Barr-coexactness for representable spaces

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Louvain-la-Neuve, July 7, 2023

[a]Based on joint work with Pedro Nora and Marco Abbadini.

Some very classic results

Theorem (Stone (1936))

$$\underline{\text{BooSp}}^{\text{op}} \simeq \underline{\text{BA}}.$$

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Remark

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Definition (Nachbin (1950))

An ordered compact Hausdorff space $(X, \leq, \tau)$ consists of a set X, an order relation $\leq$ on X and a compact Hausdorff topology on X so that the set $\{(x, y) \in X \times X \mid x \leq y\}$ is closed in $X \times X$.

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Bottom line

The categories $\underline{\text{BooSp}}^{\text{op}}$ and $\underline{\text{Priest}}^{\text{op}}$ are Barr-exact.

About the algebraic character of CompHaus$^{\mathrm{op}}$

- $\underline{\mathsf{CompHaus}}^{\mathrm{op}} \xrightarrow{\hom(-,[0,1])} \underline{\mathsf{Set}}$ is monadic.

References

Duskin, John (1969). "Variations on Beck's tripleability criterion". In: Reports of the Midwest Category Seminar III. Ed. by Saunders MacLane. Springer Berlin Heidelberg, pp. 74–129.

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- $\underline{\mathsf{CompHaus}}^{\mathrm{op}} \xrightarrow{\hom(-,[0,1])} \underline{\mathsf{Set}}$ is monadic.
- $[0,1]$ is $\aleph_1$-ary copresentable in $\underline{\mathsf{CompHaus}}$. More general, the $\aleph_1$-ary copresentable compact Hausdorff spaces are precisely the metrisable ones.

References

Gabriel, Peter and Ulmer, Friedrich (1971). Lokal präsentierbare Kategorien. Vol. 221. Lecture Notes in Mathematics. Berlin: Springer-Verlag. v + 200.

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References

Isbell, John R. (1982). "Generating the algebraic theory of $C(X)$". In: Algebra Universalis 15.(2), pp. 153–155.

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- The algebraic theory of $\underline{\mathsf{CompHaus}}^{\mathrm{op}}$ can be generated by 5 operations.
- A complete description of the algebraic theory of $\underline{\mathsf{CompHaus}}^{\mathrm{op}}$ was obtain by V. Marra and L. Reggio based on the theory of MV-algebras.

References

Marra, Vincenzo and Reggio, Luca (2017). "Stone duality above dimension zero: Axiomatising the algebraic theory of $C(X)$". In: Advances in Mathematics 307, pp. 253–287.

Partially ordered compact spaces

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- $[0,1]$ is injective with respect to embeddings.
- $[0,1]$ is a cogenerator with respect to embeddings.

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About the algebraic character of $\underline{\mathsf{PosComp}}^{\mathrm{op}}$

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- finitely copresentable = finite,
  $\aleph_1$-ary copresentable = metrizable;
  hence, $[0,1]$ is $\aleph_1$-ary copresentable.

References

Hofmann, Dirk, Neves, Renato, and Nora, Pedro (2018). "Generating the algebraic theory of $C(X)$: the case of partially ordered compact spaces". In: Theory and Applications of Categories 33.(12), pp. 276–295.

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- $\underline{\mathsf{PosComp}}^{\mathrm{op}}$ is exact, hence a $\aleph_1$-ary variety.

References

Abbadini, Marco (2019). "The dual of compact ordered spaces is a variety". In: Theory and Applications of Categories 34.(44), pp. 1401–1439.

Abbadini, Marco and Reggio, Luca (2020). "On the axiomatisability of the dual of compact ordered spaces". In: Applied Categorical Structures 28.(6), pp. 921–934.

About the dual of $\underline{\mathsf{PosComp}}$

Recall

$$\underline{\mathsf{Priest}}^{\mathrm{op}} \xrightarrow{\quad \mathsf{hom}(-,2) \quad} \underline{\mathsf{DL}}$$

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We consider

$$?? = \mathsf{LaxMon}([0,1]\text{-}\underline{\mathsf{FinSup}})^{\mathrm{op}},$$

that is: finitaly cocomplete metric spaces with a commutative monoid structure which preserves finite colimits in each variable.

A first result

Theorem

The functor

$$C\colon \underline{\mathsf{PosCompDist}}^{\mathrm{op}} \longrightarrow \mathsf{LaxMon}([0,1]\text{-}\underline{\mathsf{FinSup}})$$

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Remark

If we add at the right-hand side

- powers from $[0,1]$,
- Cauchy completeness (à la Lawvere), and
- enough characters into $[0,1]$;

then C is an equivalence.

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Idea.

- $1 \multimap X$ ($A \subseteq X$ closed) $\quad\rightsquigarrow\quad \Phi\colon CX \longrightarrow [0,1]$.
- A is irreducible $\quad\iff\quad \Phi$ is in $\mathsf{Mon}([0,1]\text{-}\underline{\mathsf{FinSup}})$.
- Every X in $\underline{\mathsf{PosComp}}$ is sober.

$\square$

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Next

Add metric to left-hand side.

Theorem (Flagg (1997))

PosComp is equivalent to the category of Eilenberg–Moore algebras for the "prime filter on upsets monad" on Pos.

References

Flagg, Robert C. (1997). "Algebraic theories of compact pospaces". In: Topology and its Applications 77.(3), pp. 277–290.

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PosComp is equivalent to the category of Eilenberg–Moore algebras for the "prime filter on upsets monad" on Pos.

Theorem (Tholen (2009))

OrdCH is equivalent to the category of Eilenberg–Moore algebras for the ultrafilter monad $\mathbb{U}$ on Ord.

Note. $\mathfrak{x}\ (U{\leq})\ \mathfrak{y})$ whenever $\forall A, B\ \exists x, y\ .\ x \leq y$.

References

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Definition

Metric compact Hausdorff space = Eilenberg–Moore algebra for the monad $\mathbb{U}$ on Met.

Note. $Ud(\mathfrak{x}, \mathfrak{y}) = \inf_{A,B} \sup_{x,y} d(x, y)$.

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Remark
More general, one defines quantale-enriched compact Hausdorff spaces as the Eilenberg-Moore algebras for the ultrafilter monad on $\mathcal{V}$-Cat.

"While listening to a 1967 lecture of Richard Swan … I noticed the analogy between the triangle inequality and a categorical composition law."[a]

- order $\leq\colon X \times X \longrightarrow \underline{2}$:

$$\top \implies x \leq x \quad \text{and} \quad (x \leq y \;\&\; y \leq z) \implies x \leq z.$$

- metric $d\colon X \times X \longrightarrow [0,\infty]$:

$$0 \geqslant d(x,x) \quad \text{and} \quad d(x,y) + d(y,z) \geqslant d(x,z).$$

- $\mathcal{V}$-category $a\colon X \times X \longrightarrow \mathcal{V}$:

$$k \leq a(x,x) \quad \text{and} \quad a(x,y) \otimes a(y,z) \leq a(x,z).$$

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[a] Lawvere, F. William (1973). "Metric spaces, generalized logic, and closed categories". In: Rendiconti del Seminario Matemàtico e Fisico di Milano 43.(1), pp. 135–166.

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Definition

A $\mathcal{V}$-categorical compact Hausdorff space X is called Priestley whenever the cone $(f \colon X \longrightarrow \mathcal{V}^{\mathrm{op}})_f$ in $\mathcal{V}$-CatCH is point-separating and initial.

A further results

Theorem

The functor

$$[0,1]\text{-}\underline{\mathsf{PriestDist}}^{\mathrm{op}} \xrightarrow{\mathsf{C}=\hom(-,1)} [0,1]\text{-}\underline{\mathsf{FinSup}}$$

is fully faithful

References

- Hofmann, Dirk and Nora, Pedro (2018). "Enriched Stone-type dualities". In: Advances in Mathematics 330, pp. 307–360.
- Hofmann, Dirk and Nora, Pedro (2023). "Duality theory for enriched Priestley spaces". In: Journal of Pure and Applied Algebra 227.(3), p. 107231.

Theorem
The functor

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is fully faithful and restricts to a fully faithful functor

$$[0,1]\text{-}\underline{\mathsf{Priest}}^{\mathrm{op}} \xrightarrow{C=\hom(-,[0,1])} [0,1]\text{-}\underline{\mathsf{FinLat}}.$$

Idea.

- $1 \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\circ\;\;} X$ ($X \to [0,1]$) $\rightsquigarrow$ $\Phi : CX \to [0,1]$.
- $1 \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\circ\;\;} X$ is irreducible $\iff$ Φ preserves finite weighted limits.
- Every X in $[0,1]\text{-}\underline{\mathsf{Priest}}$ is sober (Cauchy complete à la Lawvere).

□

Quotients in $\underline{\mathrm{MetCH}}_{\mathrm{sep}}$

Proposition

For a $\mathcal{V}$-category (X, a) and a compact Hausdorff space (X, α) with the same underlying set X, the following assertions are equivalent.

(i) $\alpha\colon \mathsf{U}(X, a) \longrightarrow (X, a)$ is a $\mathcal{V}$-functor.

(ii) $a\colon (X, \alpha) \times (X, \alpha) \longrightarrow (\mathcal{V}, \xi_{\leq})$ is continuous.

Compare with

For an order relation $\leq$ and a compact Hausdorff topology α on a set X, $\alpha\colon U(X, \leq) \longrightarrow (X, \leq)$ is monotone if and only if the order relation is closed with respect to the product topology of $X \times X$.

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Lemma

For $f \colon X \longrightarrow Y$ in $\underline{\text{MetCH}}_{\text{sep}}$,

$$\begin{aligned} \gamma_f \colon X \times X &\longrightarrow [0, \infty] \\ (x, y) &\longmapsto d_X(f(x), f(y)). \end{aligned}$$

is a metric, is continuous with respect to the upper topology of $[0, \infty]$ and is below d_X, i.e., for all $x, y \in X$, $\gamma_f(x, y) \leq d_X(x, y)$.

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Proposition

There is a order-isomorphism between such metrics on X and (isomorphism classes of) quotients $X \longrightarrow Y$ in $\underline{\mathsf{MetCH}}_{\mathrm{sep}}$.

Lemma

For embeddings $f_0\colon X \longrightarrow Y_0$, $f_1\colon X \longrightarrow Y_1$ and their pushout,

$$\begin{array}{ccc} X & \overset{f_1}{\hookrightarrow} & Y_1 \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle \lambda_1} \\ Y_0 & \underset{\lambda_0}{\longrightarrow} & P \end{array}$$

for all $i, j \in \{0,1\}$, $u \in Y_i$ and $v \in Y_j$,

$$d_P(\lambda_i(u), \lambda_j(v)) = \begin{cases} d_{Y_i}(u, v) & \text{if } i = j, \\ \inf_{x \in X}(d_{Y_i}(u, f_i(x)) + d_{Y_j}(f_j(x), v)) & \text{if } i \neq j. \end{cases}$$

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Theorem

$\underline{\mathrm{MetCH}}_{\mathrm{sep}}^{\mathrm{op}}$ is a regular category.

Equivalence corelations

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For a separated metric compact Hausdorff space X, a **binary corelation** on X is a quotient $\binom{q_0}{q_1} \colon X + X \longrightarrow S$.

A binary corelation on X is called respectively reflexive, symmetric, transitive provided that it satisfies the properties:

$$\begin{array}{ccccc} & & X+X & & \\ & \swarrow^{\binom{q_0}{q_1}} & & \searrow^{\binom{1_X}{1_X}} & \\ S & & \overset{d}{\dashrightarrow} & & X \end{array}$$

reflexivity

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symmetry

$$\begin{array}{ccc} X & \xrightarrow{q_1} & S \\ {\scriptstyle q_0}\downarrow & & \downarrow{\scriptstyle \lambda_1} \\ S & \xrightarrow[\lambda_0]{} & P \end{array} \quad \Longrightarrow \quad \begin{array}{ccccc} & & X+X & & \\ & \swarrow^{\binom{q_0}{q_1}} & & \searrow^{\binom{\lambda_0 \circ q_0}{\lambda_1 \circ q_1}} & \\ S & & \overset{t}{\dashrightarrow} & & P \end{array}$$

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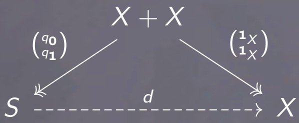
Notation

We denote the elements of $X+X$ by (x,i), where x varies in X and i varies in $\{0,1\}$. Further, i^* stands for $1-i$.

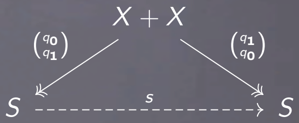
Equivalence corelations

For a separated metric compact Hausdorff space X, a **binary corelation** on X is a quotient $\binom{q_0}{q_1} : X + X \longrightarrow S$ (which can be described by a "quotient metric" γ on $X + X$).

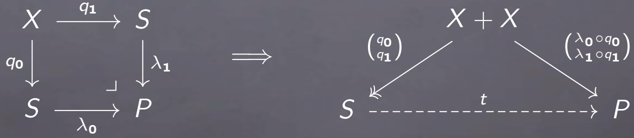
A binary corelation on X is called respectively reflexive, symmetric, transitive provided that it satisfies the properties:



reflexivity



symmetry



transitivity

Notation

We denote the elements of $X + X$ by (x, i), where x varies in X and i varies in $\{0, 1\}$. Further, i^* stands for $1 - i$.

Lemma

A binary corelational structure γ on a separated metric compact Hausdorff space X is reflexive if and only if, for all $x, y \in X$ and $i, j \in \{0, 1\}$,

$$d_X(x, y) \leq \gamma((x, i), (y, j)).$$

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Lemma
A reflexive binary corelational structure γ on a separated metric compact Hausdorff space X is transitive if and only if for all $x, y \in X$ and all $i \in \{0, 1\}$, we have

$$\gamma((x, i), (y, i^*)) = \inf_{z \in X} \gamma((x, i), (z, i^*)) + \gamma((z, i), (y, i^*)).$$

Lemma

An equivalence corelational structure γ on a separated metric compact Hausdorff space X is effective if and only if for all $x, y \in X$ and $i \in \{0,1\}$, we have

$$\gamma((x,i),(y,i^*)) := \inf_{\substack{z \in X, \\ \gamma((z,i),(z,i^*))=0}} (d_X(x,z) + d_X(z,y)).$$

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Theorem

Every equivalence corelation in $\underline{\mathrm{MetCH}}_{\mathrm{sep}}$ is effective.

Theorem

The category $\underline{\mathsf{MetCH}}^{\mathrm{op}}$ is exact.

Remark
In the sequel, we consider "sufficiently nice" quantales $\mathcal{V}$.

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Proposition

For every regular cardinal λ, the forgetful functor $\mathcal{V}\text{-}\underline{\mathsf{CatCH}}_{(\mathrm{sep})} \longrightarrow \underline{\mathsf{CompHaus}}$ preserves λ-copresentable objects (since its left adjoint preserves cofiltered limits).

In particular:

1. Every finitely copresentable (separated) $\mathcal{V}$-enriched compact Hausdorff space is finite
2. Every $\aleph_1$-copresentable (separated) $\mathcal{V}$-enriched compact Hausdorff space has a metrizable topology.

Proposition

$\mathcal{V}$-$\underline{\text{CatCH}}_{(\text{sep})}$ is the model category of a countable $\aleph_1$-ary limit sketch in $\underline{\text{CompHaus}}$.

Idea.

Use the bijection between the sets

$$\{X \longrightarrow \mathcal{V} \text{ continuous}\} \quad \text{and} \quad \{(B_u)_{u\in D} \mid B_u \subseteq X \text{ closed } \& \ B_u = \bigcap_{v \lll u} B_v\};$$

$$(\varphi\colon X \longrightarrow \mathcal{V}) \longmapsto (\varphi^{-1}(\uparrow u)_{u\in D})$$

$$(B_u)_{u\in D} \longmapsto (\varphi\colon X \to \mathcal{V}, x \mapsto \bigvee\{u \in D \mid x \in B_u\})$$

then a continuous map $a\colon (X,\alpha)\times(X,\alpha) \longrightarrow (\mathcal{V},\xi_{\leq})$ corresponds to a family $(R_u)_{u\in D}$ of closed binary relations R_u on X. □

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Corollary

The category $\left(\mathcal{V}\text{-}\underline{\mathrm{CatCH}}_{(\mathrm{sep})}\right)^{\mathrm{op}}$ is the model category of a colimit sketch in the locally $\aleph_1$-presentable category $\underline{\mathrm{CompHaus}}^{\mathrm{op}}$ and therefore locally presentable (we don't know the rank).

Proposition

$\mathcal{V}$-$\underline{\mathsf{CatCH}}_{(\mathrm{sep})}$ is the model category of a countable $\aleph_1$-ary limit sketch in $\underline{\mathsf{CompHaus}}$.

Lemma

Let λ be a regular cardinal and let $\mathcal{S} = (\underline{\mathsf{C}}, \mathcal{L}, \sigma)$ be a λ-small limit sketch. Then a model of $\mathcal{S}$ in a category $\underline{\mathsf{X}}$ is λ-copresentable in $\mathsf{Mod}(\mathcal{S}, \underline{\mathsf{X}})$ provided that each component is λ-copresentable in $\underline{\mathsf{X}}$.

Using limit sketches

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Corollary

An object is $\aleph_1$-ary copresentable in $\mathcal{V}$-$\underline{\mathsf{CatCH}}_{(\mathrm{sep})}$ if and only if its underlying compact Hausdorff space is metrizable.

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An object is $\aleph_1$-ary copresentable in $\mathcal{V}$-$\underline{\mathsf{CatCH}}_{(\mathrm{sep})}$ if and only if its underlying compact Hausdorff space is metrizable. In particular, $\mathcal{V}^{\mathrm{op}}$ is $\aleph_1$-ary copresentable.

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If the quantale $\mathcal{V}$ is finite, then the finitely copresentable objects of $\mathcal{V}$-$\underline{\mathsf{CatCH}}$ (respectively $\mathcal{V}$-$\underline{\mathsf{CatCH}}_{\mathrm{sep}}$) are precisely the finite ones.

Proposition

The reflection functor $\pi_0 \colon \mathcal{V}\text{-}\underline{\mathsf{CatCH}} \longrightarrow \mathcal{V}\text{-}\underline{\mathsf{Priest}}$ preserves $\aleph_1$-cofiltered limits (and even cofiltered limits if $\mathcal{V}$ is finite).

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Theorem

The category $\mathcal{V}\text{-}\underline{\mathsf{Priest}}$ is locally $\aleph_1$-ary copresentable (and even locally finite copresentable if $\mathcal{V}$ is finite).