Ternary semidirect products in semi-abelian categories

CT2023

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Definition

Let A, B, C be subobjects of X. We say that X is a ternary semidirect product of A, B, C if

- X is the join of A, B and C in its lattice of subobjects;
- A and $A \vee B$ are normal in $\mathcal{C}$;
- $A \wedge B$ and $(A \vee B) \wedge C$ are trivial.

In groups [1] and Lie algebras [2], these semidirect products are determined by

- actions $C \curvearrowright B$, $C \curvearrowright A$ and $B \curvearrowright A$.
- a function $\Phi\colon C \times B \to A$.

In groups

Φ must satisfy

$$\Phi(c,1) = 1 = \Phi(1,b)$$
$$\Phi(c,bb') = \Phi(c,b)^{({}^{c}b)}\Phi(c,b')$$
$$\Phi(cc',b) = {}^{c}\Phi(c',b)\Phi(c,{}^{c}b)$$
$${}^{c}({}^{b}a)\Phi(c,b) = \Phi(c,b)^{({}^{c}b)}({}^{c}a).$$

The action of C on $A \rtimes B$ is then defined by

$${}^{c}(a,b) = ({}^{c}a\Phi(c,b), {}^{c}b),$$

In Lie algebras

Φ must satisfy

$$\Phi(c,[b,b']) = {}^{b}\Phi(c,b') - {}^{b'}\Phi(c,b)$$
$$\Phi([c,c'],b) = {}^{c}\Phi(c',b) + \Phi(c,{}^{c'}b) - {}^{c'}\Phi(c,b) - \Phi(c',{}^{c}b)$$
$${}^{c}({}^{b}a) = [\Phi(c,b),a] + {}^{({}^{c}b)}a + {}^{b}({}^{c}a).$$

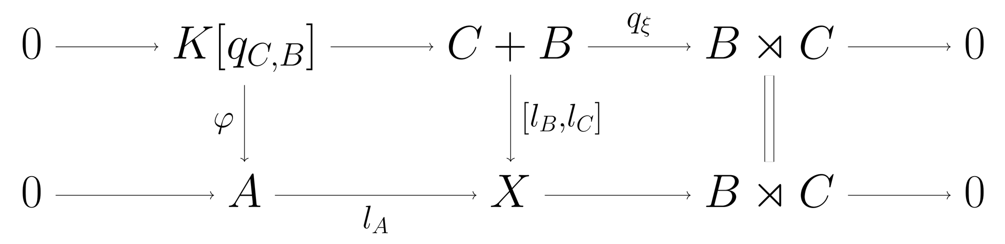
The action of C on $A \rtimes B$ is then defined by

$${}^{c}(a,b) = ({}^{c}a + \Phi(c,b), {}^{c}b)$$

Properties

A ternary semidirect product may be seen as :

- an object of the form $(A \rtimes B) \rtimes C$ with A normal;
- a lifting of the short exact sequence $A \xrightarrow{j_A} A \rtimes B \underset{s_B}{\overset{p_B}{\rightleftarrows}} B$ in $\mathbf{Act}_C(\mathcal{C})$.
- a K_C-split epimorphism $X \to B \rtimes C$ in $\mathbf{Pt}_B(\mathcal{C})$, whose kernel (in $\mathcal{C}$) is A.
- a *pushforward* in $\mathcal{C}$:



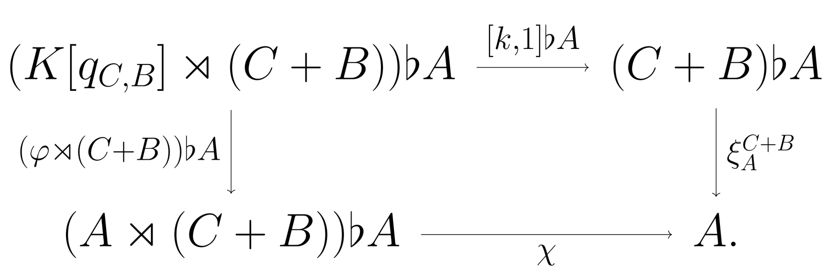
Thus for every ternary semidirect product, we have an action $(C+B)\flat A \to A$ and a $(C+B)$-equivariant morphism $\varphi\colon K[q_{C,B}] \to A$.

Main result

Theorem. *A ternary semidirect product A, B, C is determined by*

- *actions $(C+B)\flat A \to A$ and $C\flat B \to B$*
- *a $(C+B)$-equivariant morphism $K[q_{C,B}] \xrightarrow{\varphi} A$*

such that the following diagram commutes:



Moreover, the semi-direct product of A, B, C can be constructed as a colimit.

Action by a semidirect product

A composition of two split epimorphisms

$$A \rtimes (B \rtimes C) \underset{t}{\overset{q}{\rightleftarrows}} B \rtimes C \underset{s_C}{\overset{p_C}{\rightleftarrows}} C$$

gives a split epimorphism in $\mathbf{Pt}_C(\mathcal{C})$. Thus we have

$$\mathbf{Pt}_{B \rtimes C}(\mathcal{C}) \simeq \mathbf{Pt}_{(B \rtimes C, p_C, s_C)}(\mathbf{Pt}_C(\mathcal{C})),$$

or equivalently

$$\mathbf{Act}_{B \rtimes C}(\mathcal{C}) \simeq \mathbf{Act}_{(B,\xi)}(\mathbf{Act}_C(\mathcal{C}))$$

Furthermore, X is then a ternary semidirect product, with

$$A \rtimes (B \rtimes C) \simeq (A \rtimes B) \rtimes C.$$

Thus a ternary semidirect product $(A \rtimes B) \rtimes C$ corresponds to a split extension in $\mathbf{Act}_C(\mathcal{C})$ (and is then isomorphic to $A \rtimes (B \rtimes C)$) if and only if $\varphi = 0$.

Algebraically coherent categories

By contrast with the cases of groups and Lie algebras, in a semi-abelian category $\mathcal{C}$ an action of $C+B$ on A is not necessarily equivalent to a pair of actions of C and B on A.

Definition ([3])**.** *A semi-abelian category is* algebraically coherent *if for every C the functor $C\flat\_$ preserve jointly strongly epimorphic pairs of morphisms.*

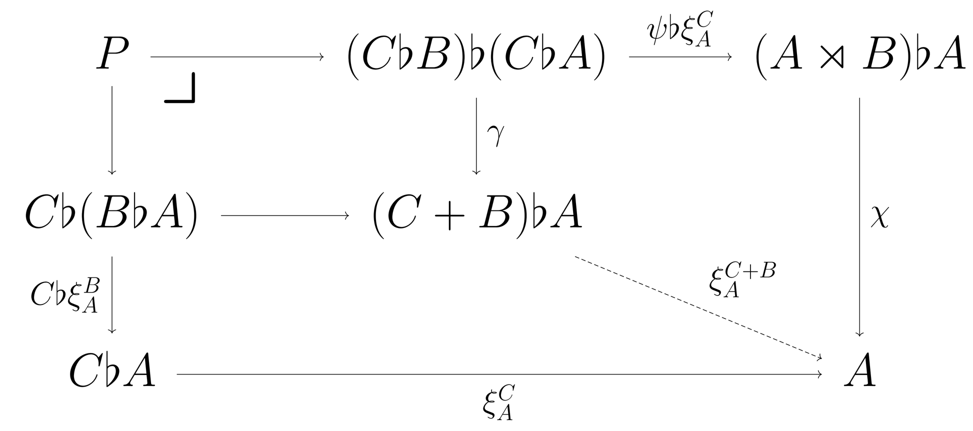
In particular, if $\mathcal{C}$ is algebraically coherent then an action $C\flat(A \rtimes B) \to A \rtimes B$ is determined by its restriction to $C\flat A$ and $C\flat B$.

Algebraically coherent categories (cont.)

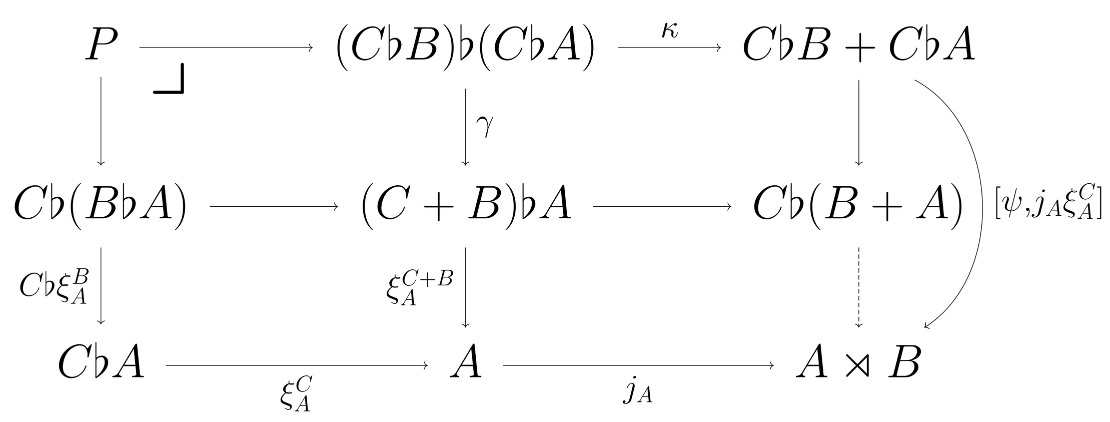
There exists a natural morphism $(C\flat B)\flat(C\flat A) \xrightarrow{\gamma} (C+B)\flat A$ which is

- a regular epimorphism iff $\mathcal{C}$ is algebraically coherent
- an isomorphism if $\mathcal{C}$ is locally algebraically cartesian closed.

As a consequence, given actions ξ_B^C, ξ_A^C and ξ_A^B and an equivariant morphism $K[q_{C,B}] \to A$, we can consider the outer rectangle in the diagram



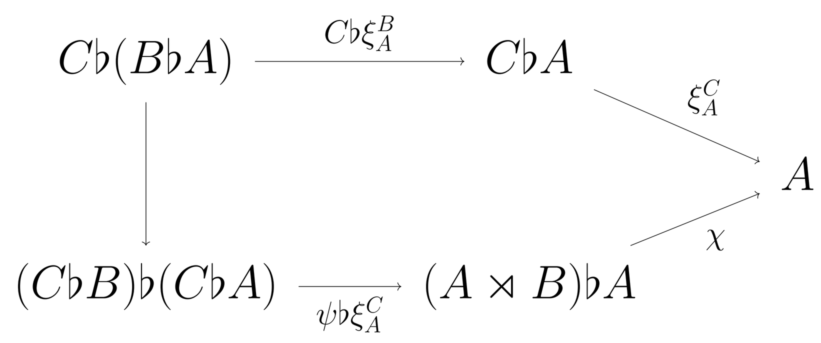
We then also have a diagram



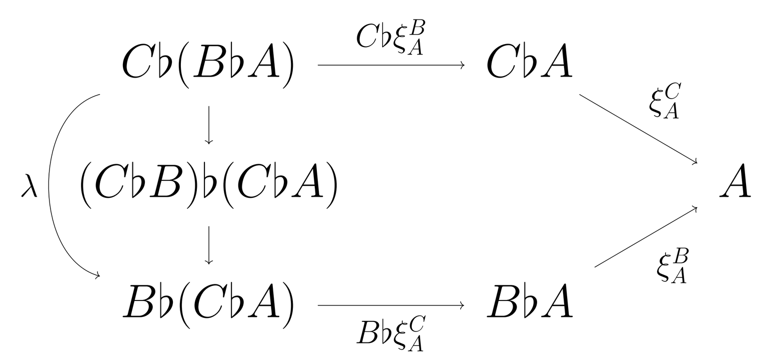
and thus an induced map $C\flat(A \rtimes B) \to A \rtimes B$.

LACC categories

When $\mathcal{C}$ is LACC [4], γ is an isomorphism, so that we have a morphism $C\flat(B\flat A) \to (C\flat B)\flat(C\flat A)$. The condition to define the action of C on $A \rtimes B$ is then simply



For the case $\varphi = 0$, this becomes



where λ is a distributive law between the monads of C-actions and B-actions.

References

[1] P. Carrasco and A. M. Cegarra. "Group-theoretic algebraic models for homotopy types". In: *Journal of Pure and Applied Algebra* 75.3 (1991), pp. 195–235.

[2] P. Carrasco and A. M. Cegarra. "A Dold-Kan theorem for simplicial Lie algebras". In: *Theory and Applications of Categories* 32 (2017), pp. 1165–1212.

[3] A. Cigoli, J. R. A. Gray, and T. Van der Linden. "Algebraically coherent categories". In: *Theory and Applications of Categories* 30 (2015), pp. 1864–1905.

[4] J. R. A. Gray. "Algebraic Exponentiation in General Categories". In: *Applied Categorical Structures* 20.6 (2011), pp. 543–567.