Slices of Higher Categories

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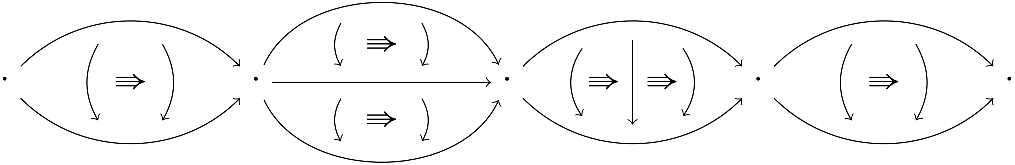

Batanin's Conjecture

If for all $k < n$, the k-dimensional string diagrams associated to some theory of n-category form a presheaf category, then those n-categories are equivalent to fully weak ones.

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Theorem (G)

For $k < n$, the k-string diagrams associated to n-categories with weak interchange laws form a presheaf category.

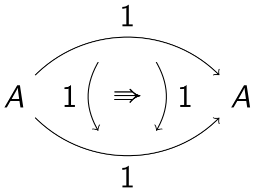
Theorem (G)

For $k < n$, the k-string diagrams associated to n-categories with weak units form a presheaf category.

Outline

Slices

The k-th slice of a theory of higher category is the symmetric operad for the algebraic structure formed by the k-cells of those higher categories whenever they contain exactly one j-cell for all $j < k$.



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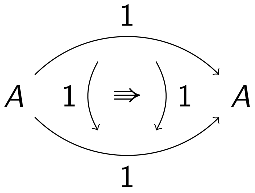

Examples

- The 0^{th} slice of any theory of higher category is the operad for sets.
- For **strict** higher categories, the first slice is the operad for monoids, and the k^{th} slice for all $k > 1$ is the operad for commutative monoids.

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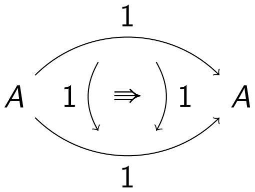

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- Slices are related to the homotopy groups of higher groupoids.

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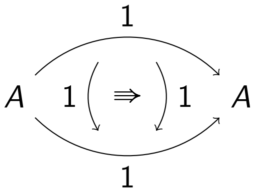

- First introduced by Michael Batanin.
- Slices are related to the homotopy groups of higher groupoids.
- Batanin conjectured that slices can tell us when semi-strict higher categories are equivalent to fully weak ones.

The Conjecture

Theorem (Batanin)

If the k^{th} slice of some theory of n-category is a *plain* operad for all $k < n$, then the ~~computads~~ k-dimensional string diagrams corresponding to those n-categories form a presheaf category.

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A plain operad is a symmetric operad with trivial group actions

- The symmetric operad for monoids is plain.
- The symmetric operad for commutative monoids is **NOT** plain.

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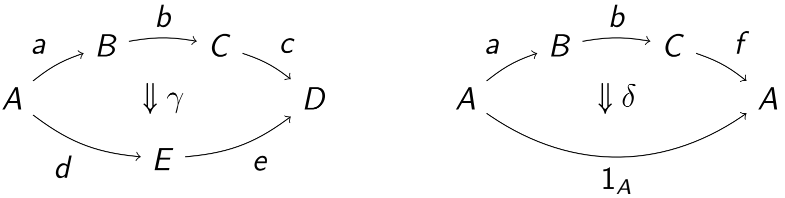
Batanin's Conjecture

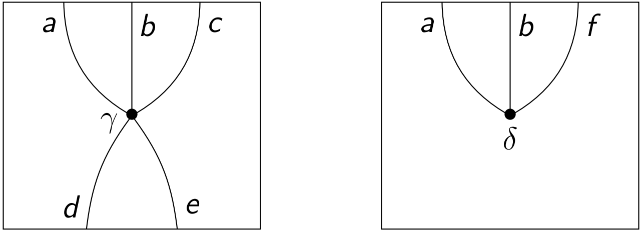
If for all $k < n$, the k-dimensional string diagrams associated to some theory of n-category form a presheaf category, then those n-categories are equivalent to fully weak ones.

Plain slices $\longrightarrow$ String diagrams are presheaves $\dashrightarrow$ Model weak n-cats

String Diagrams

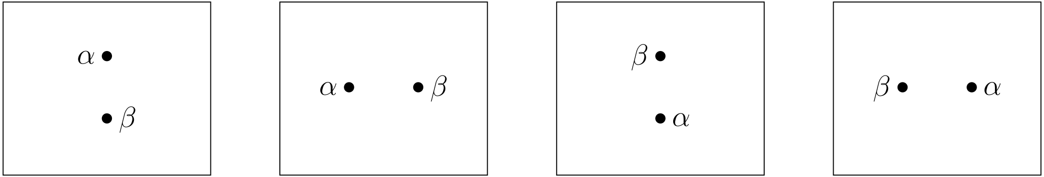


The 2-cells above are represented by the string diagrams:



String Diagrams

Given any 2-cells $\alpha, \beta : 1_A \longrightarrow 1_A$ in a **strict** *n*-category, the composites $\alpha \cdot \beta$, $\alpha * \beta$, $\beta \cdot \alpha$, $\beta * \alpha$, are all equal, so the string diagrams below all represent the *same* 2-cell.



$\Longrightarrow$ the 2-dimensional string diagrams associated to strict *n*-categories do not form a presheaf category.

Questions

k^{th} slice is plain for all $k < n$ $\longrightarrow$ k-string diagrams are presheaves for all $k < n$ $\dashrightarrow$ Model weak n-cats

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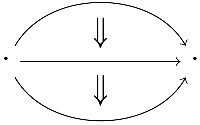
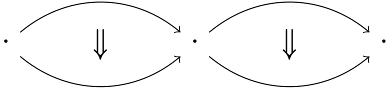
Questions

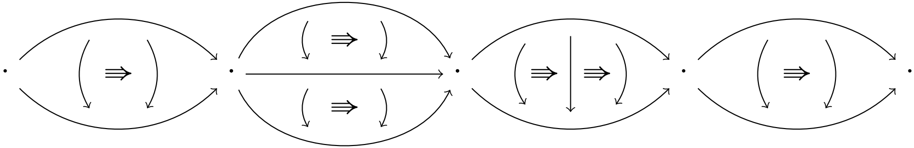
- How are slices calculated?
- Can string diagrams be presheaves when the slices are not plain operads?

Globular Operads

A globular operad is an operad where the arities of the operations are given by shapes of globular pasting diagrams:







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Definition

The k^{th} slice of an n-globular operad $\boldsymbol{G}$ is the symmetric operad determined by the k-cell generators and relations of a presentation for $\boldsymbol{G}$.

First Main Results

Theorem (G)

The k^{th} slice of an n-globular operad $\boldsymbol{G}$ does not depend on the choice of presentation for $\boldsymbol{G}$.

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For $k < n$, the k^{th} slice of the n-globular operad for n-categories with weak interchange laws is the **plain** operad for (many-pointed) k-tuple monoids with shared unit.

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An n-category with weak units is an n-category with weak identity laws in dimensions $\leqslant n - 2$.

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An n-category with weak units is an n-category with weak identity laws in dimensions $\leqslant n-2$.

For $k > 1$, the k^{th} slice of the globular operad for n-categories with weak units is **not** a plain operad. But...

Theorem (G)

For $k < n$, the k-string diagrams associated to n-categories with weak units form a presheaf category.

String Diagrams

The second slice of the globular operad for n-categories with weak units is the symmetric operad for 4-pointed semigroup-monoids $X = (X, *, \cdot, 1)$, satisfying

- $1 * 1 = 1$, and
- $(w \cdot x) * (y \cdot z) = (w * y) \cdot (x * z)$

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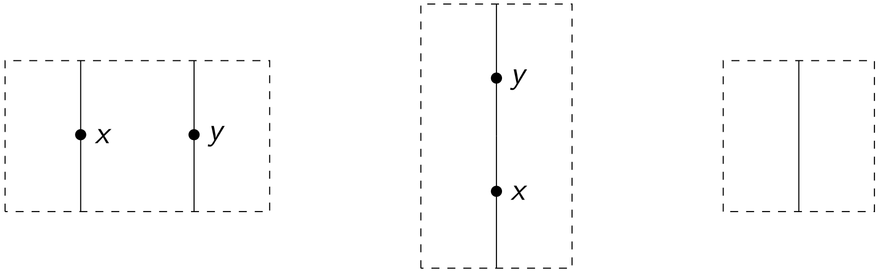
We represent $x * y$, $x \cdot y$ and 1 by the diagrams:



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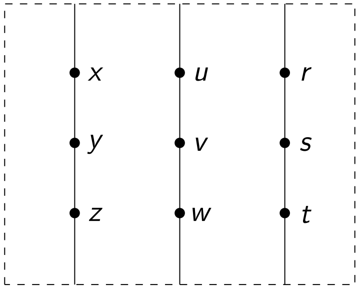
Then the axioms mean that *any* diagram represents a unique element of X:



Future Work

Using Garner's notion of homomorphisms between higher categories, it is possible to construct a weak homomorphism into any semi-strict higher category from a fully weak one.

These homomorphisms can be understood in terms of string diagrams.

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Conjecture

If the k-string diagrams associated to a theory of n-category form a presheaf category in dimensions $k < n$, then these homomorphisms are locally surjective in dimensions $k < n$, and full and faithful in dimension n.