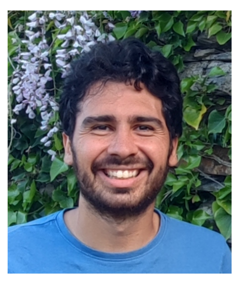
2-classifiers via dense generators and the case of stacks

Luca Mesiti
University of Leeds
mmlme@leeds.ac.uk

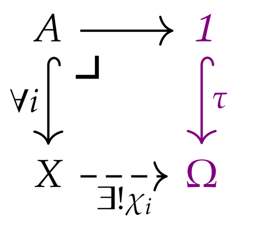


Abstract. Subobject classifiers are a fundamental notion of category theory, as they are the main ingredient of elementary toposes. A 2-dimensional generalization of subobject classifiers has been proposed by Weber, upgrading the monomorphisms (fibres of dimension 0) to discrete opfibrations (fibres of dimension 1). The classification process then shifts from pullbacks to comma objects. Our main result is that the study of the 2-dimensional classifiers can be reduced to dense generators. As an application, we find a 2-dimensional classifier in 2-presheaves and in stacks.

2-dimensional classifiers

In dimension 1, the archetypal subobject classifier is the set $\{T, F\}$ of truth values. It allows to encode subsets (and hence propositions) via characteristic functions, exhibiting $\mathcal{S}et$ as the archetypal elementary topos. We identity a subset $A \subseteq X$ with $\chi_A\colon X \to \{T, F\}$ that sends x to T if $x \in A$ and to F otherwise.

Definition 1. Let $\mathcal{C}$ be a 1-category with pullbacks. A **subobject classifier** in $\mathcal{C}$ is a monomorphism $\tau\colon 1 \hookrightarrow \Omega$ in $\mathcal{C}$ that is universal in the following sense: every subobject $i\colon A \hookrightarrow X$ in $\mathcal{C}$ is the pullback of τ along a unique morphism $\chi_i\colon X \to \Omega$ (called the characteristic morphism of i).



Equivalently, for every $X \in \mathcal{C}$ the function

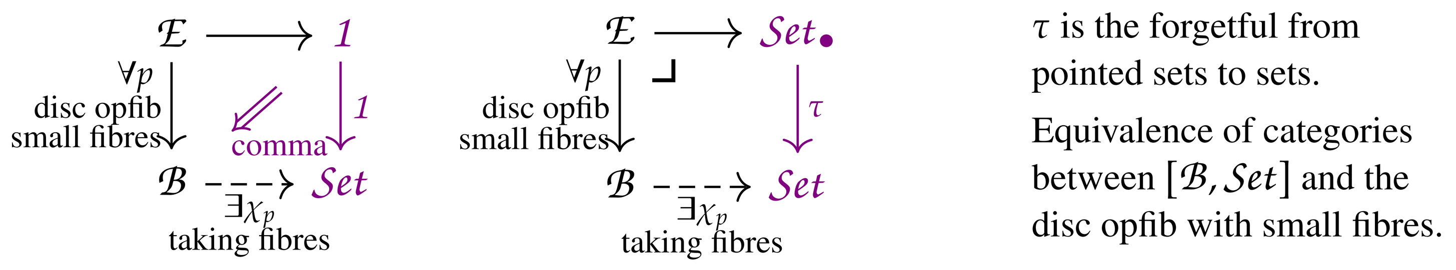
$$\mathcal{G}_{\tau,X}\colon \mathcal{E}(X,\Omega) \to \operatorname{Sub}(X)$$

given by pulling back τ is a bijection.

If $\mathcal{E}$ has a subobject classifier, has all finite limits and is cartesian closed, then $\mathcal{E}$ is called an **elementary topos**. In particular, it has an internal logic.

Remark 2. The archetypal characteristic functions are secretly considering the fibres of the inclusion $A \hookrightarrow X$. Such fibres are either singletons or empty and thus have dimension 0. In dimension 2, Weber proposes to classify discrete opfibrations. Such morphisms have fibres of dimension 1, since their fibres are (essentially) general sets.

Example 3. The archetypal 2-dimensional classification process is the **construction of the category of elements** (Grothendieck construction), that exhibits $\mathcal{CAT}$ as the archetypal elementary 2-topos. This construction can be captured either with a comma object from $1\colon 1 \to \mathcal{S}et$ or with a pullback along a replacement, that is precisely the **lax limit of the arrow** $1\colon 1 \to \mathcal{S}et$ (comma from $1\colon 1 \to \mathcal{S}et$ to the identity).

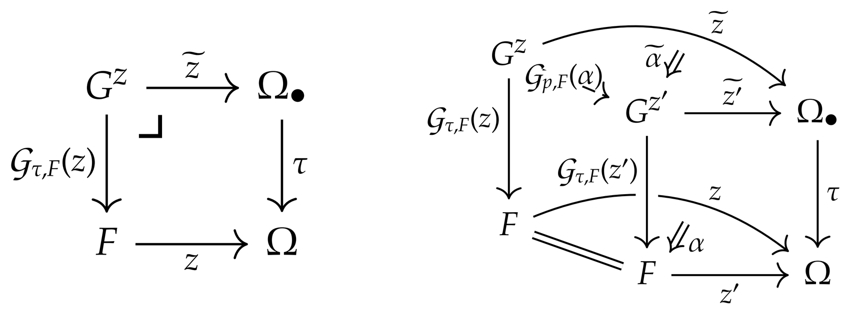


Moving to dimension 2, we can either upgrade the classification process to be regulated by commas or still take pullbacks but now of a discrete opfibration. Weber chose the latter, which is slightly more general.

Definition 4 (Weber)**.** Let $\mathcal{K}$ be a 2-category. A **2-classifier** in $\mathcal{K}$ is a discrete opfibration $\tau\colon \Omega_\bullet \to \Omega$ in $\mathcal{K}$ with small fibres such that for every $F \in \mathcal{K}$ the functor

$$\mathcal{G}_{\tau,F}\colon \mathcal{K}(F,\Omega) \to \mathcal{DO}p\mathcal{F}ib_{\mathcal{K}}(X)$$

given by pulling back $\tau\colon \Omega_\bullet \to \Omega$ and calculating liftings of 2-cells along τ is **fully faithful**.



We say that a discrete opfibration $\varphi\colon G \to F$ in $\mathcal{K}$ is **classified by** τ if φ is in the essential image of $\mathcal{G}_{\tau,F}$.

Remark 5. The functors $\mathcal{G}_{\tau,F}$ are automatically pseudonatural in F. We are asking $\mathcal{G}_{\tau,F}$ to be an equivalence of categories with its essential image, that could be smaller than all discrete opfibrations with small fibres.

Reduction to dense generators

We reduce the study of the conditions for a discrete opfibration $\tau\colon \Omega_\bullet \to \Omega$ in a 2-category $\mathcal{K}$ to be a 2-classifier to dense generators. The study of what gets classified is reduced as well.

Definition 6. A 2-functor $J\colon \mathcal{Y} \to \mathcal{K}$ is **dense** if the restricted Yoneda embedding $\widetilde{J}\colon \mathcal{K} \to [\mathcal{Y}^{\mathrm{op}}, \mathcal{CAT}]$ that sends $F \in \mathcal{K}$ to $\mathcal{K}(J(-), F)$ is fully faithful.

If J is fully faithful, this is equivalent to ask each object $F \in \mathcal{K}$ to be a weighted 2-colimit of objects of $\mathcal{Y}$ that is preserved by $\widetilde{J}$ (we say J-**absolute** 2-colimit).

Example 7. The inclusion y of the representables inside 2-presheaves is a dense generator, as $\widetilde{y}$ is the identity. Every 2-presheaf is a weighted 2-colimit of representables.

Assume $\mathcal{K}$ has terminal object, pullbacks along discrete opfibrations and comma objects. Let then $J\colon \mathcal{Y} \to \mathcal{K}$ be a fully faithful dense generator of $\mathcal{K}$.

Remark 8. We prove that we can check the faithfulness of the functors $\mathcal{G}_{\tau,F}$ just on those $F \in \mathcal{Y}$. This first reduction actually only needs F to be any 2-colimit of objects of $\mathcal{Y}$ (i.e. J a naive generator).

Proposition 9 (M.)**.** *If for every $Y \in \mathcal{Y}$ the functor $\mathcal{G}_{\tau,Y}\colon \mathcal{K}(Y,\Omega) \to \mathcal{DO}p\mathcal{F}ib(Y)$ is faithful, then for every $F \in \mathcal{K}$ the functor $\mathcal{G}_{\tau,F}\colon \mathcal{K}(F,\Omega) \to \mathcal{DO}p\mathcal{F}ib(F)$ is faithful.*

Idea. Given $\alpha, \alpha'\colon z \Rightarrow z'\colon F \to \Omega$ such that $\mathcal{G}_{\tau,F}(\alpha) = \mathcal{G}_{\tau,F}(\alpha')$, we want to show that $\alpha = \alpha'$. It suffices to look at the 2-cells $\alpha * \Lambda_A(X)$ given by whiskering α with the morphisms of the universal cocylinder that exhibits F as a 2-colimit of objects of $\mathcal{Y}$. But $\mathcal{G}_{\tau,D(A)}(\alpha\,\Lambda_A(X))$ is essentially given by $\Lambda_A(X)^*\left(\mathcal{G}_{\tau,F}(\alpha)\right)$. □

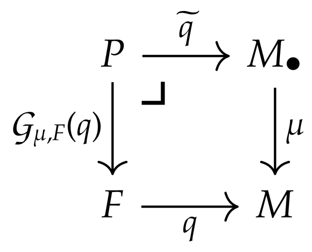
Remark 10. The reduction of the fullness of the functors $\mathcal{G}_{\tau,F}$ and of what gets classified requires more technology. We need to consider the kind of 2-colimits described below. These give a theory equivalent to that of weighted 2-colimits [Street, new proof M.[2]], but they are **essentially conical**.

Definition 11. Consider 2-functors $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{CAT}$ (a marking) with $\mathcal{A}$ small and $F\colon \int W \to \mathcal{K}$ (a diagram) where $\int W$ is the Grothendieck construction of W. The **oplax normal conical 2-colimit** of F is given by a universal oplax normal natural transformation

$$\Delta 1 \xRightarrow[\text{oplax}^{\mathrm{n}}]{\Lambda} \mathcal{K}(F(-), \text{oplax}^{\mathrm{n}}\text{-colim}F),$$

where oplax normal means oplax with the request that $\Lambda_{(f,\mathrm{id})} = \mathrm{id}$ for every morphism (f, id) in $\left(\int W\right)^{\mathrm{op}}$.

Construction 12 (M.[1])**.** Given $F \in \mathcal{K}$, there exists a 2-diagram $K\colon \int W \to \mathcal{K}$ which factors through $\mathcal{Y}$ and makes F into an J-absolute oplax normal conical 2-colimit of K. Call Λ the universal oplax normal cocone. Consider now $q\colon F \to M$, a discrete opfibration $\mu\colon M_\bullet \to M$ and the 2-pullback in $\mathcal{K}$



We express P as an oplax normal conical colimit of a diagram constructed from K and Λ. First we exhibit $q = \text{oplax}^{\mathrm{n}}\text{-colim}K'$ in $\mathcal{K}/_{\mathrm{lax}}M$. This generalizes the well known fact that a colimit in a 1-dimensional slice is precisely the map from the colimit of the domains which is induced by the universal property. We need the lax slice because we only have essentially conical 2-colimits. Then we need to consider a change of base 2-functor μ^* between lax slices. The 2-functor

$$\mathcal{K}/_{\mathrm{lax}}M \xrightarrow{\mu^*} \mathcal{K}/_{\mathrm{lax}}M_\bullet \xrightarrow{\mathrm{dom}} \mathcal{K}$$

preserves the colimit q, exhibiting $P = \text{oplax}^{\mathrm{n}}\text{-colim}(\mathrm{dom} \circ \mu^* \circ K')$. We can also apply this to $q = \mathrm{id}_F$.

Theorem 13 (M.)**.** *If for every $Y \in \mathcal{Y}$ the functor $\mathcal{G}_{\tau,Y}\colon \mathcal{K}(Y,\Omega) \to \mathcal{DO}p\mathcal{F}ib(Y)$ is fully faithful, then for every $F \in \mathcal{K}$ the functor $\mathcal{G}_{\tau,F}\colon \mathcal{K}(F,\Omega) \to \mathcal{DO}p\mathcal{F}ib(F)$ is full (and faithful).*

Idea. Consider $z, z'\colon F \to \Omega$ in $\mathcal{K}$ and $h\colon \mathcal{G}_{\tau,F}(z) \to \mathcal{G}_{\tau,F}(z')$. We produce $\alpha\colon z \Rightarrow z'$ such that $\mathcal{G}_{\tau,F}(\alpha) = h$. We write F as an oplax normal conical 2-colimit of objects of $\mathcal{Y}$ and define the "components" of α by fullness of the functors $\mathcal{G}_{\tau,Y}$. The faithfulness of the $\mathcal{G}_{\tau,Y}$'s guarantees that they induce a 2-cell $\alpha\colon z \Rightarrow z'$. To prove $\mathcal{G}_{\tau,F}(\alpha) = h$, we use the universal property of a colimit, thanks to Construction 12. □

Remark 14. We now know that we can check on a dense generator if τ is a 2-classifier in $\mathcal{K}$. We show that we can also reduce the study of what τ classifies. This would be smooth in the bicategorical context, considering bicolimits. In the context of strict 2-categories, we further ask that a normalization process is possible. Such extra assumption is true for 2-presheaves.

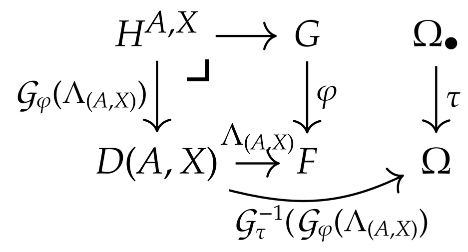
Theorem 15 (M.)**.** *Let τ be a 2-classifier in $\mathcal{K}$ and $\varphi\colon G \to F$ be a discrete opfibration in $\mathcal{K}$. Consider a 2-diagram $D\colon \int W \to \mathcal{K}$ that factors through $\mathcal{Y}$ such that $F = \text{oplax}^{\mathrm{n}}\text{-colim}D$ and this is J-absolute (guaranteed by density); call Λ the universal oplax normal cocone. The following facts are equivalent:*

(i) φ is classified by τ, i.e. φ is in the essential image of $\mathcal{G}_{\tau,F}$;

(ii) for every $(A, X) \in \int W$ the change of base of φ along $\Lambda_{A,X}$ is in the essential image of $\mathcal{G}_{\tau,D(A,X)}$ and the operation of normalization described below is possible.

Idea. $(i) \Rightarrow (ii)$: take as classifying morphism just a classifying morphism for φ precomposed with $\Lambda_{A,X}$.
$(ii) \Rightarrow (i)$: we **induce the classifying morphism $z\colon F \to \Omega$ for φ by the universal property of the colimit** $F = \text{oplax}^{\mathrm{n}}\text{-colim}D$. As every $\mathcal{G}_{\varphi,D(A,X)}(\Lambda_{A,X})$ is in the essential image of $\mathcal{G}_{\tau,D(A,X)}$, we can consider the oplax natural transformation χ given by the composite below on the left

$$\Delta 1 \xRightarrow[\text{oplax}^{\mathrm{n}}]{\Lambda} \mathcal{K}(D(-), F) \xRightarrow[\text{pseudo}]{\mathcal{G}_{\varphi,D(-)}} \mathcal{DO}p\mathcal{F}ib(D(-)) \xRightarrow[\text{pseudo}]{\mathcal{G}_{\tau,D(-)}^{-1}} \mathcal{K}(D(-), \Omega)$$



This is a sigma natural transformation, which would be enough to induce a morphism $z\colon F \to \Omega$ if we were in a bicategorical context. In our strict context, we assume that we can find an oplax normal natural transformation $\aleph$ isomorphic to χ. Then the morphism $z\colon F \to \Omega$ induced by $\aleph$ is a classifying morphism for φ with respect to τ. The strategy is to show that G is isomorphic over F to the 2-pullback of τ along z, by Construction 12. □

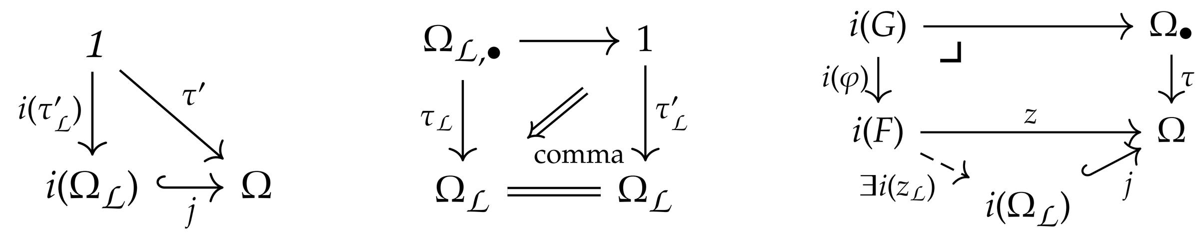
Example 16. A fortiori, the study of the construction of the category of elements, that is the archetypal 2-dimensional classification process (see Example 3), can be greatly reduced. Indeed the terminal 1 is a dense generator in $\mathcal{CAT}$. So we can just look at the discrete opfibrations over 1.

$$\mathcal{G}_{\tau,1}\colon \mathcal{CAT}(1, \mathcal{S}et) \to \mathcal{S}et$$

sends a functor $1 \to \mathcal{S}et$ to the set it picks, so it is an equivalence of categories. By the theorems above, the construction of the category of elements is fully faithful and classifies all discrete opfibrations with small fibres. Moreover, by the proof of Theorem 15, the classifying morphism for a φ is obtained by collecting all its fibres, since the pullback of φ along $B\colon 1 \to \mathcal{B}$ gives precisely the fibre over B.

The following proposition helps restrict a 2-classifier to a 2-classifier in a subcategory.

Proposition 17. *Let $\tau'\colon 1 \to \Omega$ in $\mathcal{K}$ such that its lax limit τ is a 2-classifier in $\mathcal{K}$. Consider $i\colon \mathcal{L} \overset{\mathrm{ff}}{\hookrightarrow} \mathcal{K}$ with $\mathcal{L}$ closed under terminal and comma objects of $\mathcal{K}$ (that is, i lifts them) and such that i preserves discrete opfibrations. Finally, let $\tau'_{\mathcal{L}}\colon 1 \to \Omega_{\mathcal{L}}$ in $\mathcal{L}$ such that $i(\Omega_{\mathcal{L}}) \overset{\mathrm{ff}}{\hookrightarrow} \Omega$ making the triangle below on the left commute. Then the lax limit $\tau_{\mathcal{L}}$ of $\tau'_{\mathcal{L}}$ is a 2-classifier in $\mathcal{L}$. Moreover, given φ a discrete opfibration in $\mathcal{L}$, if $i(\varphi)$ is classified by τ via a **classifying morphism z that factorizes through $i(\Omega_{\mathcal{L}})$** then φ is classified by $\tau_{\mathcal{L}}$.*



Applications (work in progress)

We can apply the theory shown above to find a **2-classifier in 2-presheaves and in stacks**. The details of the ideas below are to be checked.

Let $\mathcal{A}$ be a 2-category and consider $\mathcal{K} = [\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]$. We search for $\tau'\colon 1 \to \Omega$ in $\mathcal{K}$ such that its lax limit τ is a 2-classifier. The representables form a dense generator, so we can just look at discrete opfibrations over representables. Wishing to classify all discrete opfibrations with small fibres, we want

$$\mathcal{G}_{\tau,y(A)}\colon [\mathcal{A}, \mathcal{CAT}](y(A), \Omega) \to \mathcal{DO}p\mathcal{F}ib(y(A))$$

to be an equivalence of categories for every $A \in \mathcal{A}$. But then we want $\Omega(A)$ to be equivalent to $\mathcal{DO}p\mathcal{F}ib(y(A))$ for every A. The assignment $A \overset{\Omega}{\mapsto} \mathcal{DO}p\mathcal{F}ib(y(A))$ only gives a pseudofunctor $\mathcal{A}^{\mathrm{op}} \to \mathcal{CAT}$. In a bicategorical context, this would be perfectly fine. In our strict context, we replace $\mathcal{DO}p\mathcal{F}ib(y(A))$ with $\left[q\text{-}\left(\mathcal{A}/_{\mathrm{oplax}\,A}\right)^{\mathrm{op}}, \mathcal{S}et\right]$, where $q\text{-}\left(\mathcal{A}/_{\mathrm{oplax}\,A}\right)$ is the quotient of the oplax slice by its 2-cells. We should be able to do such replacement by an **indexed version of the Grothendieck construction** that does not seem to appear in the literature:

$$\mathcal{O}p\mathcal{F}ib_{[\mathcal{A},\mathcal{CAT}]}(F) \simeq \left[\int F, \mathcal{CAT}\right] \quad \text{that restricts to} \quad \mathcal{DO}p\mathcal{F}ib_{[\mathcal{A},\mathcal{CAT}]}(F) \simeq \left[q\text{-}\int F, \mathcal{S}et\right]$$

If $\mathcal{A}$ is a 1-category and $F\colon \mathcal{A} \to \mathcal{S}et$ then the right hand side equivalence becomes the well known

$$[\mathcal{A}^{\mathrm{op}}, \mathcal{S}et]/_F \simeq \left[\int F, \mathcal{S}et\right].$$

Ω should then be a 2-classifier in $[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]$. When $\mathcal{A}$ is a 1-category, Ω becomes the Hofmann-Streicher universe.

Taking $\mathcal{A}$ to be a 1-category $\mathcal{C}$ (as it is usually done in geometry) with a Grothendieck topology on it, we would like to **restrict Ω to a 2-classifier in stacks**. The idea is to restrict $\Omega(A) = \left[\left(\mathcal{A}/_A\right)^{\mathrm{op}}, \mathcal{S}et\right]$ to the sheaves on $\mathcal{A}/_A$ for every $A \in \mathcal{A}$. As written by Hofmann and Streicher, this does not form a sheaf. So it does not form a 2-classifier in 2-dimensional sheaves. Nonetheless, it is a stack that is strictly functorial. The factorization assumed in Proposition 17 is at least true on representables. From this, it should follow that $A \mapsto \mathcal{S}h\left(\mathcal{A}/_A\right)$ is a 2-classifier in strictly functorial stacks.

References

[1] Luca Mesiti. Colimits in 2-dimensional slices. *arXiv:2305.01494*, 2023.
[2] Luca Mesiti. The 2-Set-enriched Grothendieck construction and the lax normal conical 2-limits. *arXiv:2302.04566*, 2023.
[3] Ross Street. Limits indexed by category-valued 2-functors. *J. Pure Appl. Algebra*, 8:149–181, 1976.
[4] Mark Weber. Yoneda structures from 2-toposes. *Appl. Categ. Struct.*, 15(3):259–323, 2007.