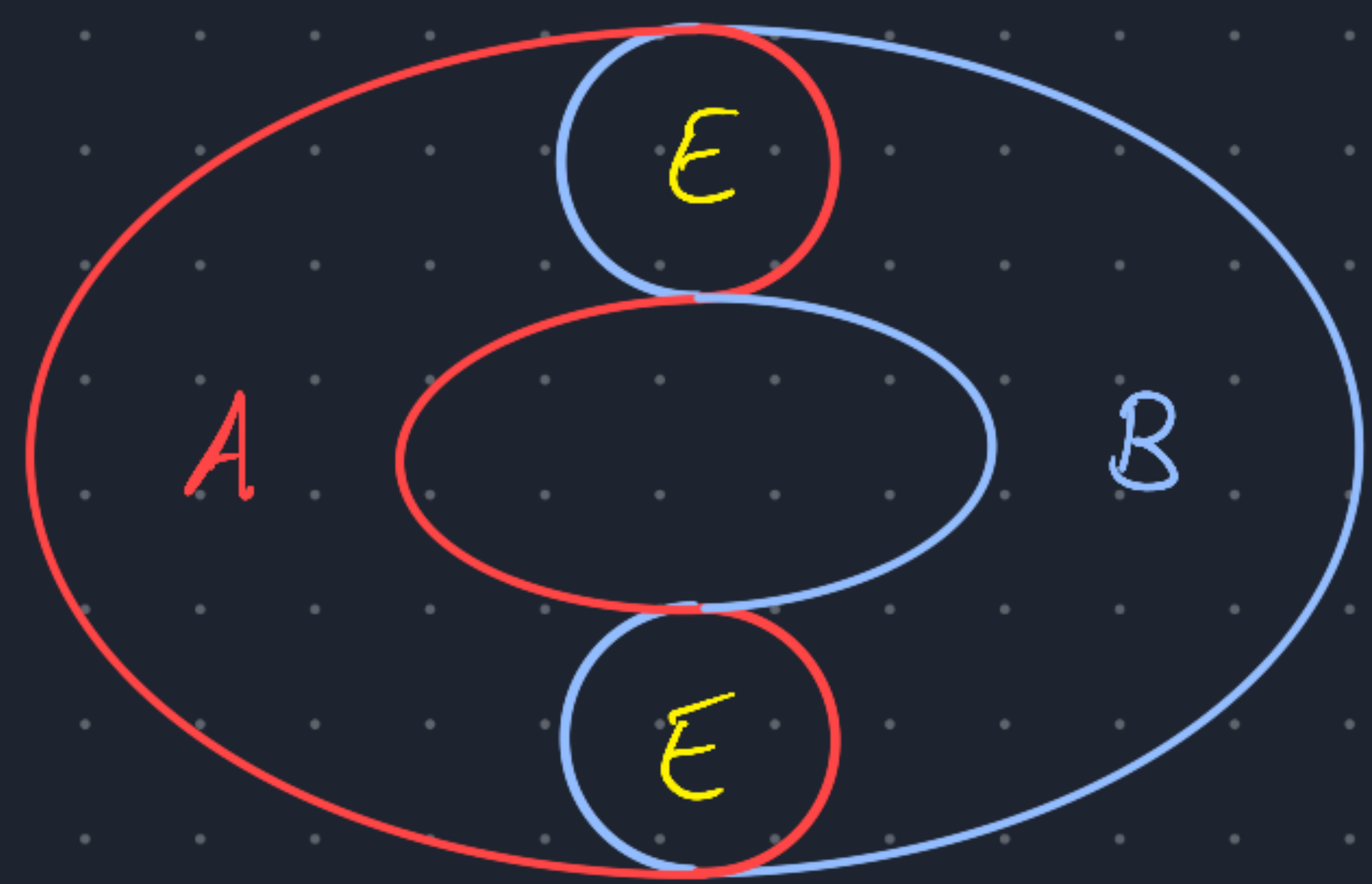

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$\Rightarrow$ Mayer-Vietoris sequence

$\Rightarrow$ Homology, derived functors ... etc.

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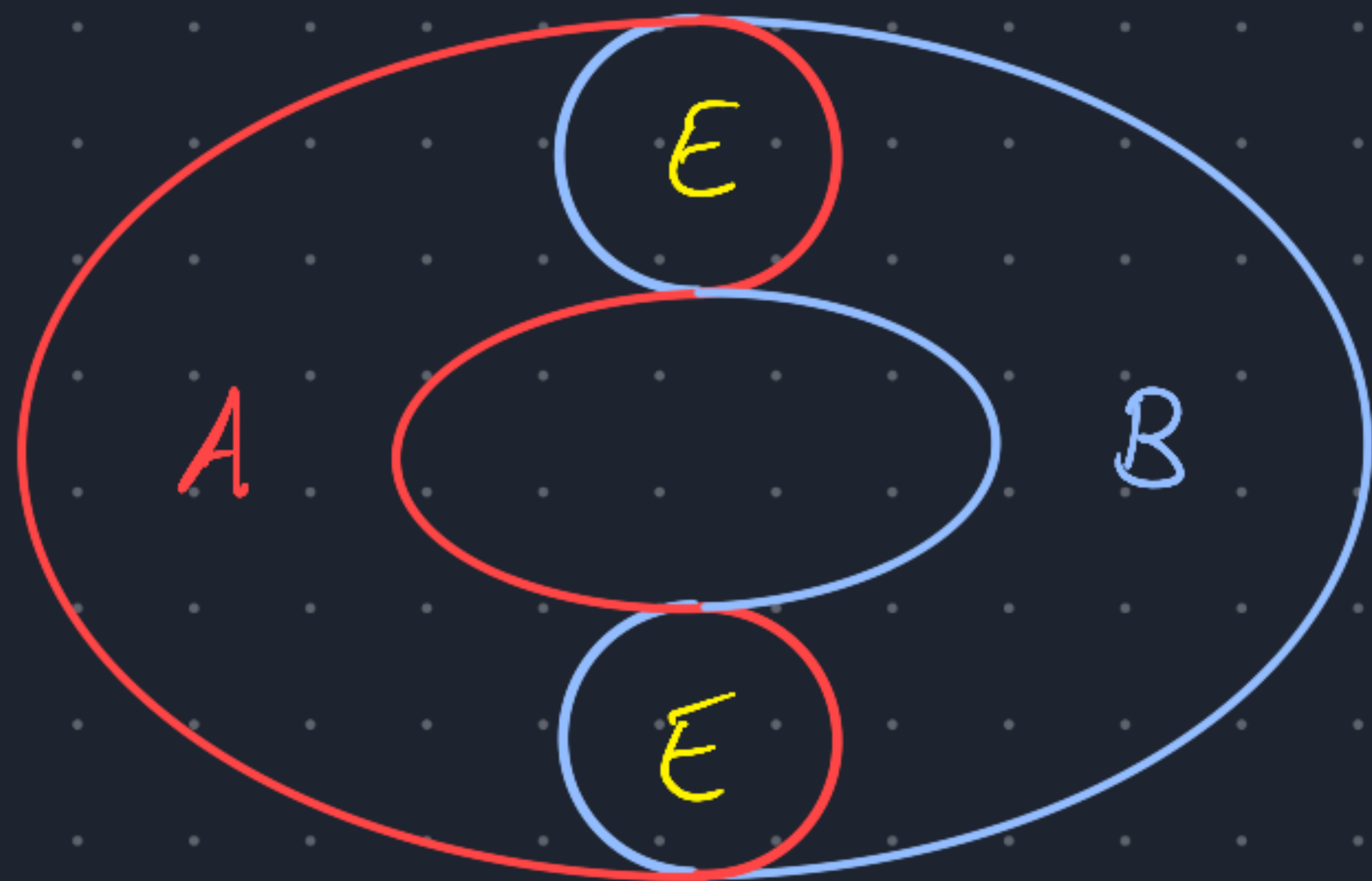

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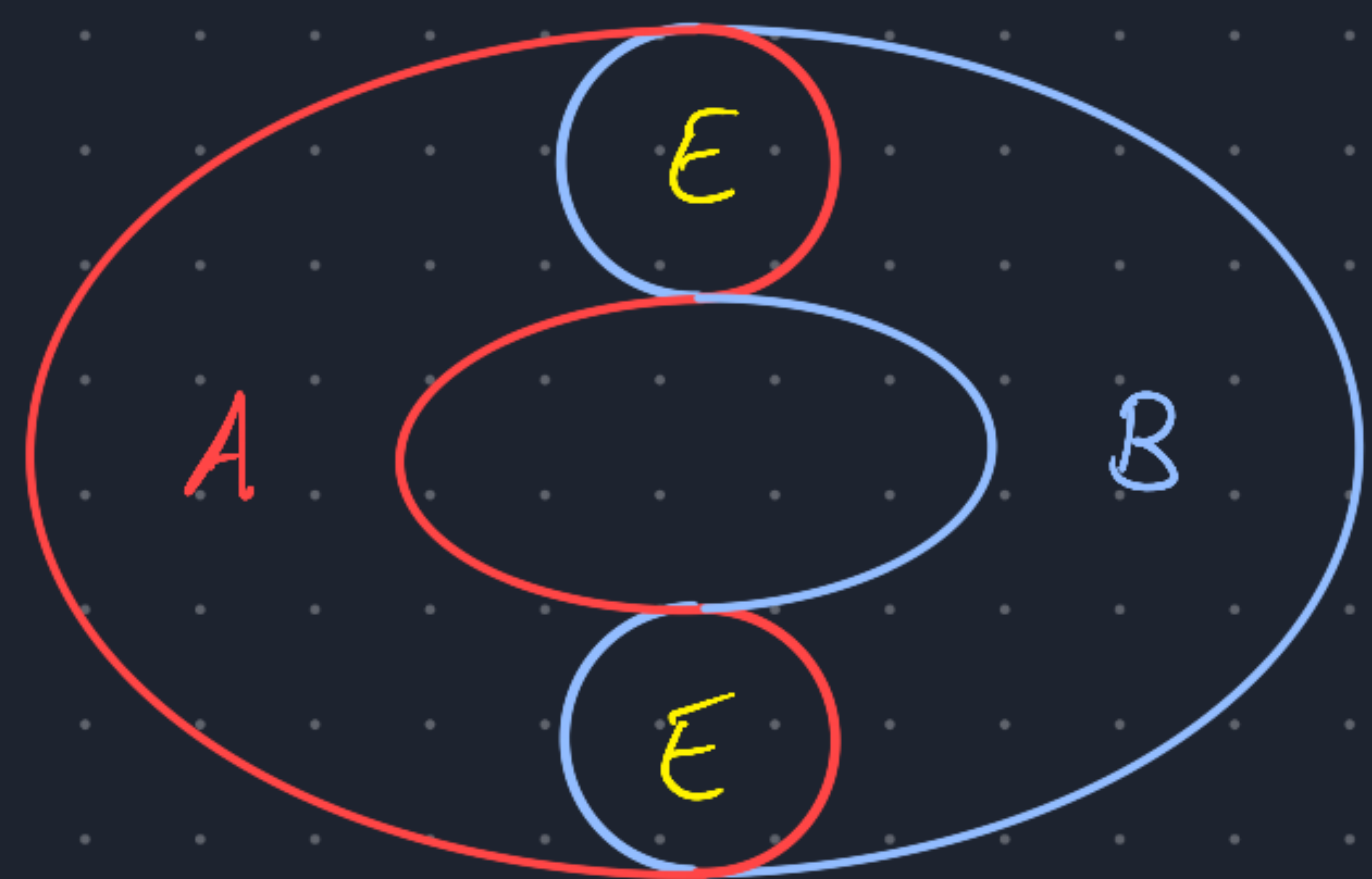

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→ Where should Probability texts start a chapter on categories?

WHAT IS PROBABILITY THEORY ?

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- Probability theory is <u>not</u> about measure spaces (X, p) with $p(X) = 1$.

Cfr. "Real Analysis is about infinite strings of digits with $0.999\ldots = 1$."

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THINGS HERE ARE RANDOM

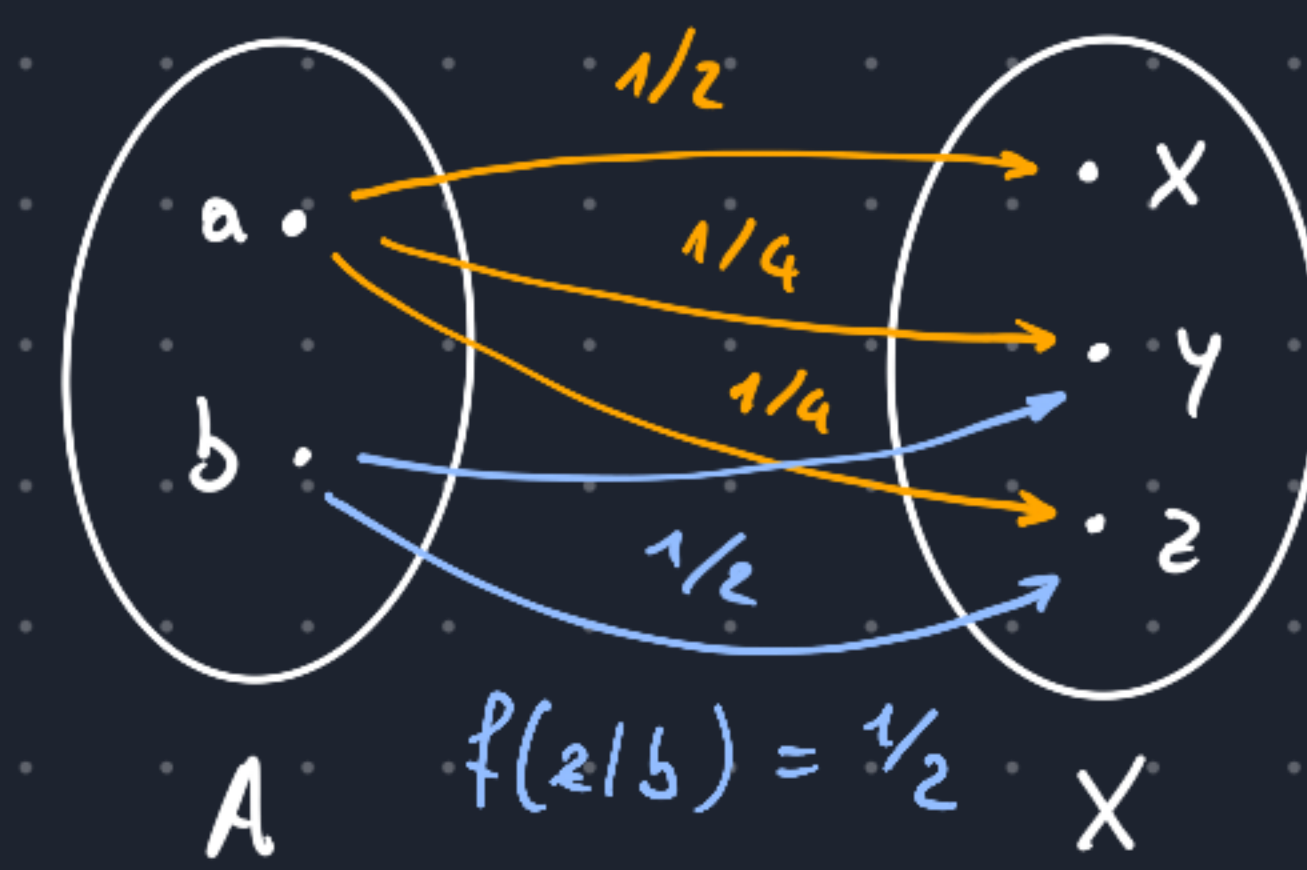
WHAT CAN WE STILL SAY ?

- Knowing X will tell us something about Y $\Rightarrow$ how things interact.

MONOIDAL CATEGORIES IN PROBABILITY

Morphism: "map with randomness"

In **Finstoch**: stochastic matrix



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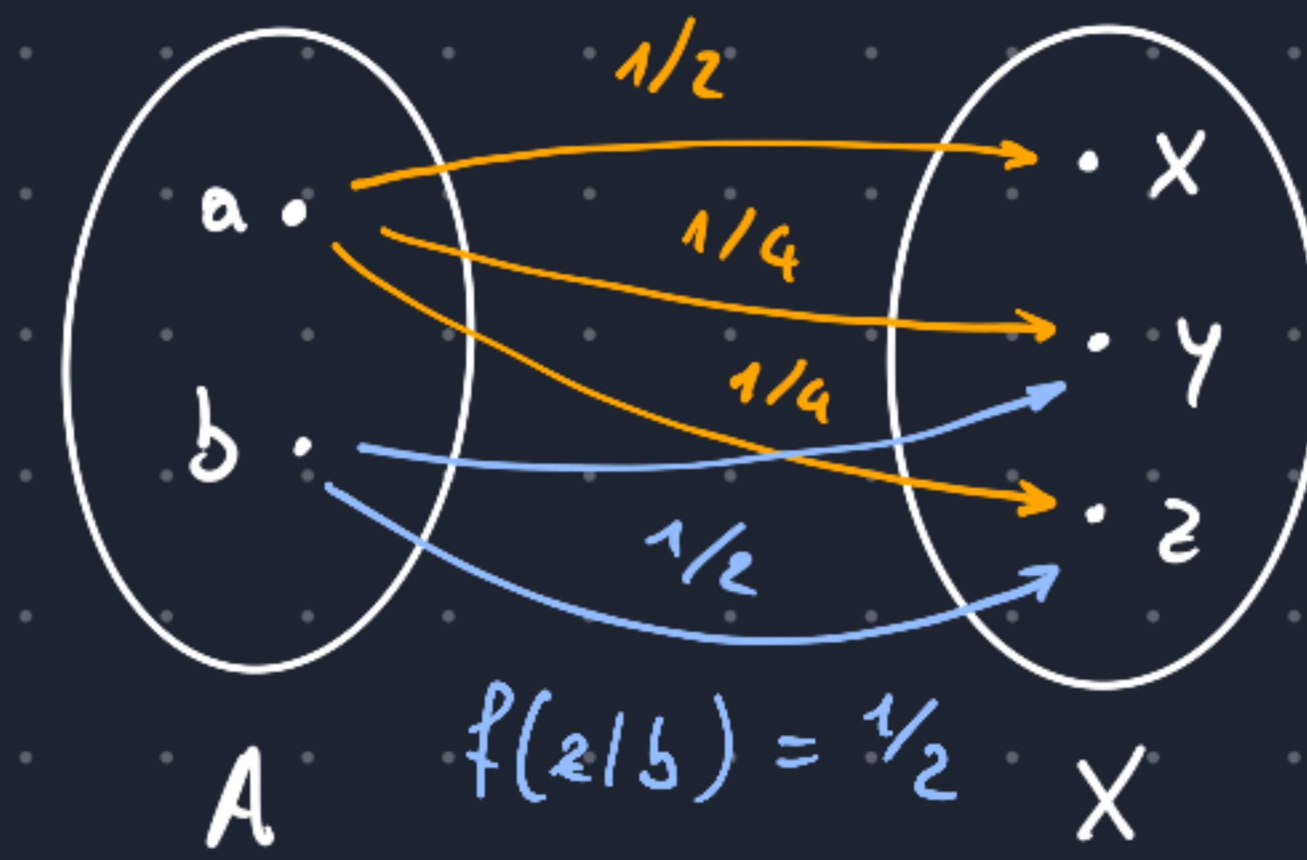
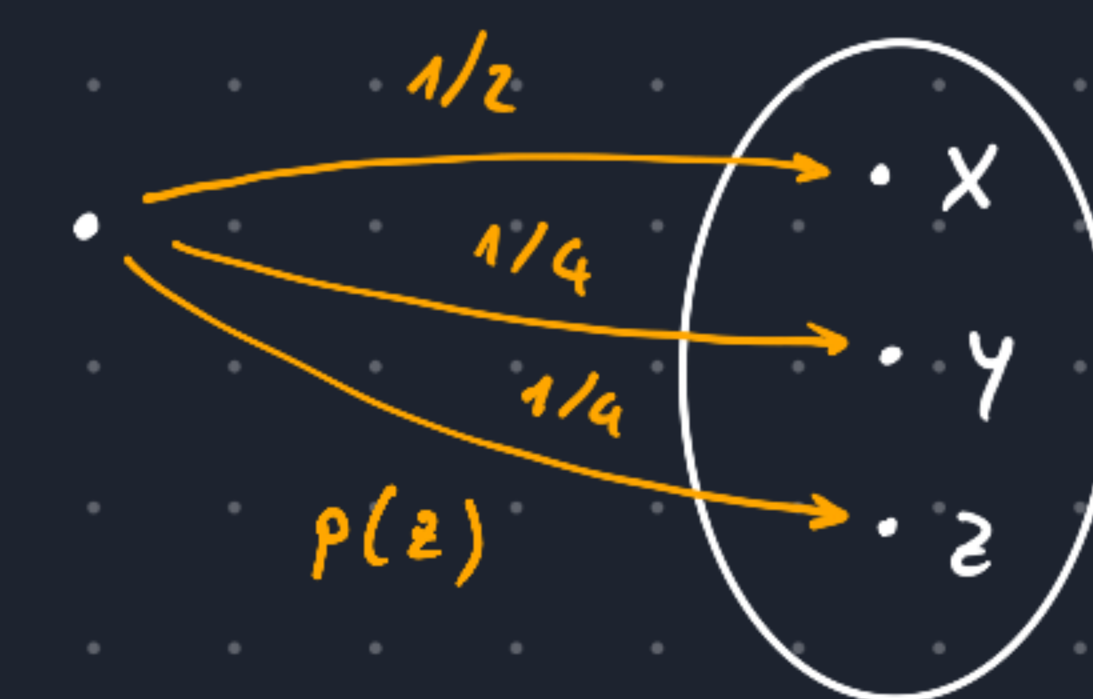


State: "probability measure"

In **FinStoch**: discrete p. measure



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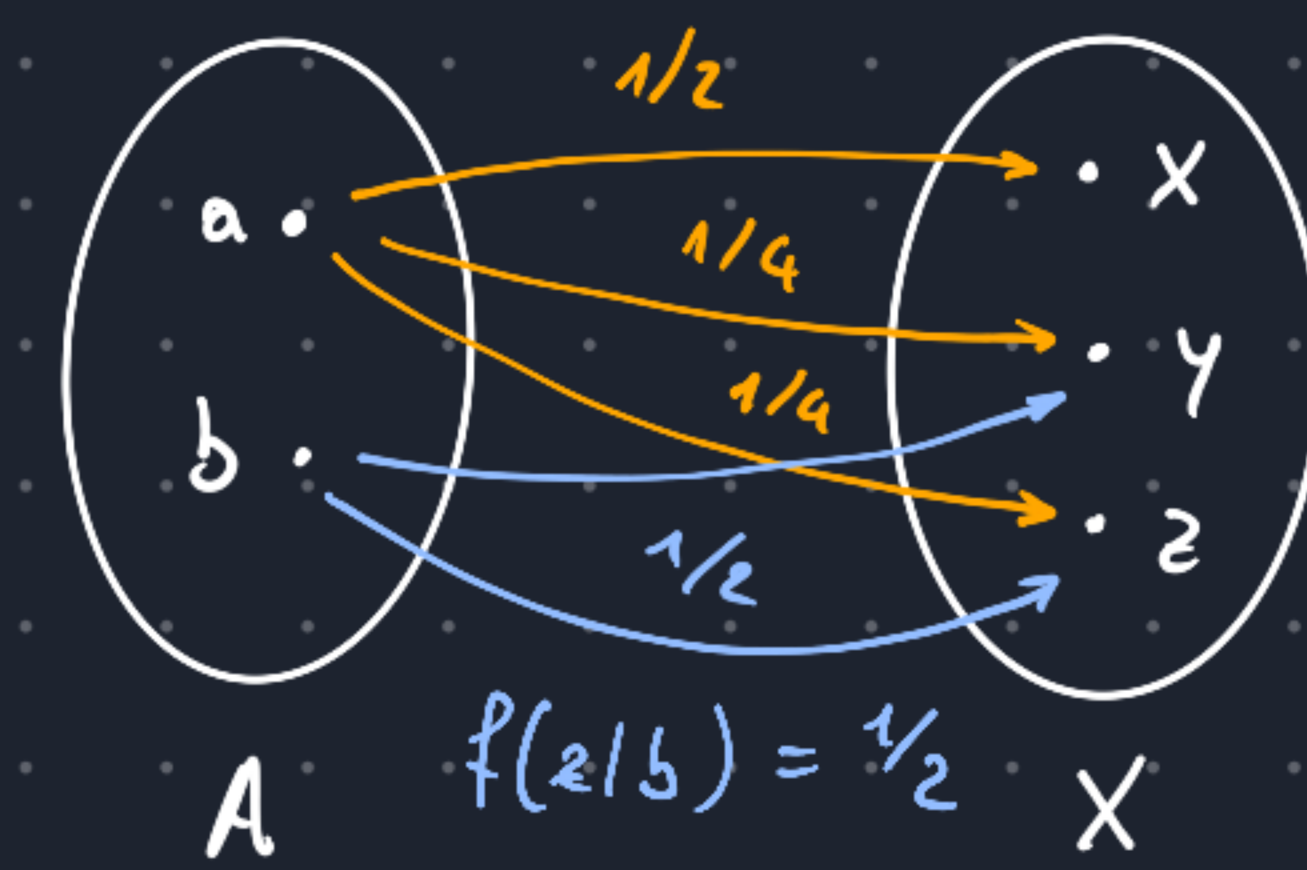

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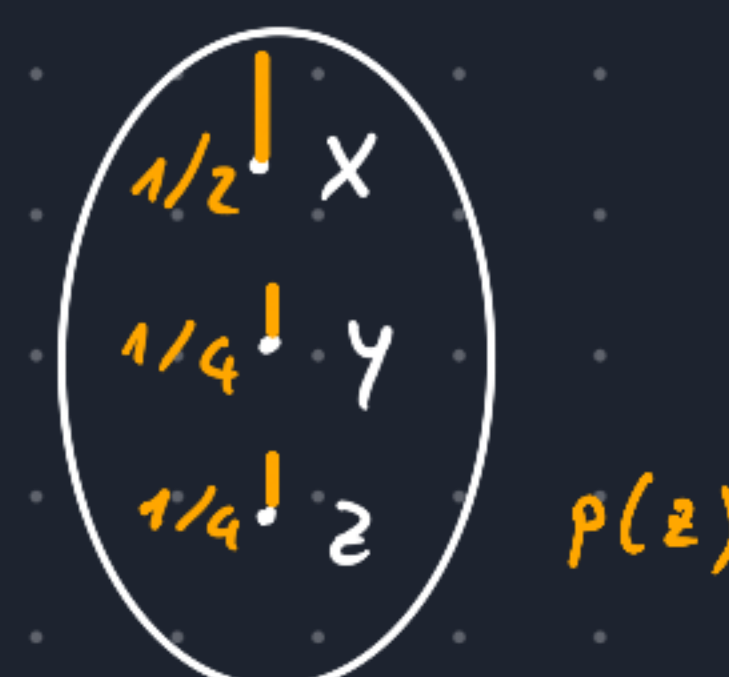

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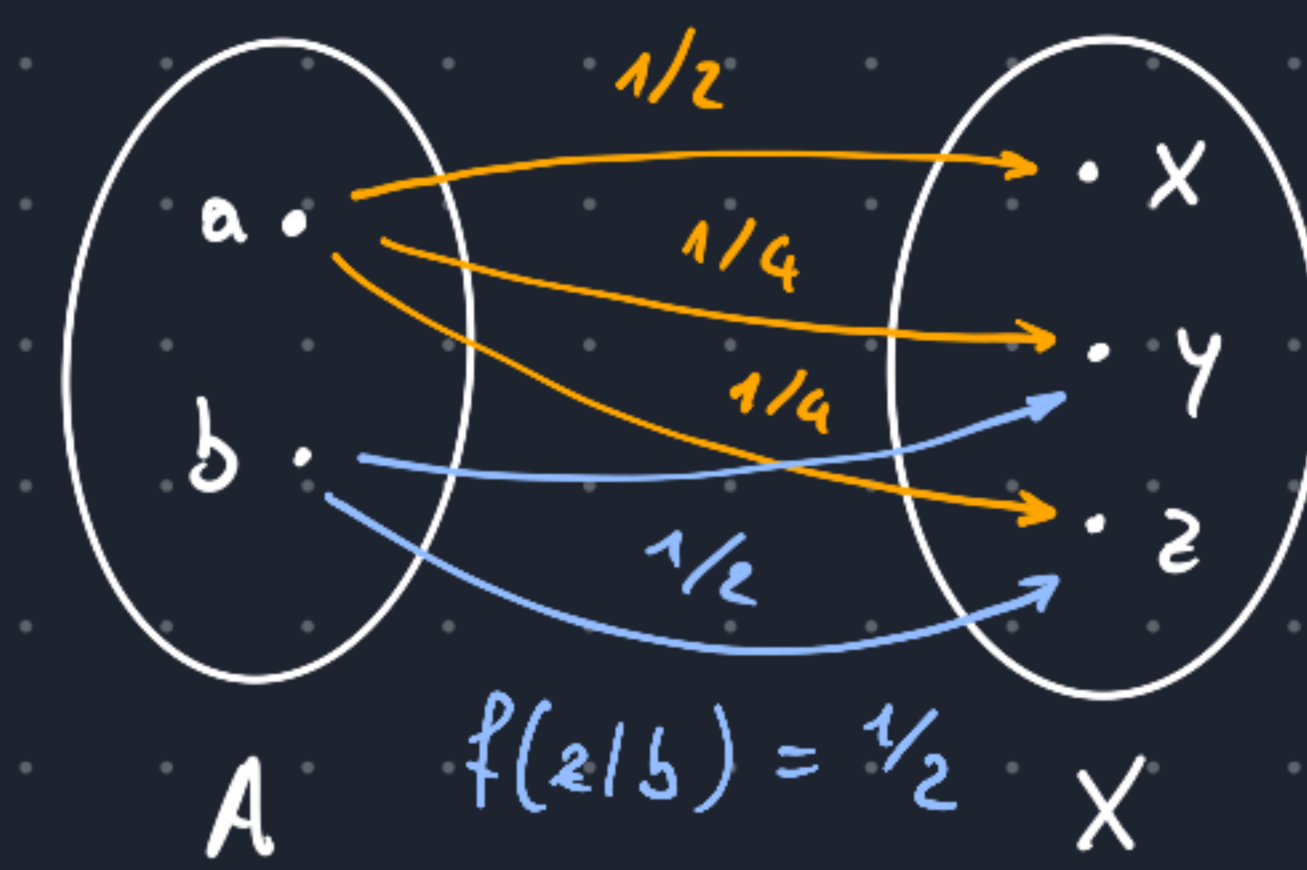

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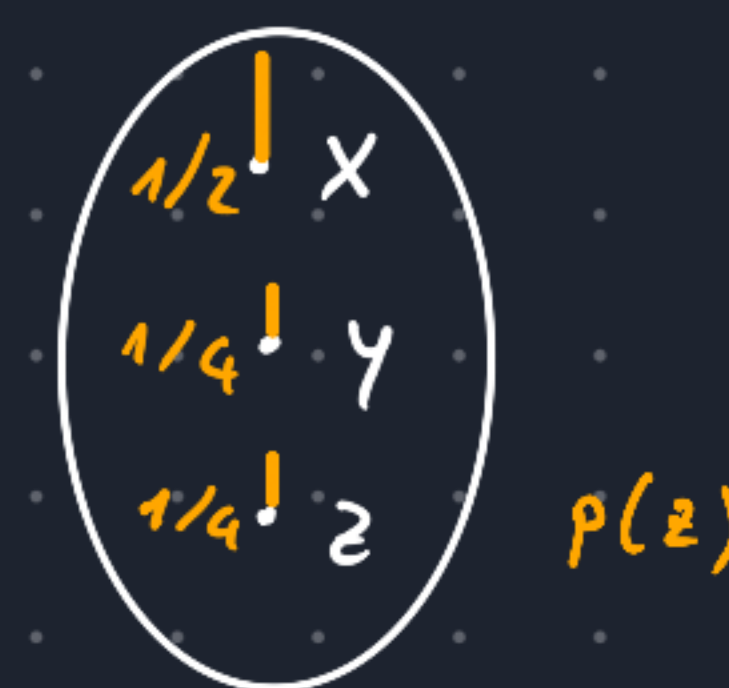
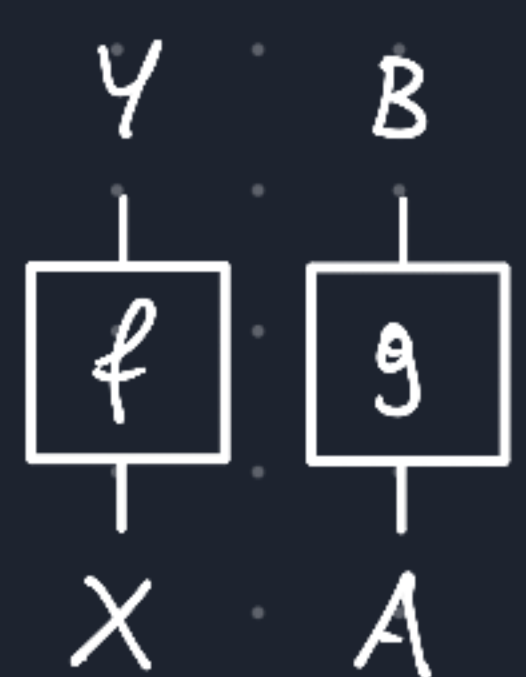




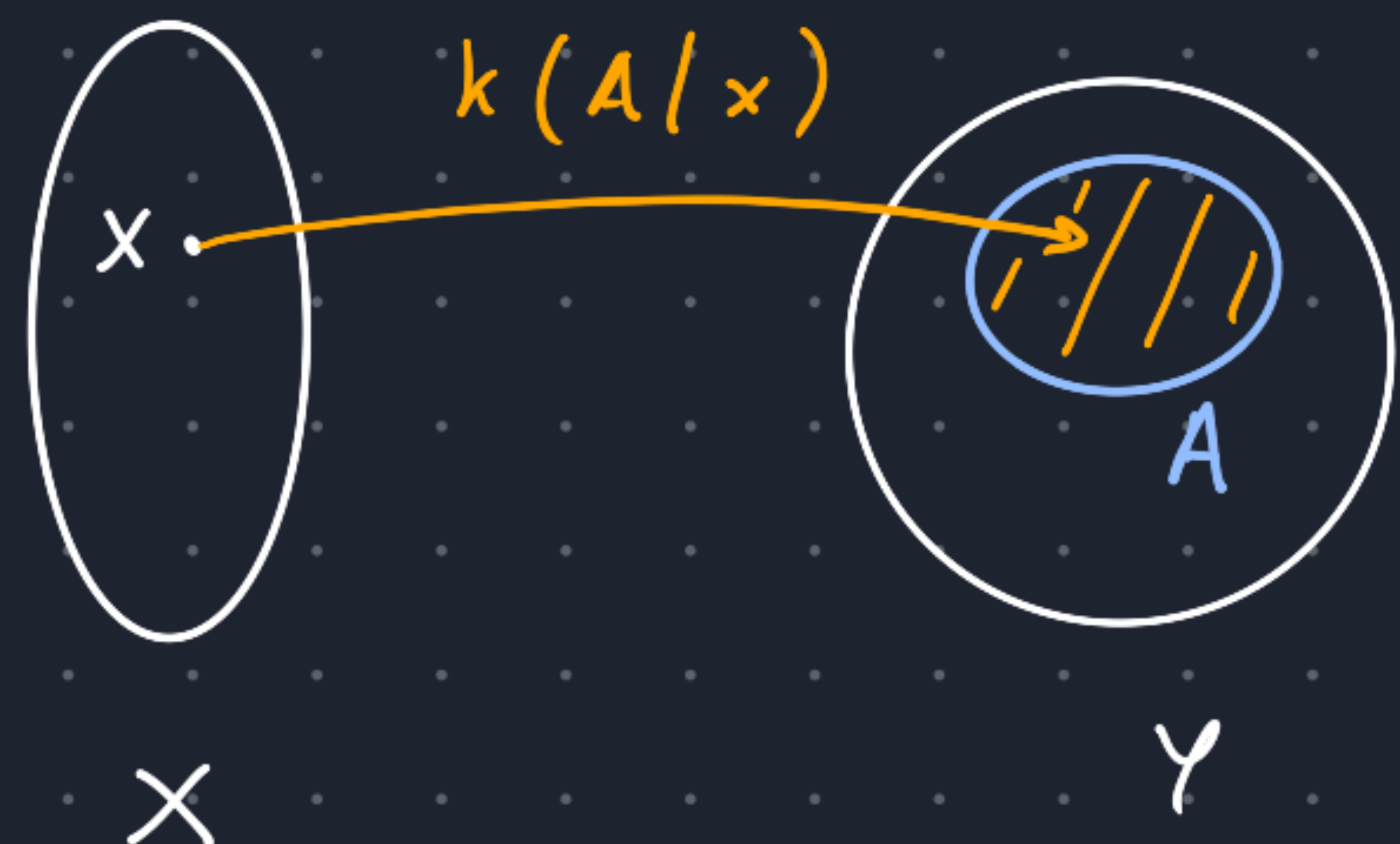
Monoidal product: "independent"

In **FinStoch**: product of probabilities

$$f\otimes g\,(y,b\mid x,a) = f(y\mid x)\; g(b\mid a)$$

Def. The category **BorelStoch*** has

- As objects, standard Borel ("Polish") measurable spaces.
- As morphisms, **Markov kernels**.



$$X \times \Sigma_Y \longrightarrow [0,1]$$
$$(x, A) \longmapsto k(A|x)$$

prob. measure

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*[Lawvere '62, Čencov '65]

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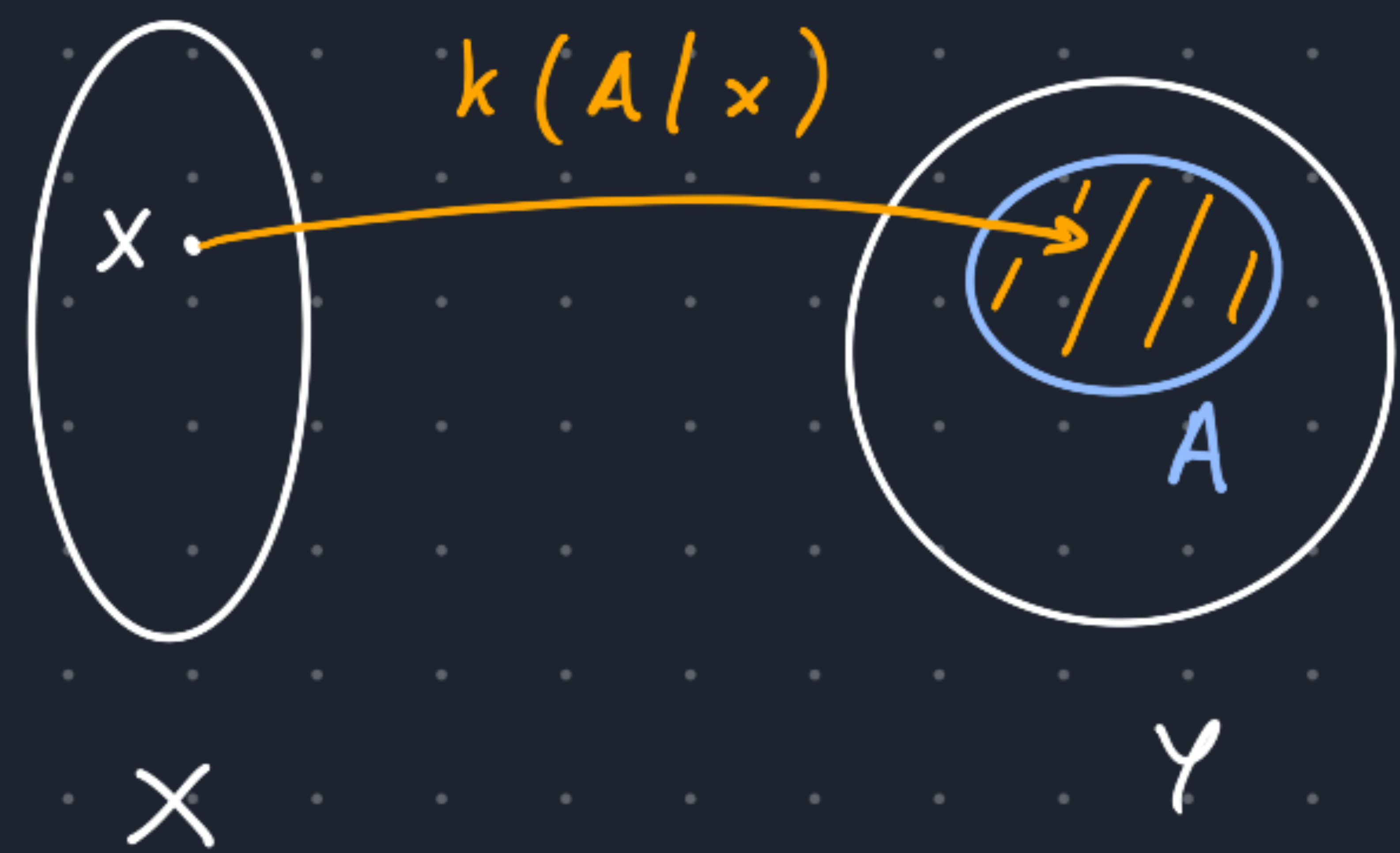

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$$h \circ k(B|x) = \int_Y h(B|y)\, k(dy|x)$$

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UNIVERSAL PROPERTIES IN PROBABILITY:

1. NO PRODUCTS
2. INFINITE SEQUENCES
3. PROBABILITY MONADS
4. DE FINETTI AS A LIMIT

(& MORE)

1. NO PRODUCTS

$$\begin{tikzcd}
& I \arrow[d, dashed] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
& X \otimes Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X & & Y
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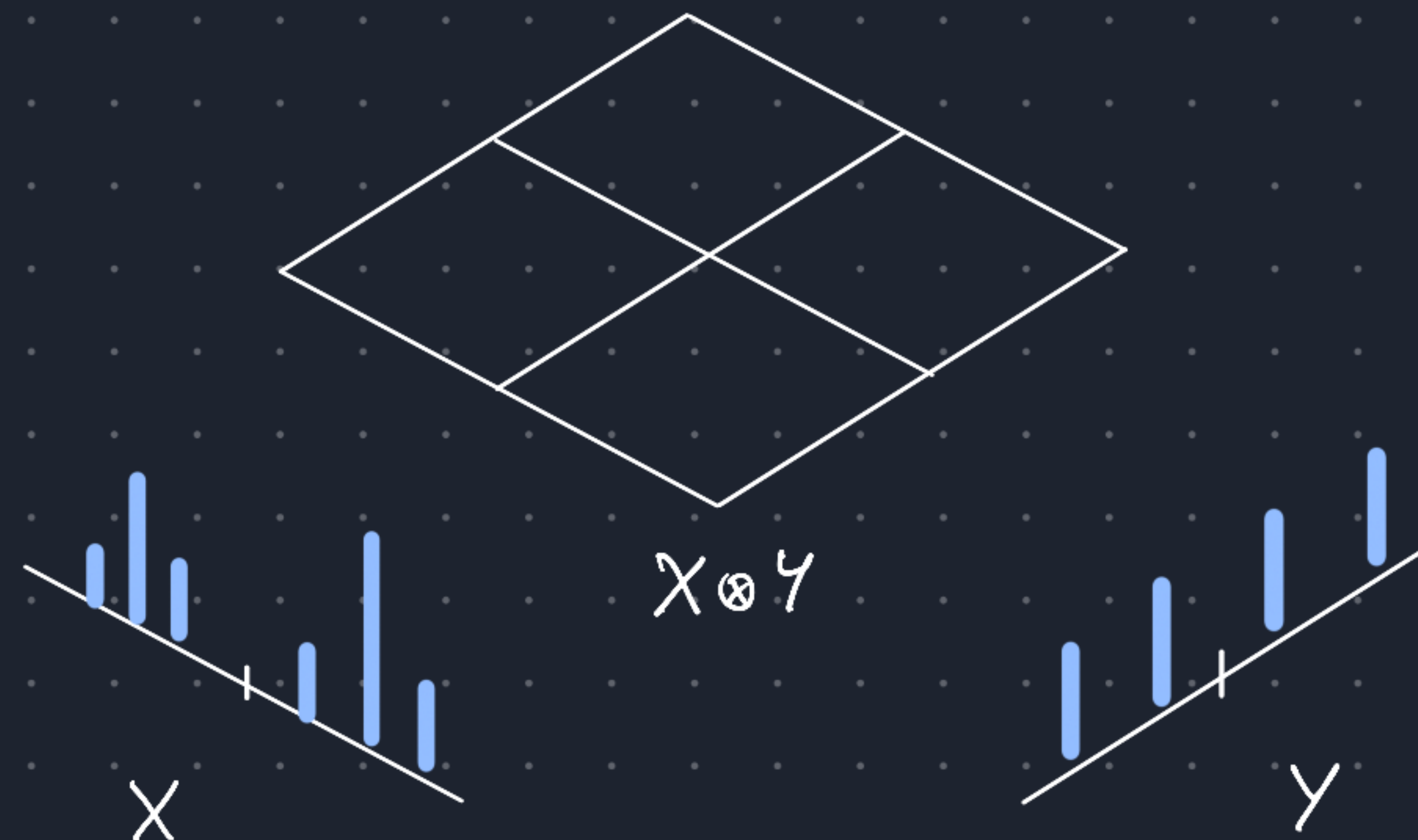

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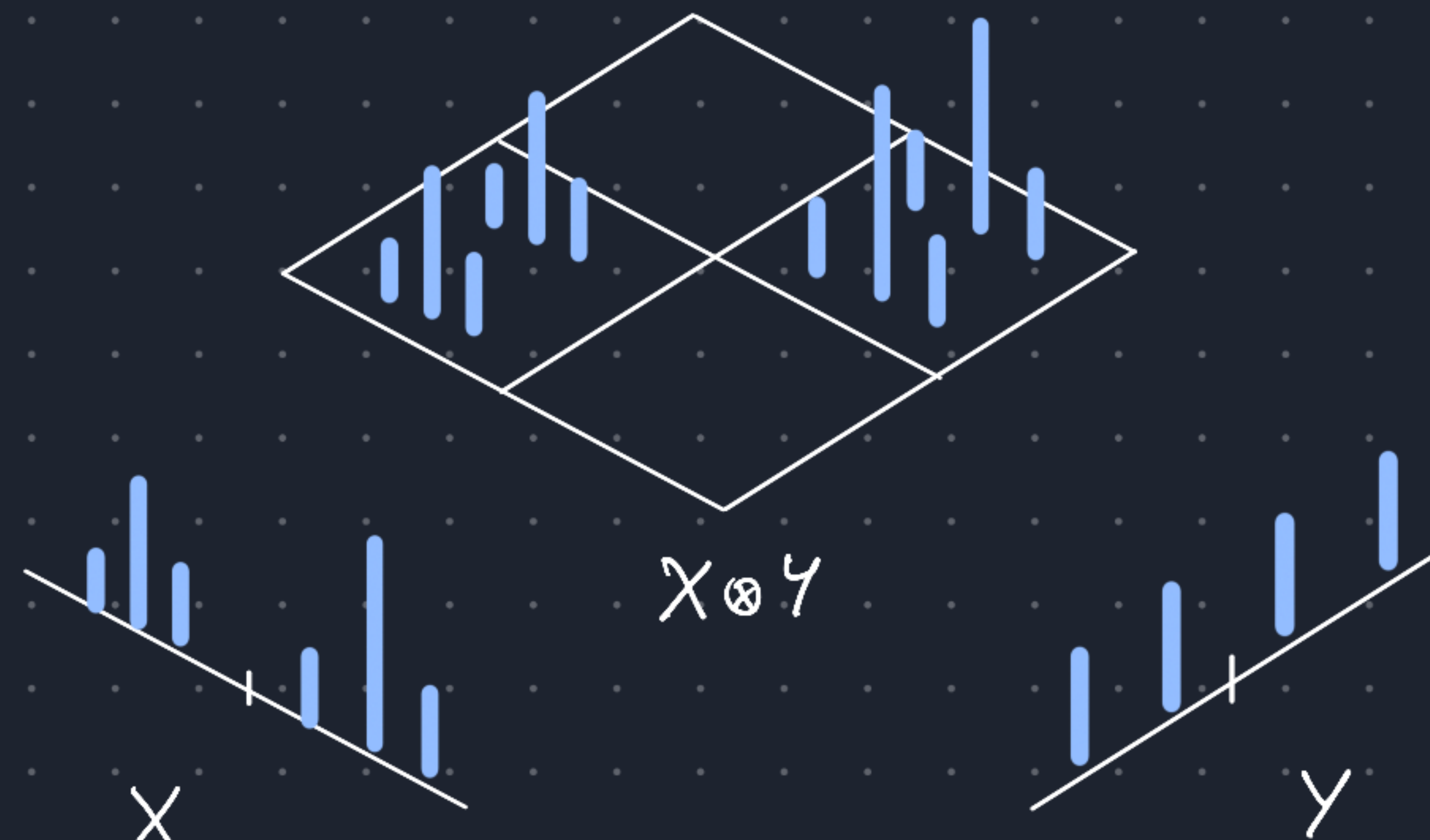

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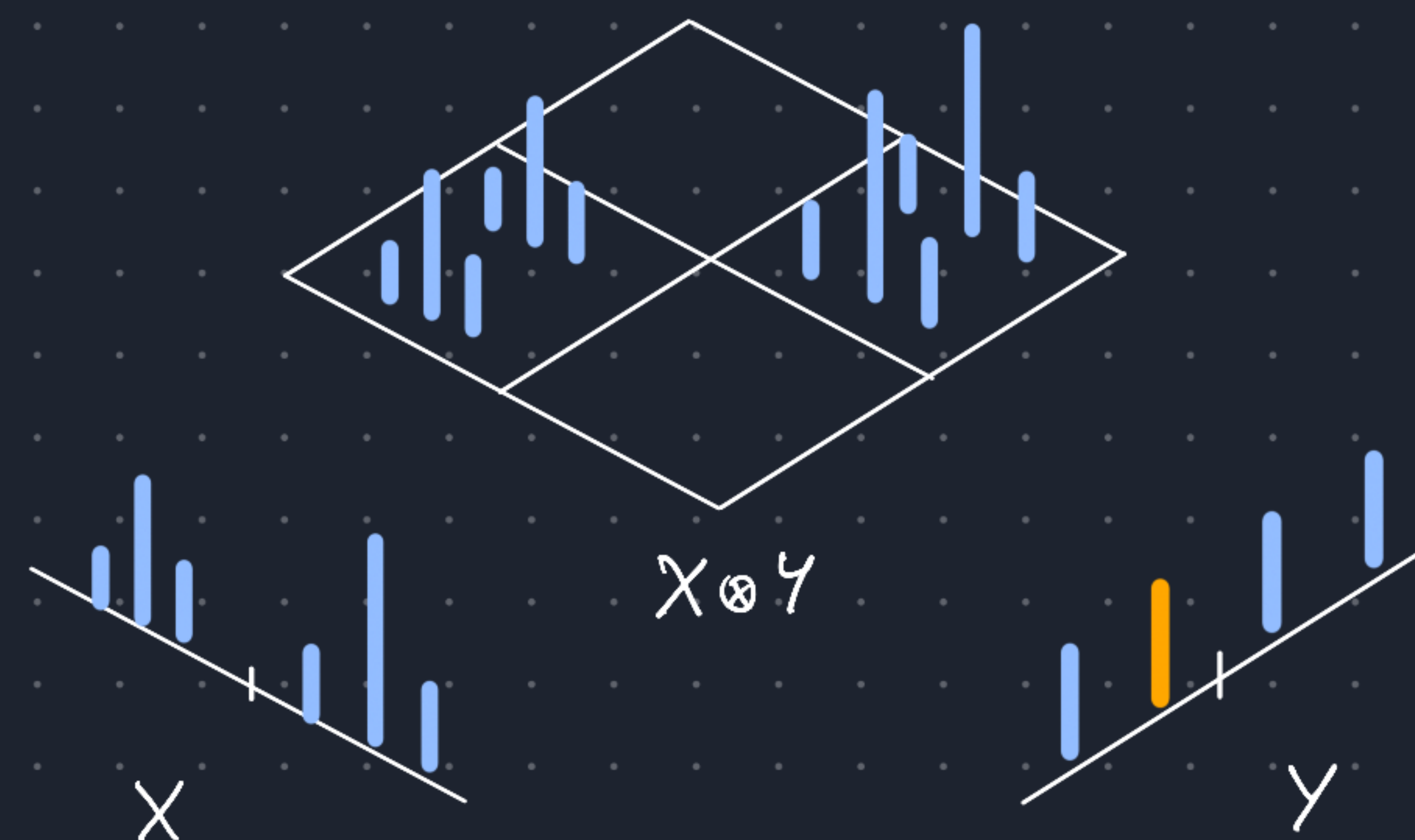

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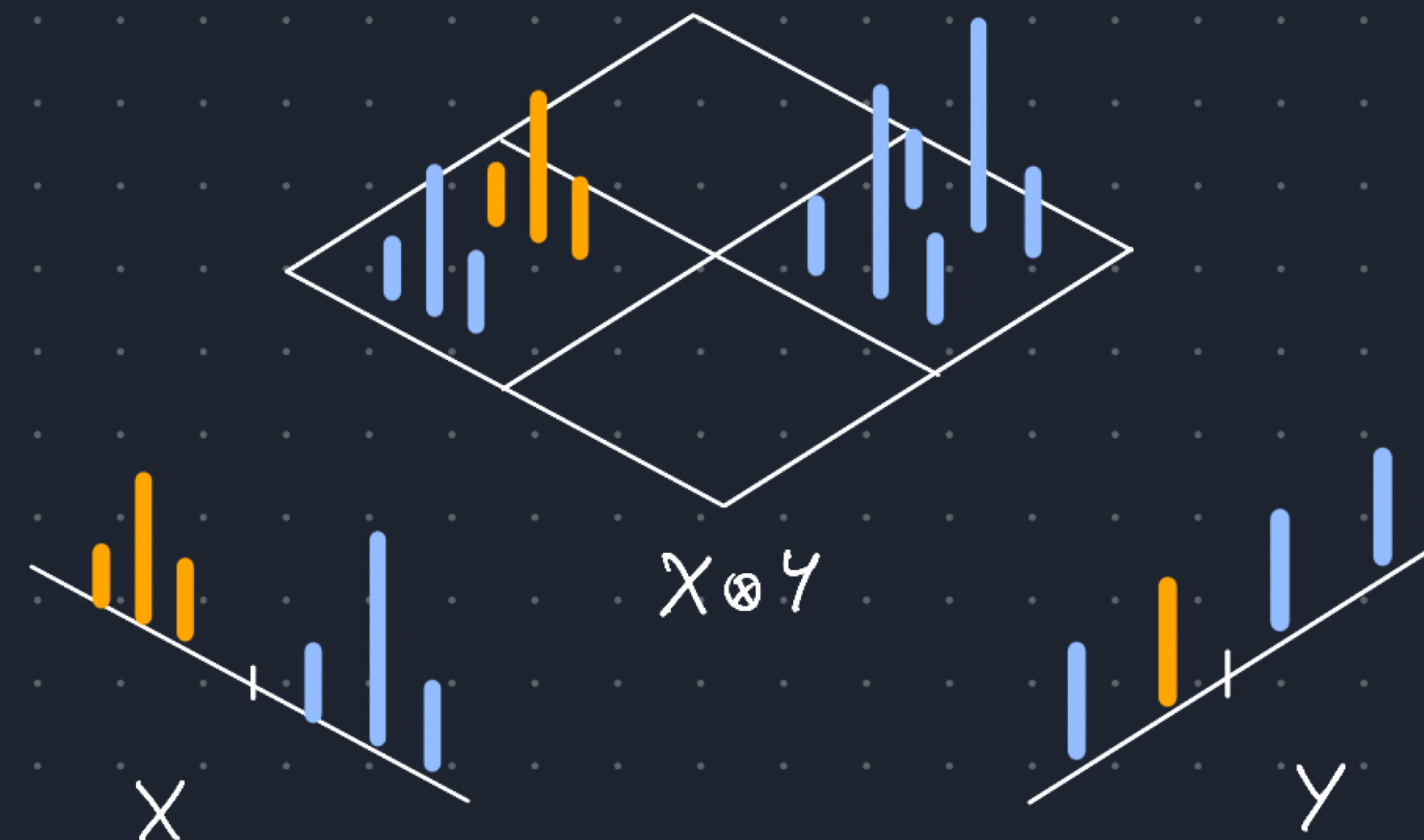

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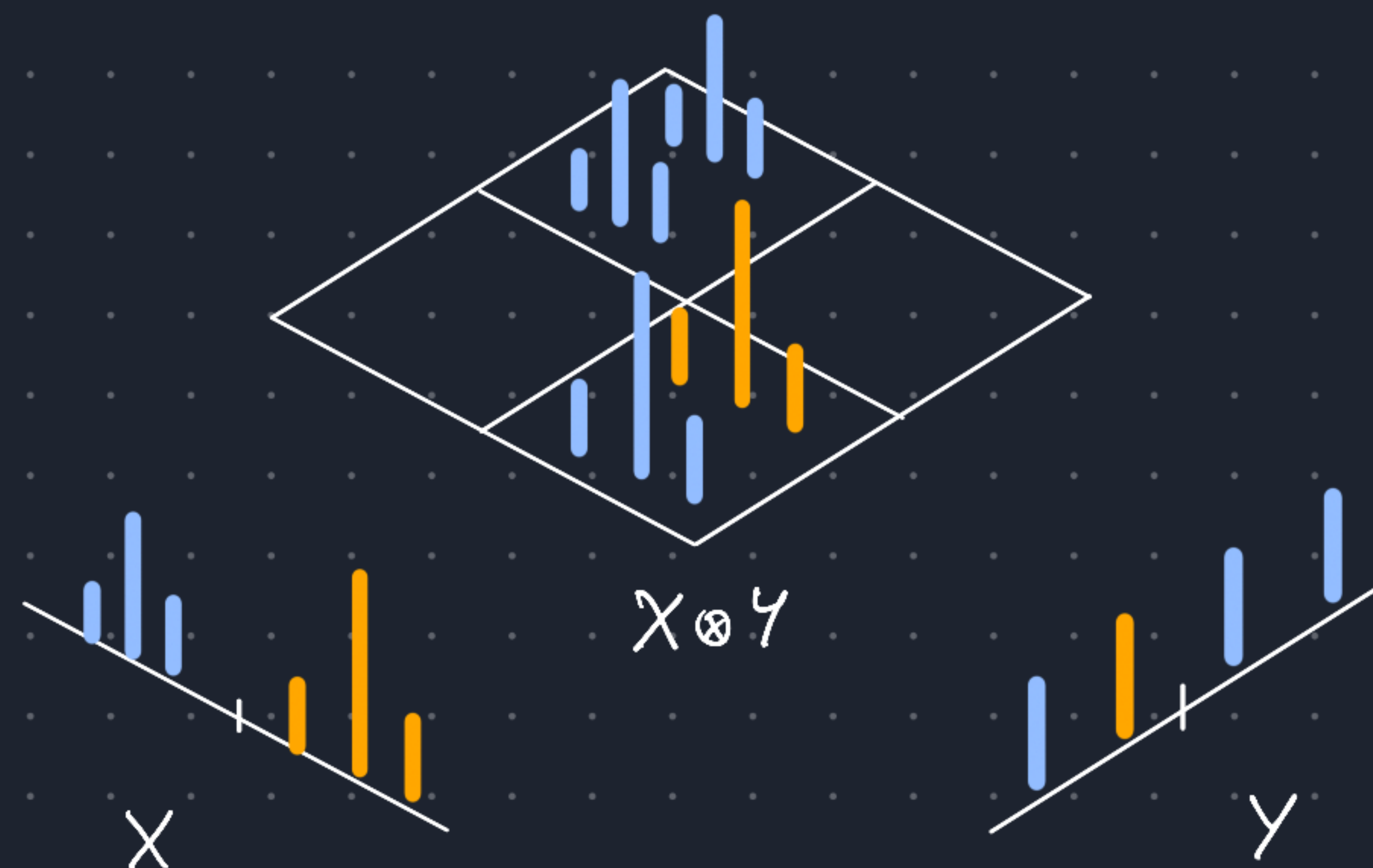

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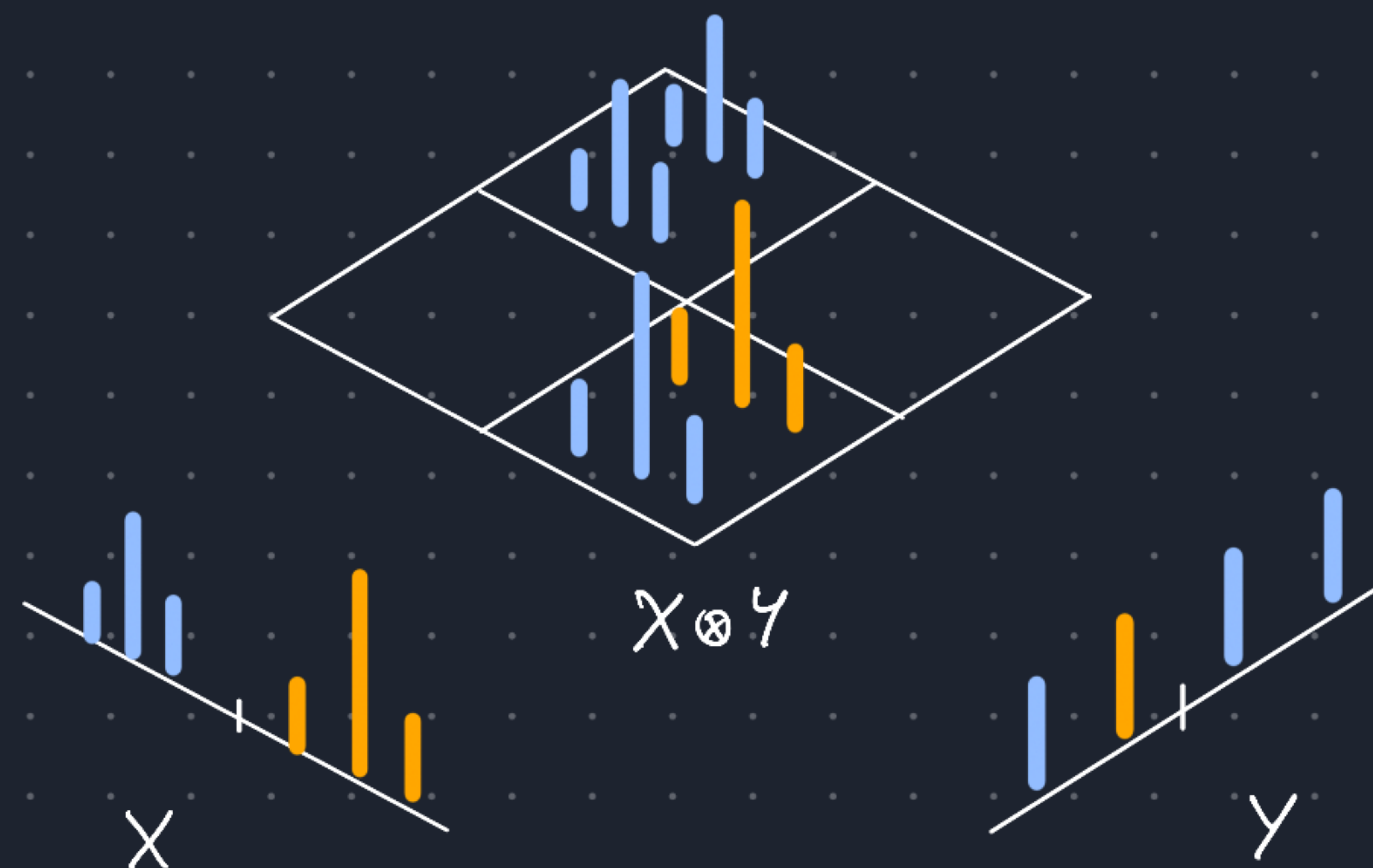

Def. A *Markov category* is a SMC where each object X has a distinguished commutative comonoid structure :

copy : 

discard: 

* [Fritz '19,
Cho-Jacobs '18,
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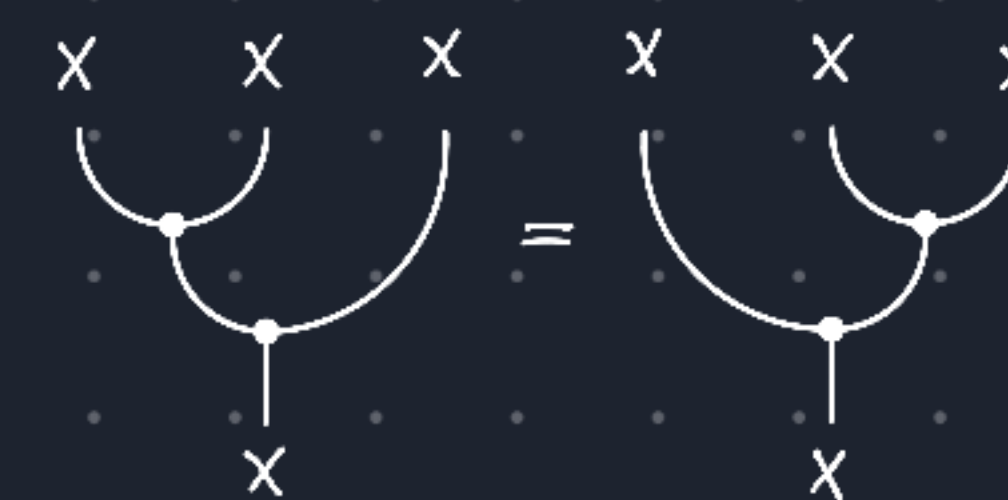

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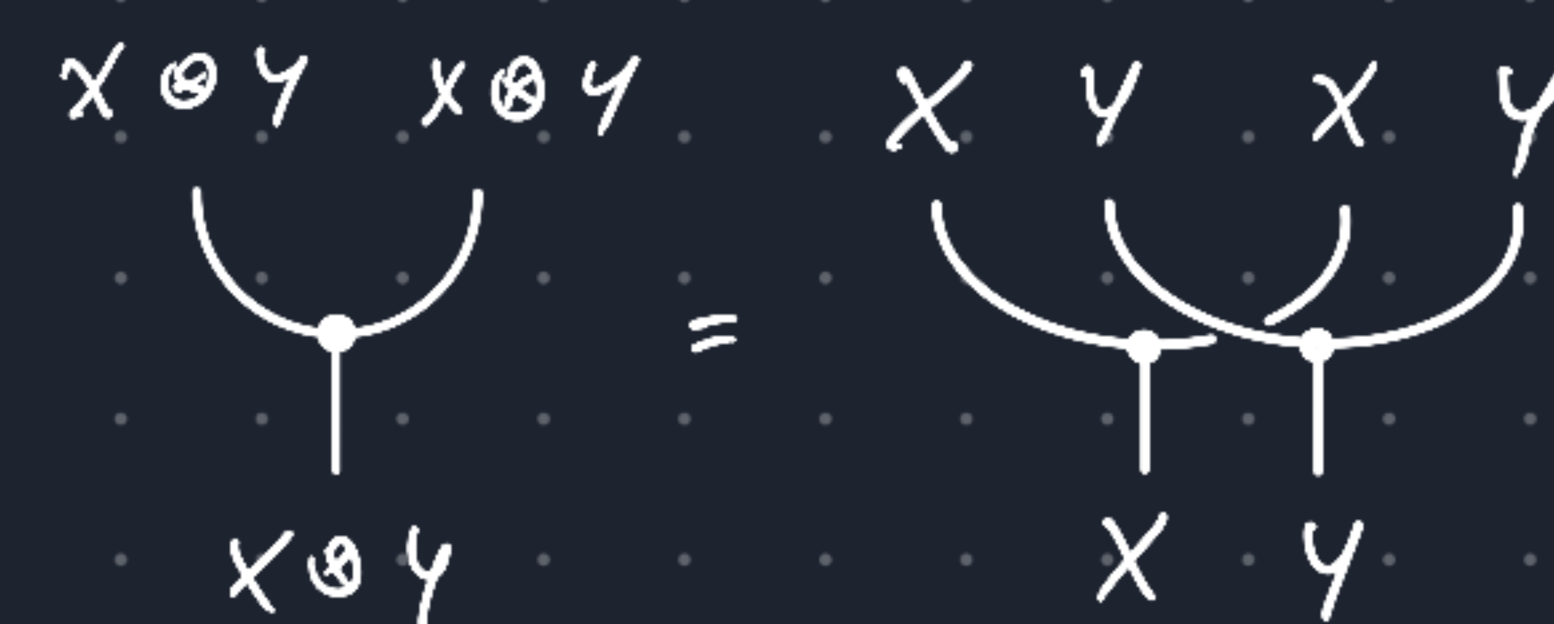
compatible with $\otimes$: 

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Examples

- In Set (cartesian), $\exists !$ comonoid structure for every object:

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\begin{tikzcd}
 & X & & \\
X \arrow[ur, "id"] \arrow[r, dashed] \arrow[dr, "id"'] & X \times X \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] & & X \arrow[r, "!"] & 1 \\
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X \xrightarrow{!} 1
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- In **FinStoch**,
  inherited from **Set**

X
x
y

$X \otimes X$	x	y
x	$\boxed{(x,x)}$	(x,y)
y	(y,x)	$\boxed{(y,y)}$

$\swarrow$	x	y
(x,x)	1	0
(x,y)	0	0
(y,x)	0	0
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	x	y
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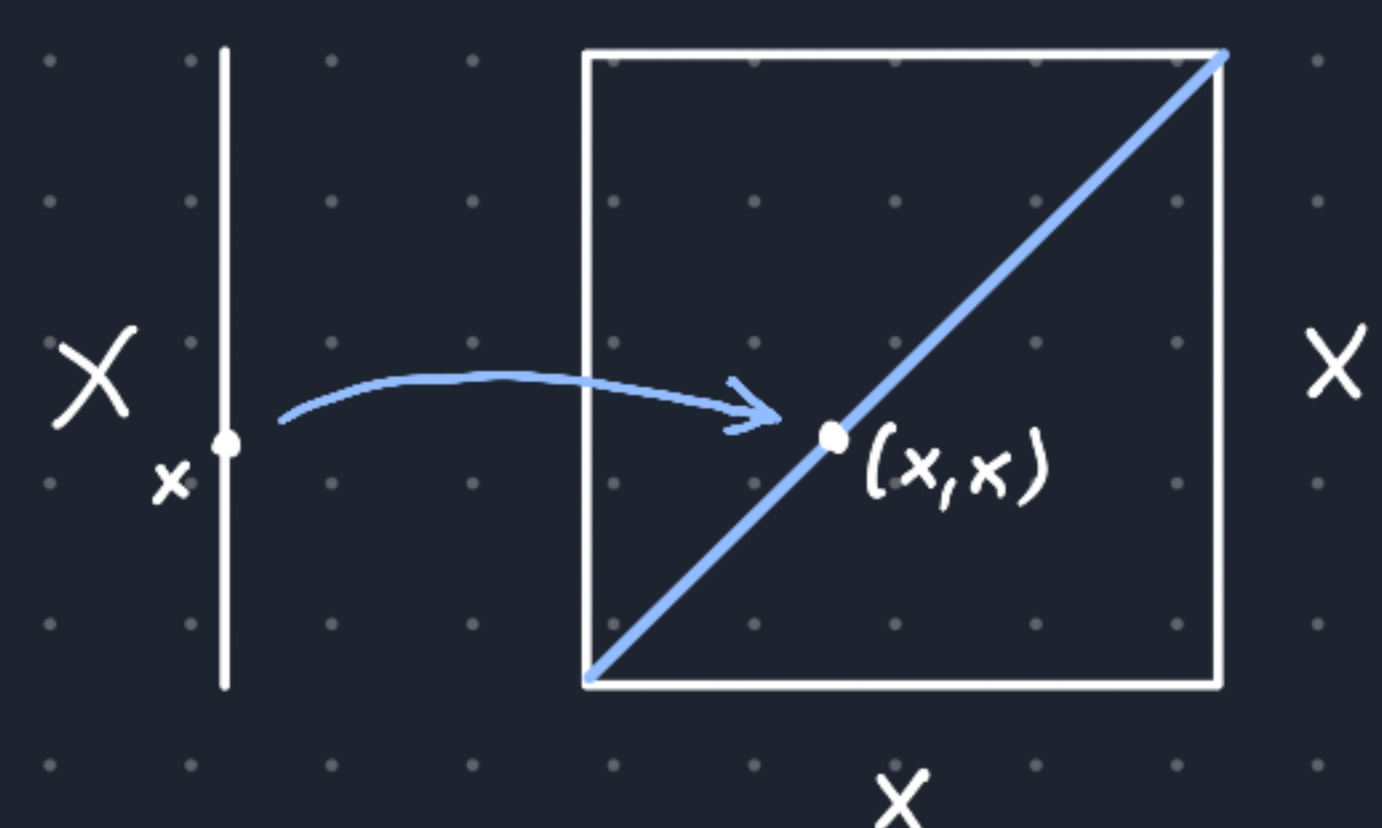
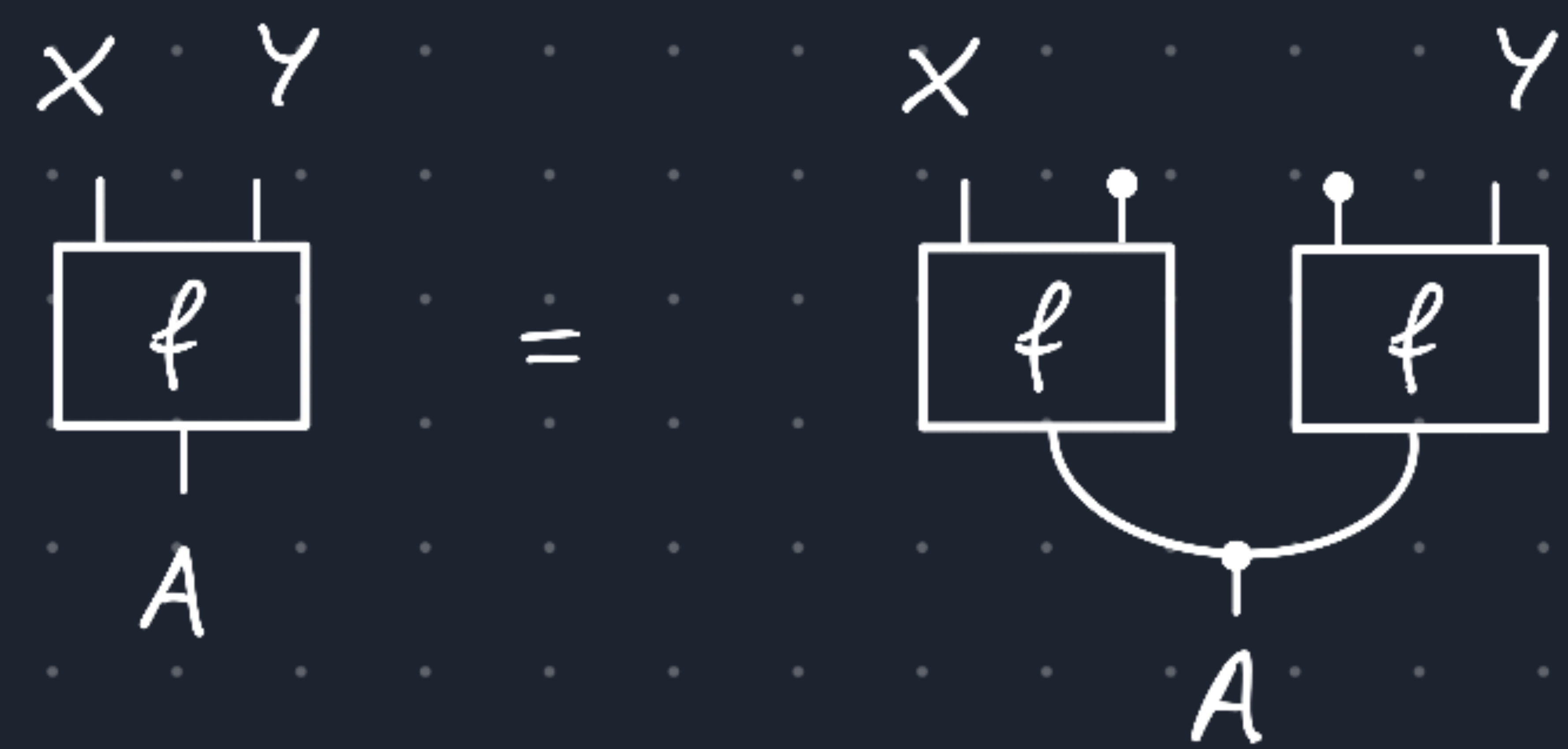
(x,y)	0	0
(y,x)	0	0
(y,y)	0	1

- In **BorelStoch**, again inherited from **Set**



$$\mathrm{copy}(A \times B \mid x) = \begin{cases} 1 & x \in A \cap B \\ 0 & x \notin A \cap B \end{cases}$$

Def. A morphism $A \xrightarrow{f} X \otimes Y$ exhibits conditional independence of X and Y given A $(X \perp Y \mid A)$ if:



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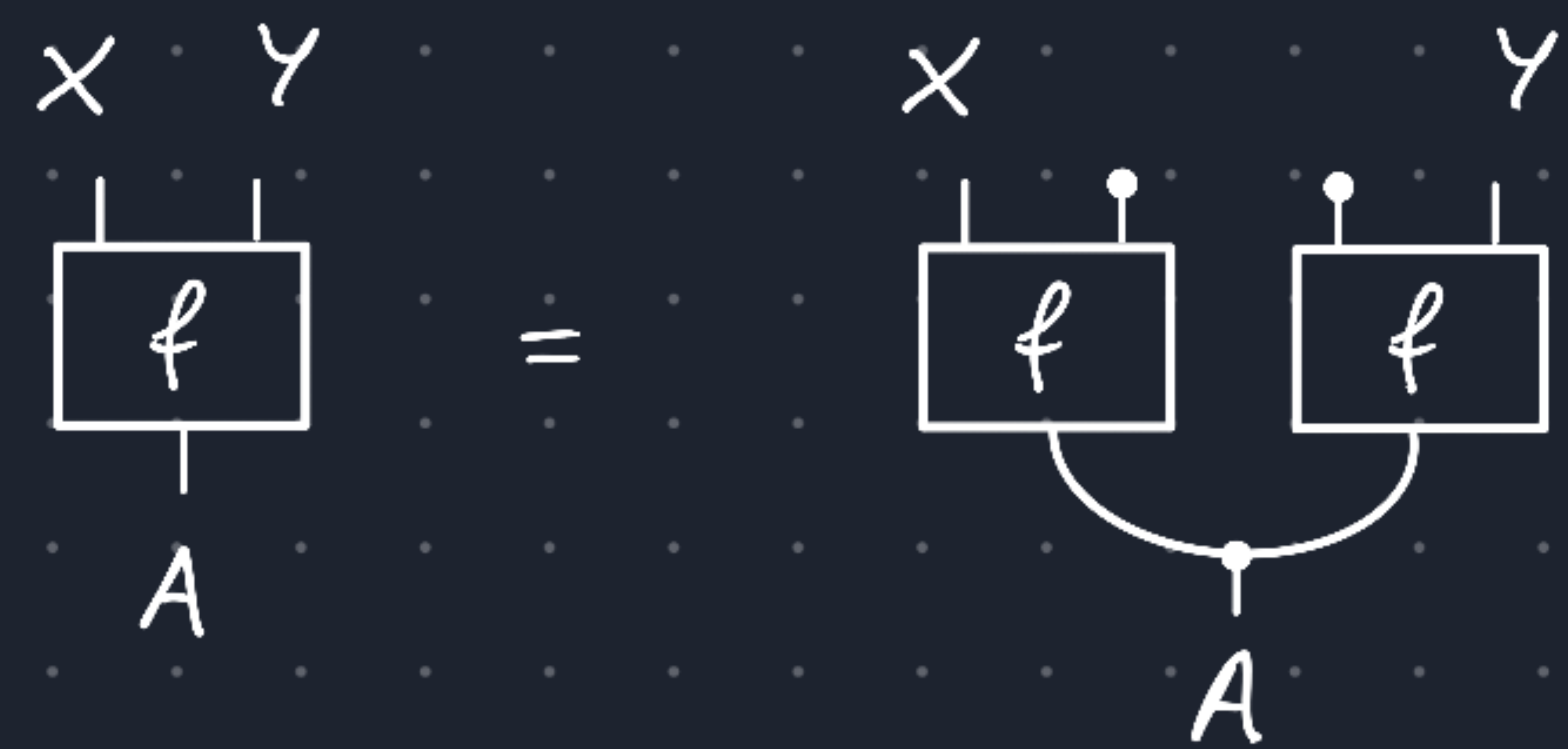

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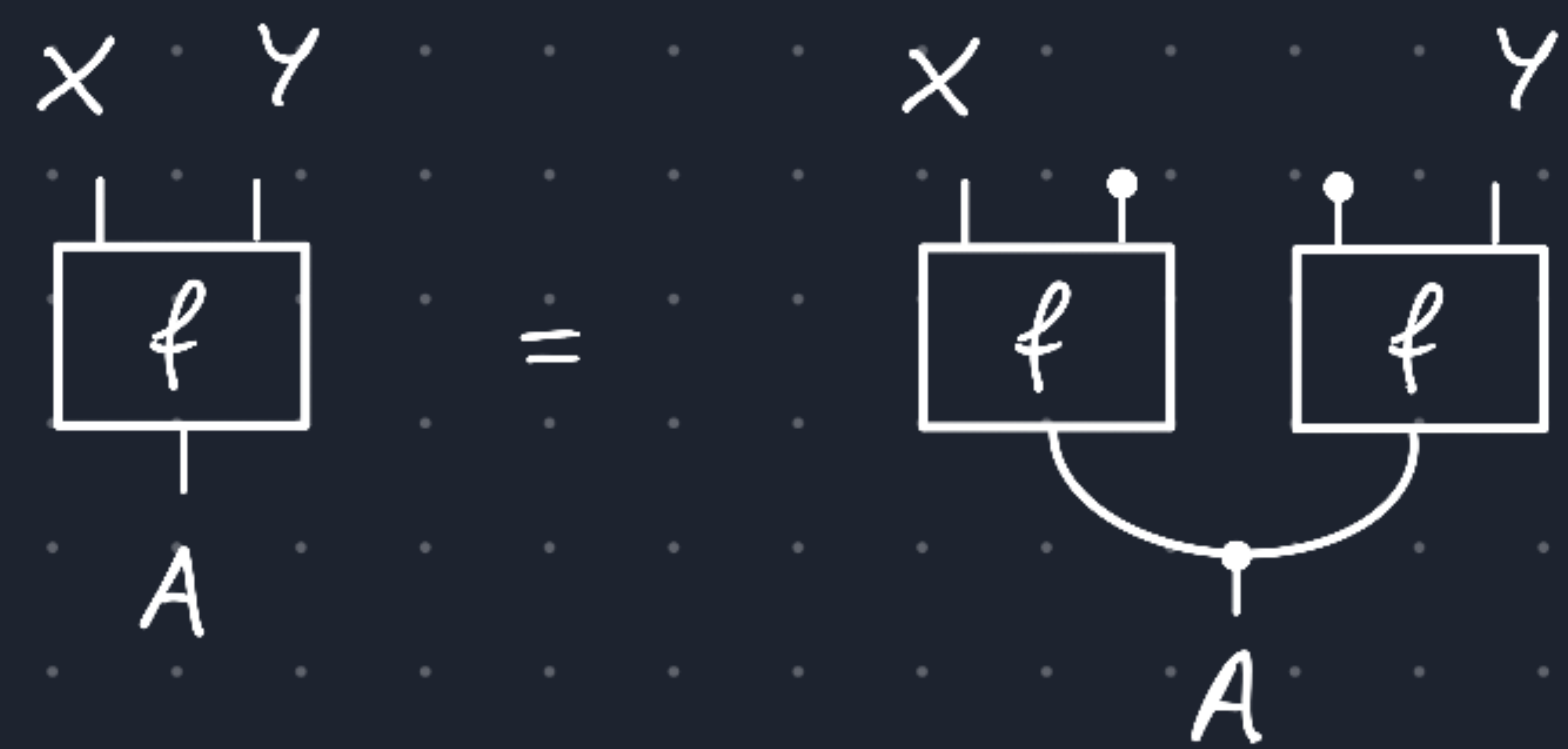

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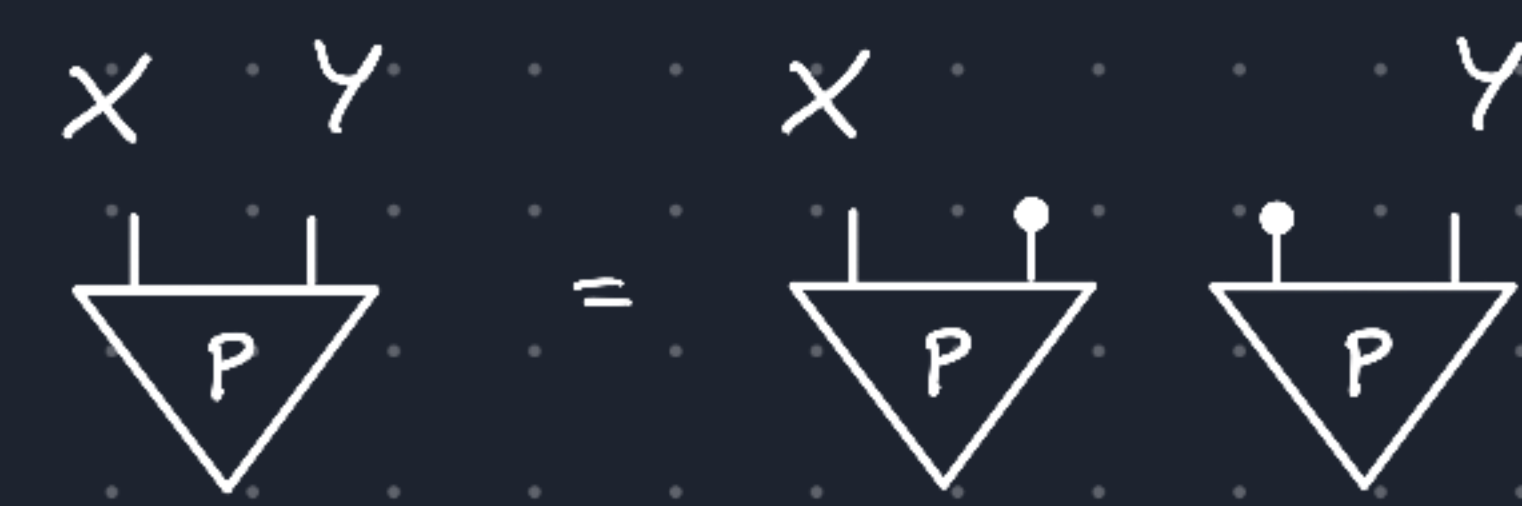

Example. In FinStoch: $p(x,y) = p(x)\,p(y)$ (e.g. coin flips)

$$f(x,y \mid a) = f(x \mid a)\, f(y \mid a)$$

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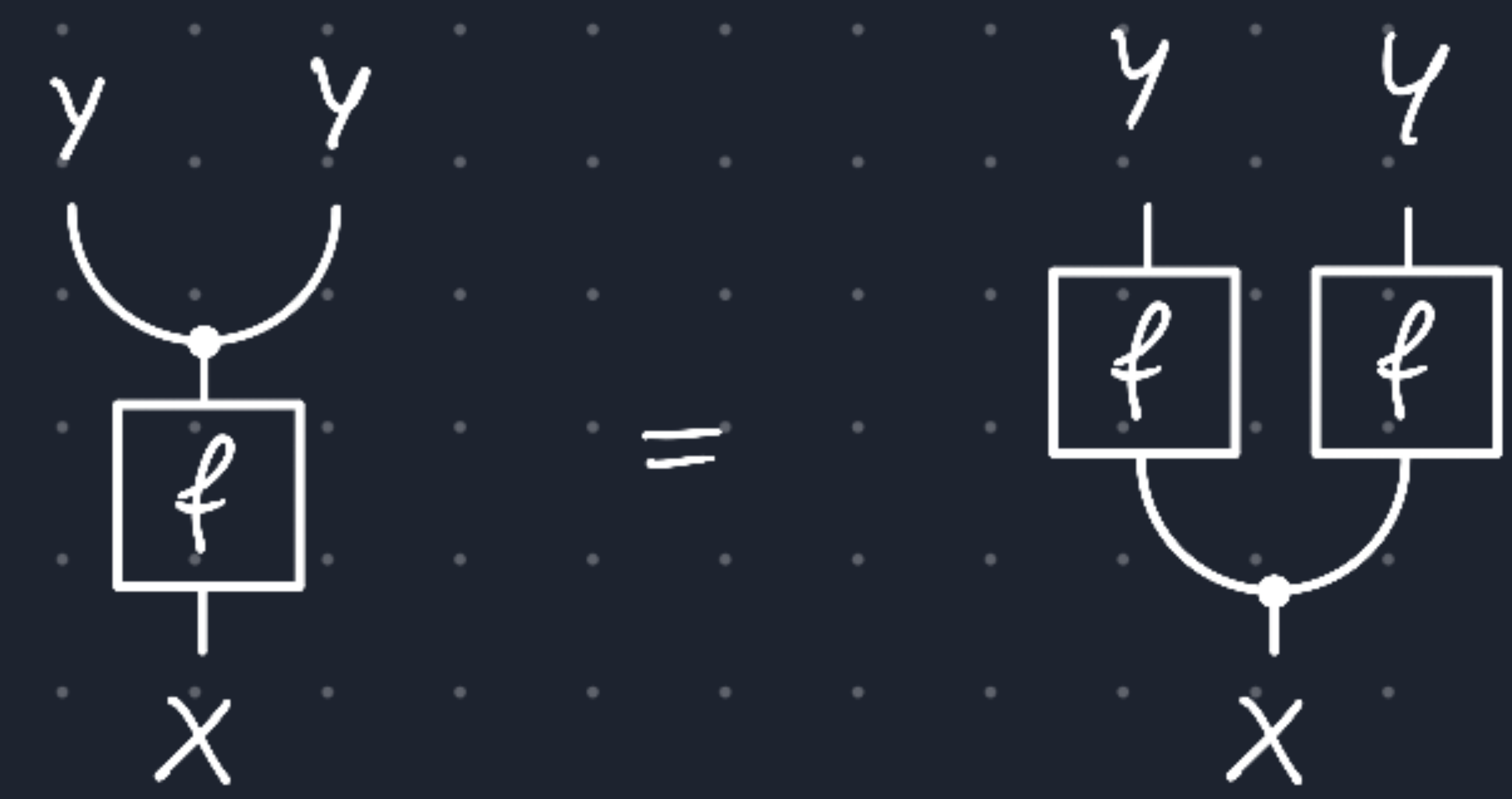

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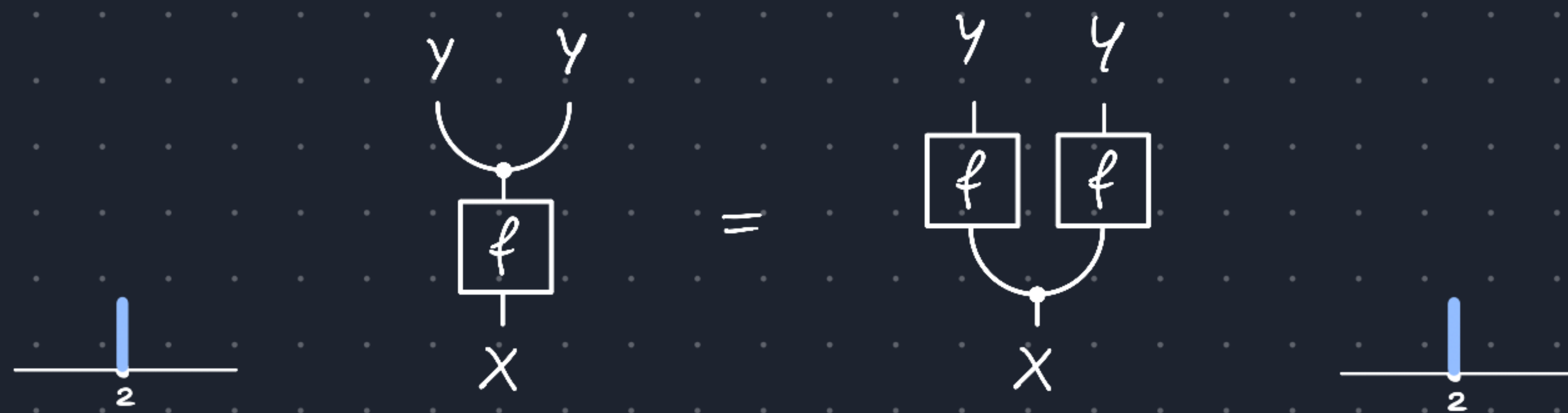

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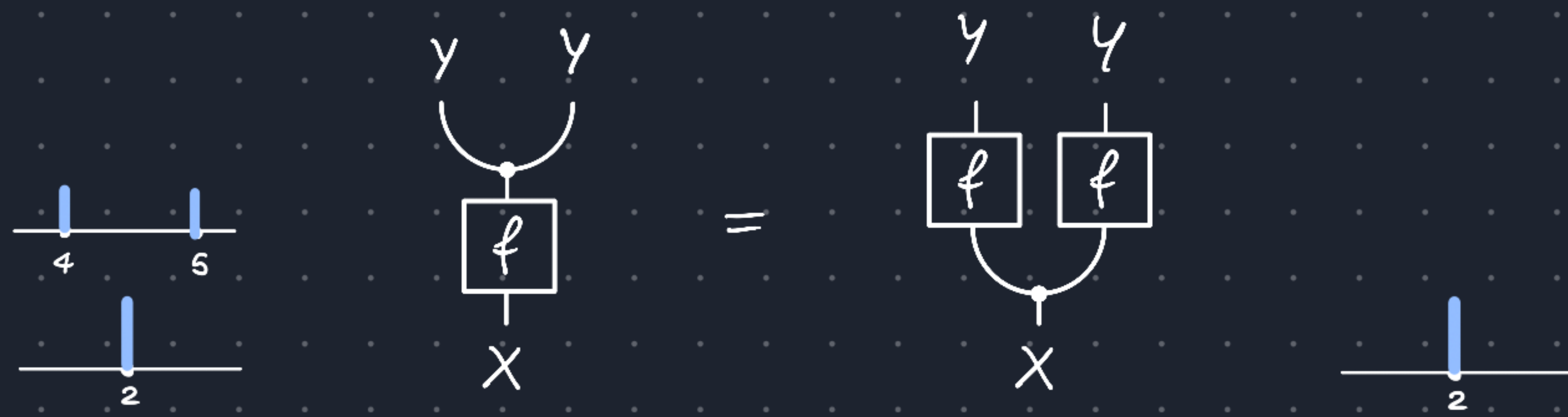

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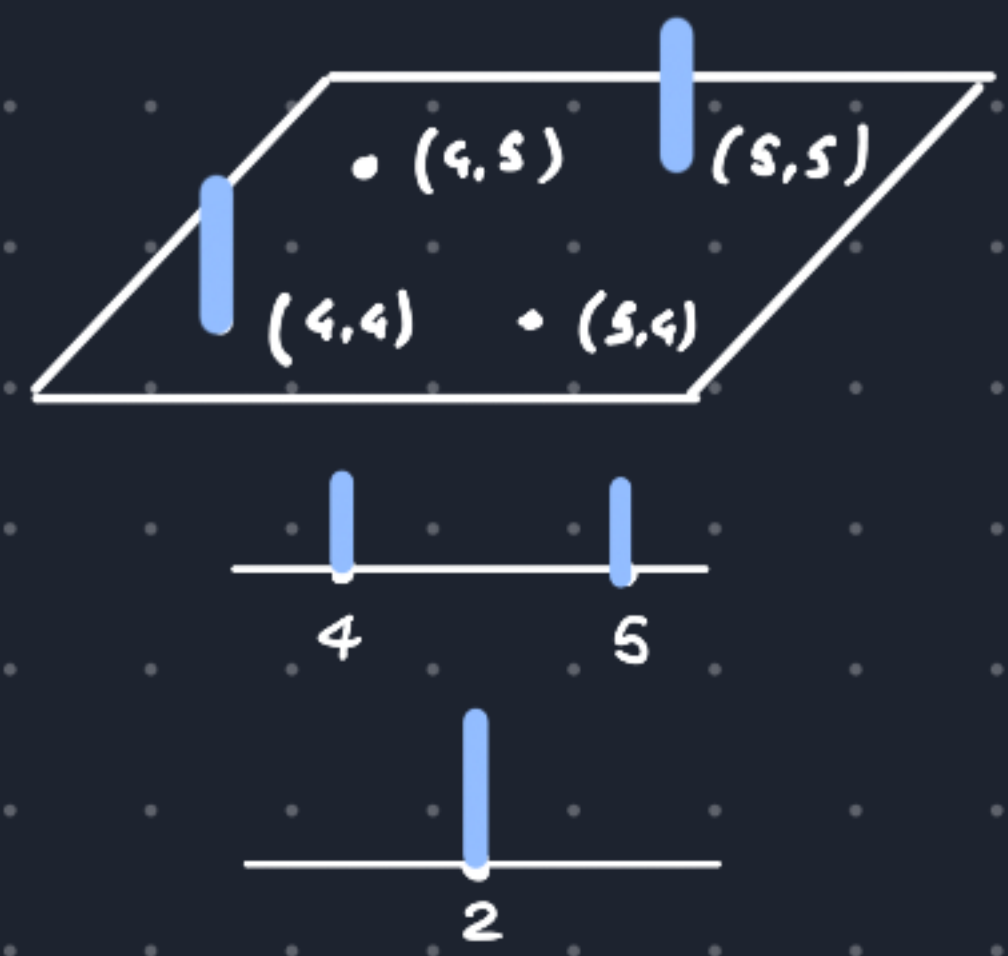

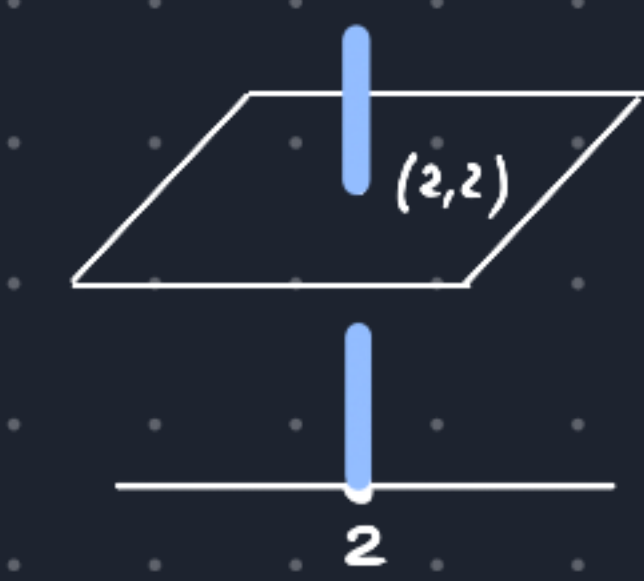

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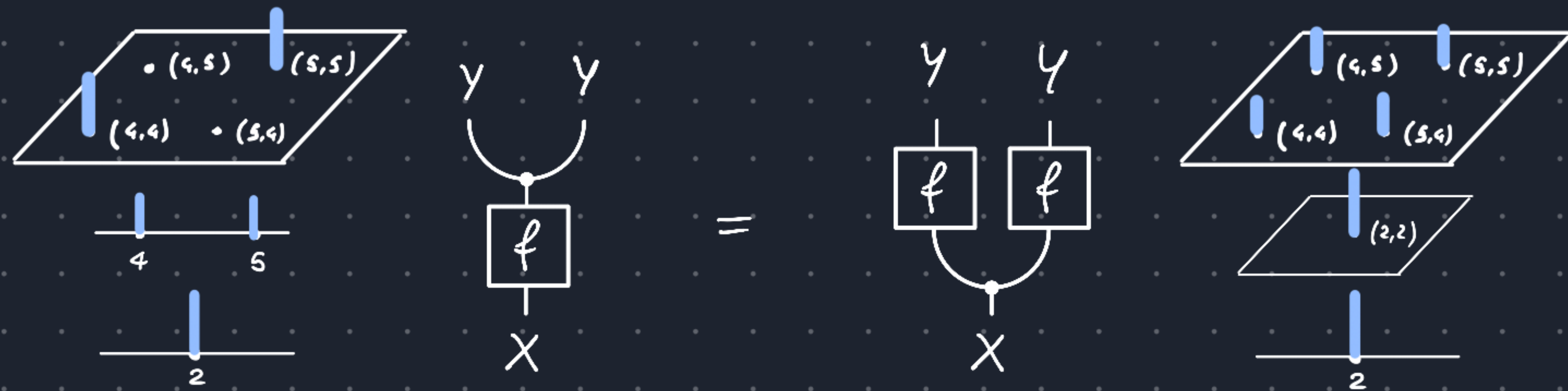

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Example. In FinStoch :

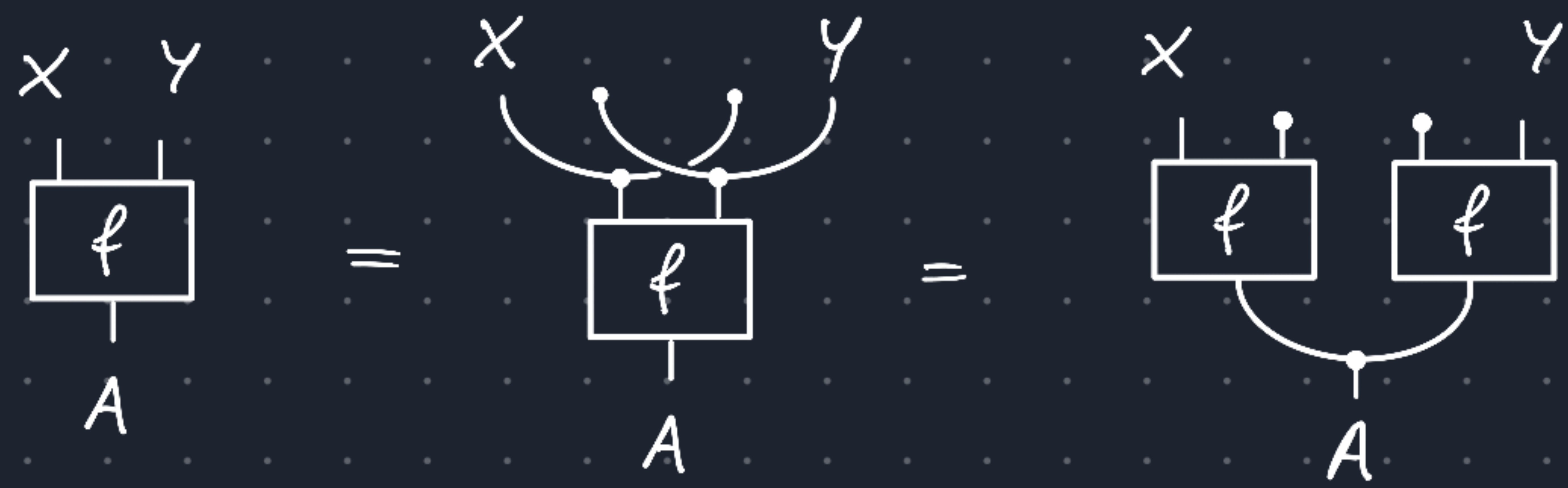
$$p(x)\,\delta_{xx'} = p(x)\,p(x')$$

$$p(x) = p(x)^2$$

$$p(x) = 0 \text{ or } 1$$

Prop. If $A \xrightarrow{f} X \otimes Y$ is deterministic, $X \perp Y \mid A$.

Proof.



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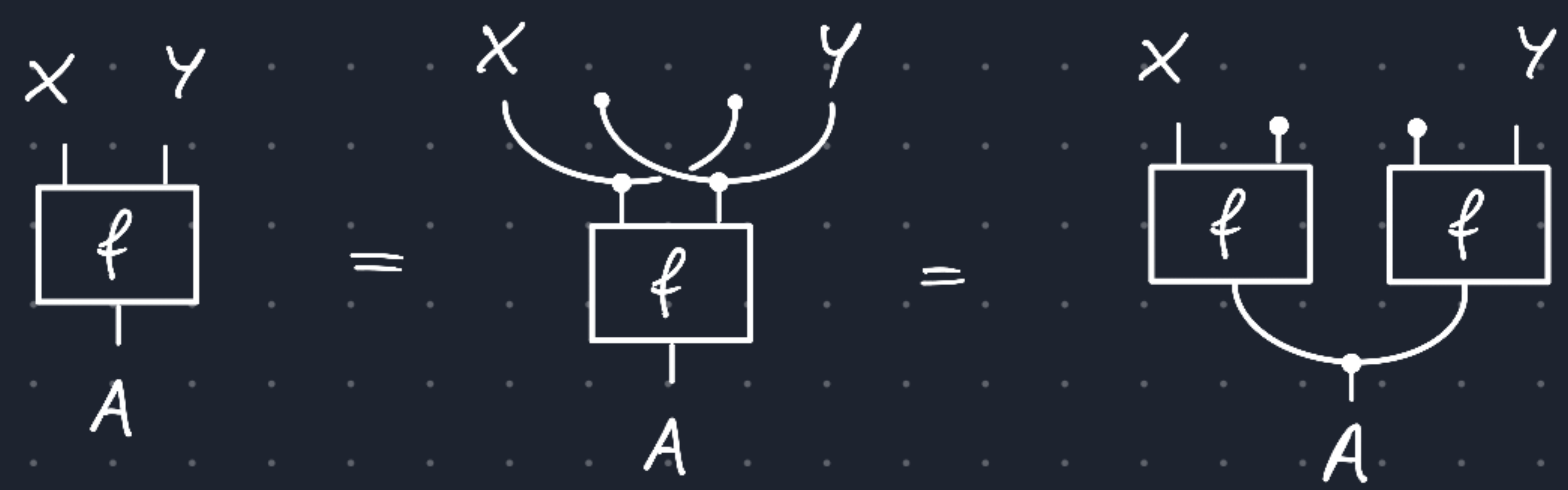

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1) Every morphism exhibits c. independence
2) Every morphism is deterministic
3) The copy maps are natural
4) $\mathcal{C}$ is cartesian ($\otimes \cong \times$)

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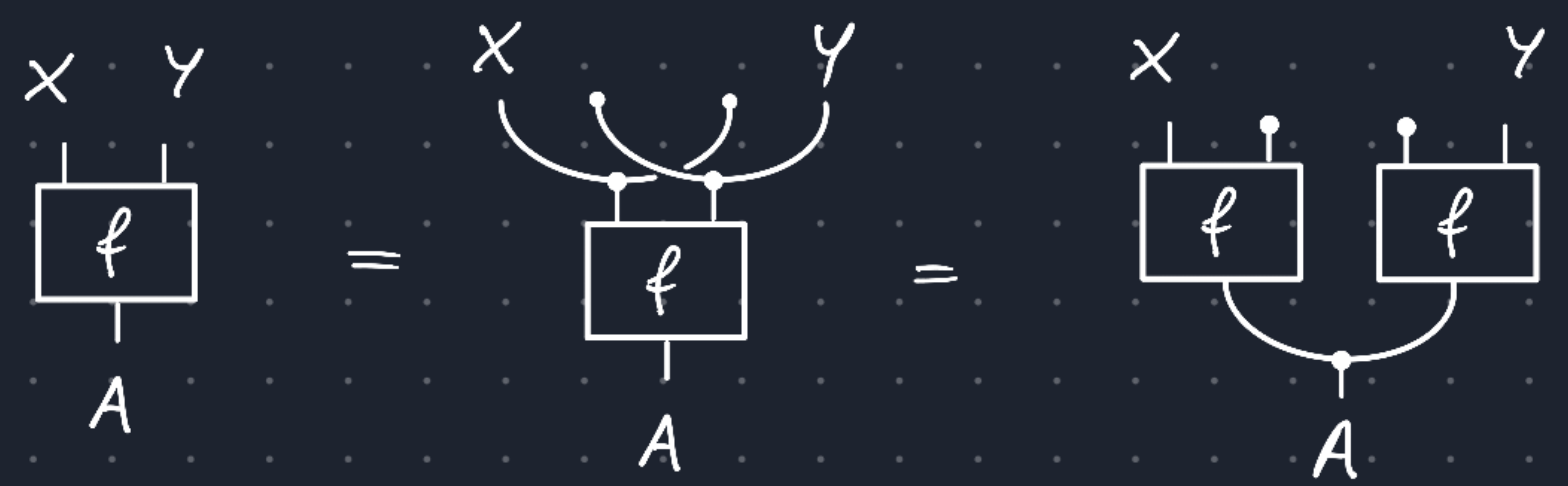

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$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{copy}"'] & Y \arrow[d, "\mathrm{copy}"] \\ X \otimes X \arrow[r, "f \otimes f"'] & Y \otimes Y \end{tikzcd}$$

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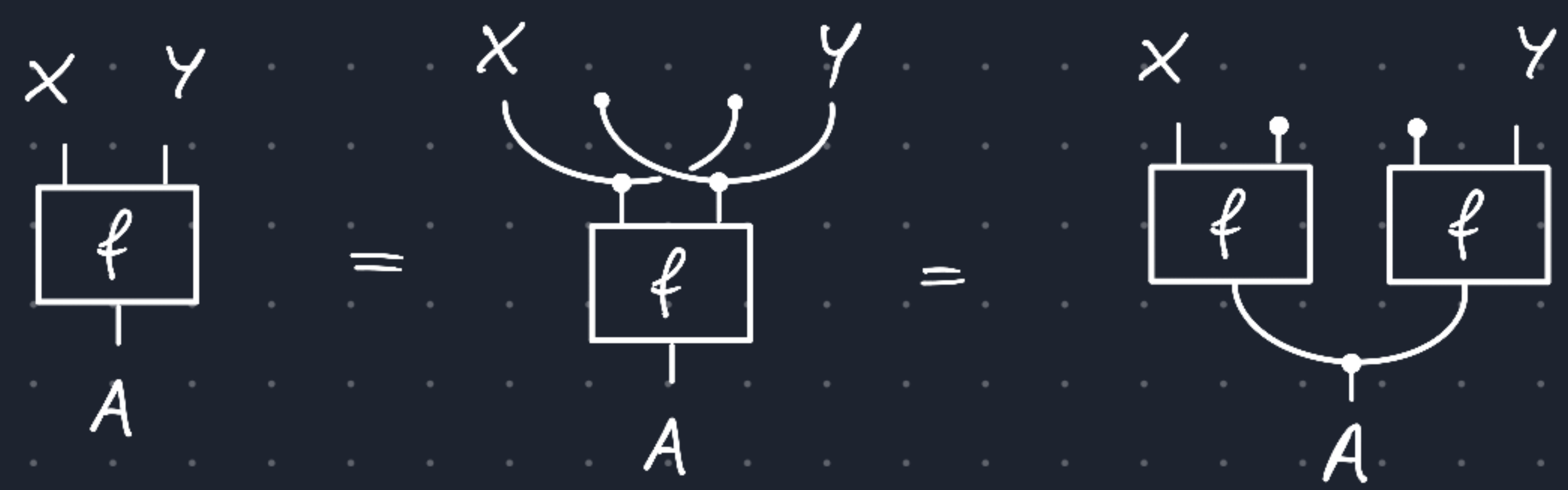

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not prob. theory!

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2. INFINITE SEQUENCES

A stochastic process is (equivalently) given by a probability measure on $X^{\mathbb{N}}$, a countably* infinite product.

*discrete time

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Def. Let $\mathcal{C}$ be a **Markov** category. Let I be an infinite set, and $\{X_i\}_{i\in I}$ as before.

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Theorem (Kolmogorov extension). BorelStoch has countable Kolmogorov products.

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Consider a monad (P, μ, η) which is

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$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

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✓

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Def. A commutative monad is called *observational* if for each X, the family

$$\left\{ \quad , \; n \in \mathbb{N} \right\}$$

is jointly monic.

*or $\mathbb{N}$, if Kolmogorov products exist.

Statistically, this means: we can tell apart two distributions by taking an arbitrarily large number of independent samples.

Example.

Flip a coin (

, )

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  So: expect 50/50.

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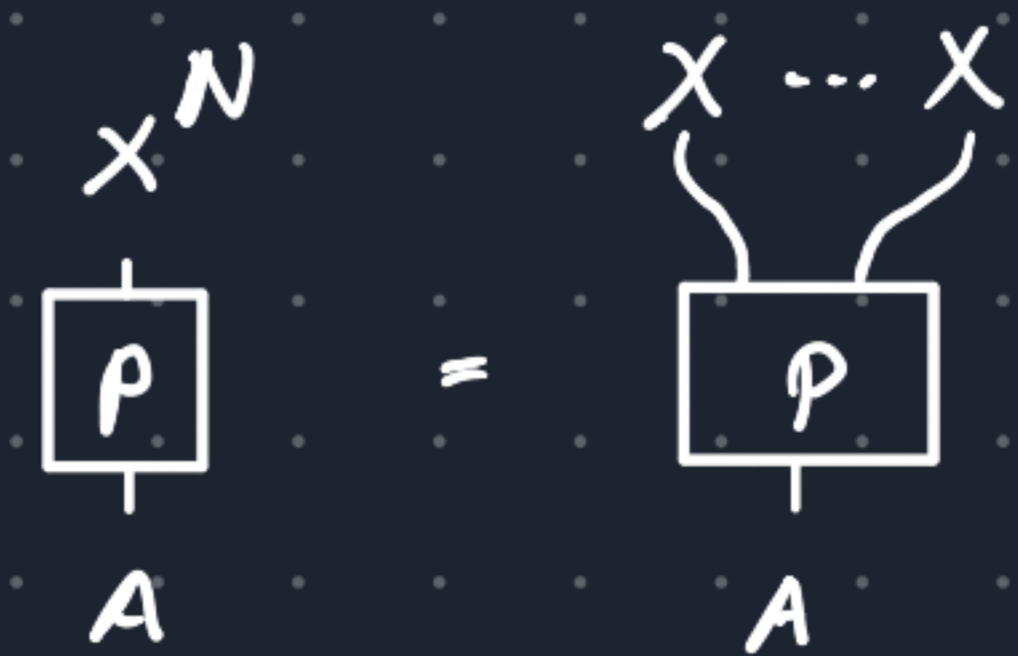
Given the distribution, the flips are independent!

4. DE FINETTI'S THEOREM AS A LIMIT

A morphism



is called **exchangeable** if it commutes with finite permutations (in the result).

$$\begin{tikzcd} & X^{\mathbb{N}} \arrow[d, "\sigma"] \\ A \arrow[ur, "p"] \arrow[r, "p"'] & X^{\mathbb{N}} \end{tikzcd}$$

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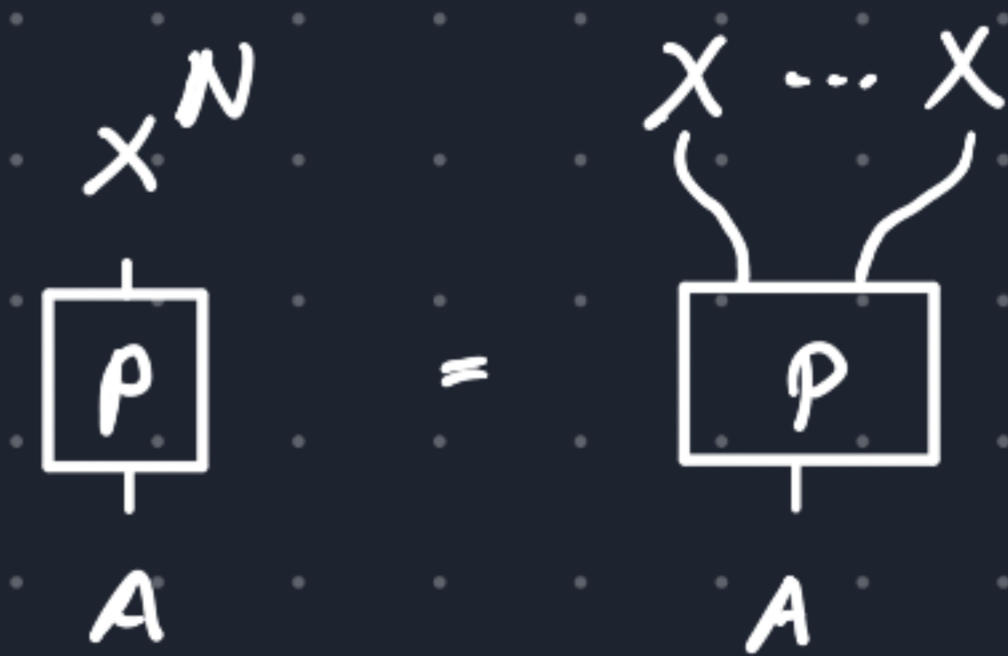
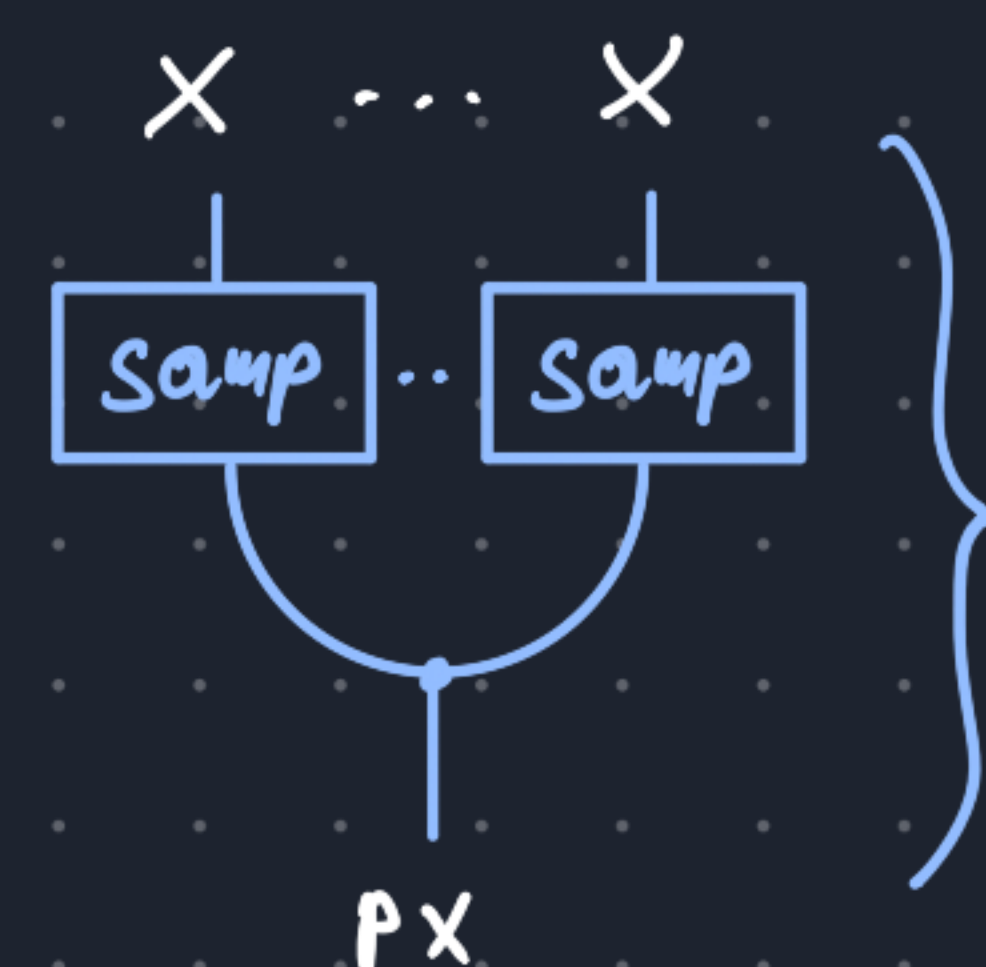
A morphism  is called *exchangeable* if it commutes with finite permutations (in the result).

$$\begin{tikzcd}[column sep=small] & X^{\mathbb{N}} \arrow[dd, "\sigma"] \\ A \arrow[ur, "p"] \arrow[dr, "p"'] & \\ & X^{\mathbb{N}} \end{tikzcd}$$

Theorem. In BorelStoch, for every A and X, there is a natural bijection

$$\begin{tikzcd} & & X^{\mathbb{N}} \arrow[dd, "\sigma"] \\ A \arrow[urr, "p"] \arrow[drr, "p"'] \arrow[r, dashed] & PX \arrow[ur] \arrow[dr] & \\ & & X^{\mathbb{N}} \end{tikzcd}$$



"iid sampling" is the limit cone.

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Now

$$\begin{tikzcd} & X^N \arrow[d, bend left, "t^\#"] \\ I \arrow[ur, "p"] \arrow[r, dashed] & PX \arrow[u, "\mathrm{samp}_N"] \end{tikzcd}$$

$$e_{det}(X^N, PX) \cong e(X^N, X)$$

$$t^\# \longleftrightarrow t$$

MORE UNIVERSAL PROPERTIES

- Invariant, tail σ-algebras: these are particular colimits
- Ergodic decompositions: these are particular factorizations
- "Positivity" of measures as a pullback condition

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Works in progress:

- Splitting idempotents in Markov categories
- Universal properties of "supports of measures"

} with T. Fritz, T. Gonda, A. Lorenzin, D. Stein

- Universal properties almost surely, "up to measure zero" - with Noé Ensarguet

IN CONCLUSION :

1. NO PRODUCTS
2. INFINITE SEQUENCES
3. PROBABILITY MONADS
4. DE FINETTI AS A LIMIT

Probability theory can be highly structured,
full of interesting universal properties,
and pleasing to a category theorist's eye,
if we look at it in the right way!

Some references:

- T. Fritz, A synthetic approach to Markov kernels, conditional independence, and theorems on sufficient statistics. Advances in Mathematics. arXiv: 1908.07021
- T. Fritz, T. Gonda, P. Perrone, E. F. Rischel, Representable Markov categories and comparison of statistical experiments in categorical probability. arXiv: 2010.07416
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... & more