Some toposes over which essential implies locally connected

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Background

We say that a topological space X is locally connected if it has a basis consisting of connected open sets. Many of the most important spaces in algebraic topology, including CW-complexes and topological manifolds, are locally connected.

Inspired by this, Grothendieck and Verdier [SGA4, Exp. IV, Ex. 7.6] introduced essential geometric morphisms between toposes: these are geometric morphisms f such that the inverse image functor f^* has a left adjoint, called $f_!$. The link to local connectedness is that a topological space X is locally connected if and only if the global sections geometric morphism for $\mathbf{Sh}(X)$ is essential.

$$\mathbf{Sh}(X) \xrightarrow{\gamma} \mathbf{Sets}$$

So for an essential geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$, would it be accurate to say that $\mathcal{F}$ is locally connected over $\mathcal{E}$?

It turns out that this does not match with geometric intuition; the correct notion was found by Barr–Paré [BP80].

There a geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$ is called molecular (or locally connected) if

- f^* has a left adjoint $f_!$; and
- this is in fact an $\mathcal{E}$-indexed left adjoint.

What does "$\mathcal{E}$-indexed left adjoint" mean here?

It boils down to this: we want that in the diagram

$$\begin{tikzcd}
\mathcal{F}/f^*Y \arrow[r, "\pi"] \arrow[d, "f"'] & \mathcal{F}/f^*X \arrow[d, "f"] \\
\mathcal{E}/Y \arrow[r, "\pi"'] & \mathcal{E}/X
\end{tikzcd}$$

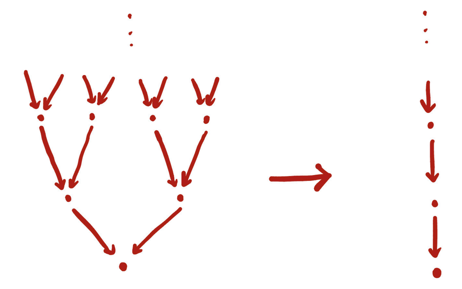
the Beck–Chevalley condition $\pi^* f_! \simeq f_! \pi^*$ holds, for any morphism $Y \to X$ in $\mathcal{E}$.

Recall that for each object X in $\mathcal{E}$, the slice category $\mathcal{E}/X$ is again a topos, and that $f : \mathcal{F} \longrightarrow \mathcal{E}$ induces a geometric morphism $\mathcal{F}/f^*X \longrightarrow \mathcal{E}/X$ on slice toposes. In the other direction, a morphism $Y \to X$ in the topos externalizes to a geometric morphism $\mathcal{E}/Y \to \mathcal{E}/X$.

Then what are some examples of essential geometric morphisms that are not locally connected?

For any functor $\phi : \mathbb{C} \to \mathbb{D}$ between small categories, the induced geometric morphism $f : \mathbf{PSh}(\mathbb{C}) \longrightarrow \mathbf{PSh}(\mathbb{D})$ between presheaf toposes is essential.

In some sense, these geometric morphisms are rarely locally connected. For a more precise example, consider the following functor between posets.



The induced geometric morphism is not locally connected. Indeed, the topos of presheaves for the poset on the right has a point at infinity, and the fiber over this point will be homeomorphic to the Cantor Set (which is not locally connected). For locally connected geometric morphisms, each fiber will be locally connected.

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EILC toposes

We will say that a topos $\mathcal{E}$ is EILC (Essential Implies Locally Connected) if any essential geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$ is also locally connected.

So if f is a geometric morphism to a EILC topos, then as soon as we have a left adjoint $f_!$ for f^*, this left adjoint is automatically $\mathcal{E}$-indexed.

It is immediate that the topos of sets is EILC, but what are some other examples? Can we characterize the EILC toposes?

This question was first asked by Menni on the category theory mailing list (May 3, 2017).

There no terminology was introduced for these toposes. The terminology 'EILC' that I use is hopefully seen as a placeholder until we know enough to give them a more geometrically meaningful name. In [Men22b], the name 'basic' is used, in light of various aspects that are studied there.

The problem of characterizing EILC toposes is related to a conjecture by Lawvere and Menni, about whether every precohesive geometric morphism is also stably precohesive. Indeed, any counterexample to this conjecture (if there are any) will have a codomain that is *not* EILC. Some negative evidence for the conjecture by Lawvere–Menni was given in [HR21] [GS21] [Men22a].

So what are some examples of EILC toposes?

Most notably, for any Hausdorff space X the topos of sheaves on X is EILC [Hem22]. To prove this, we consider any point $x \in X$ and the associated pullback diagram

$$\begin{tikzcd}
\mathcal{E}_p \arrow[r, "q"] \arrow[d, "g"'] & \mathcal{E} \arrow[d, "f"] \\
\mathbf{Sets} \arrow[r, "p"'] & \mathbf{Sh}(X)
\end{tikzcd}$$

In a Hausdorff space, any point is a closed subset, so the associated geometric morphism $\mathbf{Sets} \to \mathbf{Sh}(X)$ is a closed inclusion. Because this is a closed inclusion (in particular tidy), we know that the Beck–Chevalley condition $f^*p_* \simeq q_*g^*$ holds, see the Elephant [Joh02, C3.4.7]. Note that this is an analogue of Proper Base Change in algebraic geometry.

<u>Exercise</u>. Show that the functor $p^* f_! q_*$ is a left adjoint for g^* using that p_* and q_* are fully faithful.

So the geometric morphism g is essential as well, and therefore locally connected, because **Sets** is EILC. Because x was arbitrary, we conclude that f has locally connected fibers.

From this, we can deduce that f itself is locally connected, again using the Beck–Chevalley conditions and the fact that the morphisms $\mathbf{Sets} \to \mathbf{Sh}(X)$ for all different points $x \in X$ form a jointly surjective family.

We conclude that $\mathbf{Sh}(X)$ is EILC, for X Hausdorff!

With the above in mind, it is not a big leap to show that $\mathbf{Sh}(X)$ is also EILC as soon as X is Jacobson. Or more generally, $\mathcal{E}$ is EILC if it is *locally* a Jacobson space (we call this a 'Jacobson étendue').

With different methods, we can show that Boolean étendues and classifying toposes of compact topological groups are EILC.

<u>Open question</u>: are all Boolean toposes EILC?

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