Coinductive equivalences in algebraic weak ω-categories

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Category Theory 2023

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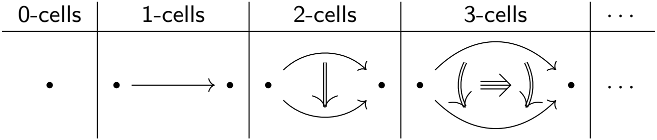
[2]Supported by JSPS Overseas Research Fellowship and Australian Research Council Discovery Project DP190102432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1-cells
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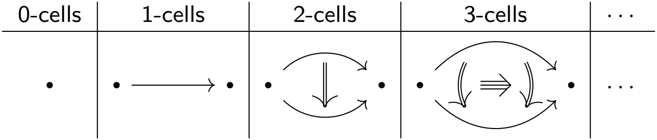
Weak ω-categories?



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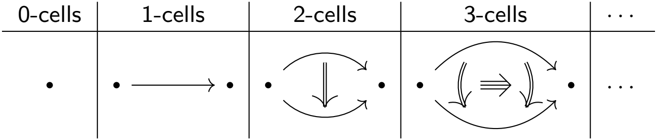

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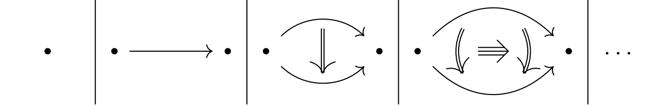

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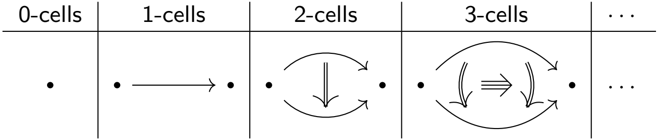
equipped with extra structure encoded by a monad $T^{\mathrm{w}k}$.



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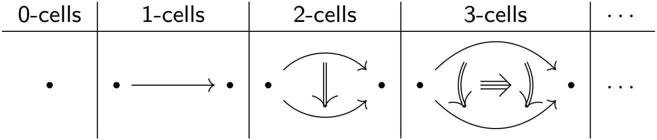

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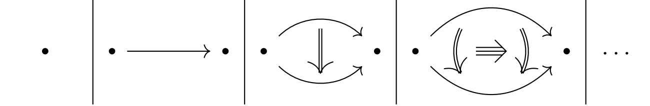

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We should have $\{$strict ω-cats$\} \subset \{$weak ω-cats$\}$, or equivalently a monad map $\alpha : T^{\mathrm{w}k} \to T^{\mathrm{st}}$.

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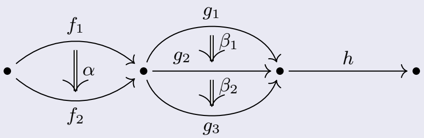

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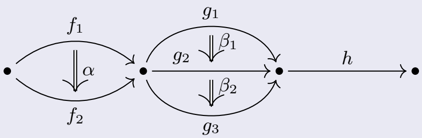

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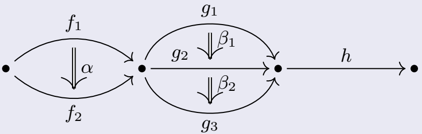

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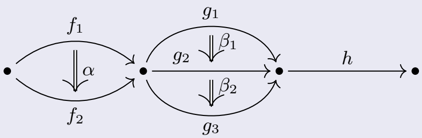

in a strict 2-category gives rise to a unique 2-cell $hg_1f_1 \to hg_3f_2$.

In a weak 2-category (bicategory), we similarly get a unique 2-cell but only after specifying what we mean by "hg_1f_1" and "hg_3f_2".

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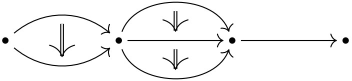
- $(T^{\mathrm{st}}1)_2$ contains cells like  .

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Existence part of Pasting Theorem

We ask that any commutative square

$$\begin{array}{ccc} \partial G^n & \longrightarrow & T^{\mathrm{wk}}1 \\ \downarrow & \nearrow & \downarrow{\scriptstyle \alpha_1} \\ G^n & \longrightarrow & T^{\mathrm{st}}1 \end{array}$$

admit a chosen diagonal lift for $n \geq 1$.

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"f is an equivalence" means "f admits such an infinite hierarchy of witnesses"

Uniqueness part of Pasting Theorem (Fujii-Hoshino-M.)

Let $(X, T^{\mathrm{wk}}X \xrightarrow{\xi} X)$ *be a weak* ω*-category. If* $u /\!\!/ v$ *in* $(T^{\mathrm{wk}}X)_n$ *and* $\alpha_X(u) = \alpha_X(v)$ *then there is an equivalence* $(n+1)$*-cell* $\xi(u) \to \xi(v)$ *in* X*.*

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Using these facts, we can treat weak ω-categories just like strict ones.

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The class of equivalence n-cells in a weak ω-category is closed under pastings.

Using these facts, we can treat weak ω-categories just like strict ones...???

Uniqueness part of Pasting Theorem (Fujii-Hoshino-M.)

Let $(X, T^{\mathrm{wk}}X \xrightarrow{\xi} X)$ be a weak ω-category. If $u /\!/ v$ in $(T^{\mathrm{wk}}X)_n$ and $\alpha_X(u) = \alpha_X(v)$ then there is an equivalence $(n+1)$-cell $\xi(u) \to \xi(v)$ in X.

Instances of this result yield:

$$h(gf) \sim (hg)f, \quad 1f \sim f \sim f1 \quad \text{etc.}$$

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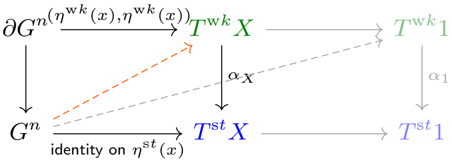
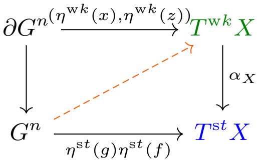
Using these facts, we can treat weak ω-categories just like strict ones...???

Thank you!

Let $(X, T^{\mathrm{wk}}X \xrightarrow{\xi} X)$ be a weak ω-category and $x \in X_{n-1}$.
We can define $1_x \in X_n$ by applying ξ to the lift in



Similarly, given n-cells $x \xrightarrow{f} y \xrightarrow{g} z$, we can define $gf \in X_n$ using

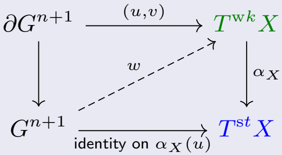


Uniqueness part of Pasting Theorem

Let $(X, T^{\mathrm{wk}}X \xrightarrow{\xi} X)$ *be a weak* ω*-category. If* $u /\!/ v$ *in* $(T^{\mathrm{wk}}X)_n$ *and* $\alpha_X(u) = \alpha_X(v)$ *then there is an equivalence* $(n+1)$*-cell* $\xi(u) \to \xi(v)$ *in* X*.*

Proof.

We proceed by coinduction. Obtain $w : u \to v$ as



and similarly $w' : v \to u$. Then we have $w'w /\!/ 1_u$ and $ww' /\!/ 1_v$ in $(T^{\mathrm{wk}}X)_{n+1}$, and $\alpha_X(w'w) = 1_{\alpha_X(u)} = \alpha_X(1_u)$ and $\alpha_X(ww') = 1_{\alpha_X(v)} = \alpha_X(1_v)$. $\square$