A Universal Kaluzhnin–Krasner Embedding Theorem

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The classical theorem (in groups)

The Kaluzhnin–Krasner Embedding Theorem [KK51]

Any group extension $0 \dashrightarrow A \dashrightarrow G \dashrightarrow B \dashrightarrow 0$ can be embedded into the wreath product $A \wr B$ via a group homomorphism $\phi_G \colon G \to A \wr B$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \phi_G} & & \| & & \\ 0 & \longrightarrow & \mathrm{Set}(B,A) & \longrightarrow & A \wr B & \rightleftarrows & B & \longrightarrow & 0 \end{array}$$

The wreath product $A \wr B$ is the group $\mathrm{Set}(B, A) \rtimes B$ and ϕ_G is constructed based on a **set-theoretical** section of the surjection $G \to B$, which makes it non-canonical in some sense.
Moreover, if our extension is split, this ϕ_G is a morphism of **split** extensions.
There exist recent analogues of this theorem for **Lie algebras** [PRS07] and for **cocommutative Hopf algebras** [BST14].

Further details for the case of groups

In groups, $A \wr B$ is the semi-direct product $\mathrm{Set}(B, A) \rtimes B$ with respect to the action given by

$$(b \cdot h)(b') \coloneqq h(bb')$$

for $h \in \mathrm{Set}(B, A)$ and $b, b' \in B$.
Given a set-theoretical section $s \colon B \to G$ of the surjection $f \colon G \to B$ of our extension $0 \dashrightarrow A \dashrightarrow G \overset{f}{\dashrightarrow} B \dashrightarrow 0$, the embedding $\phi_G \colon G \to A \wr B$ is given, for $g \in G$, by

$$\phi_G(g) \coloneqq (h_g, f(g)),$$

where h_g is defined by

$$h_g(b) \coloneqq s(b)\, g\, s(bf(g))^{-1}$$

for $b \in B$.

The condition (LACC)

Let $\mathbb{X}$ be a semi-abelian category and $B \in \mathbb{X}$. We consider the forgetful functor $K \colon \mathrm{ExtS}(B) \to \mathbb{X}$ that sends each split extension over B to its kernel object.

$$K(0 \dashrightarrow A \dashrightarrow G \rightleftarrows B \dashrightarrow 0) \coloneqq A$$

When this functor K has a right adjoint R, we say that the category $\mathbb{X}$ is **locally algebraically cartesian closed** (LACC) [Gra12, BG12].
Groups and Lie algebras are (LACC) but very few other categories are known to satisfy this condition—for instance, among algebras over an infinite field, only the variety of Lie algebras is (LACC). On the contrary, most of the standard examples of semi-abelian categories—such as rings or loops, for example—are known **not** to be (LACC).
In the category of groups, the right adjoint R of K is well known and it is precisely the wreath product extension we have seen earlier:

$$R(A) = (0 \dashrightarrow \mathrm{Set}(B, A) \dashrightarrow A \wr B \rightleftarrows B \dashrightarrow 0).$$

The case of split extensions

For each $A \in \mathbb{X}$, we would like to have a split extension $R(A)$ universal in the following sense: given $S \in \mathrm{ExtS}(B)$ with $K(S) = A$, we have the universal property

$$\begin{array}{ccc} S & \xrightarrow{\eta_S} & R(A) \\ & {\scriptstyle \forall\alpha} \searrow & \downarrow{\scriptstyle R(\overline{\alpha})} \\ & & R(C) \end{array} \qquad \begin{array}{c} A \\ \downarrow{\scriptstyle \exists!\overline{\alpha}} \\ C \end{array},$$

i.e. $R \colon \mathbb{X} \to \mathrm{ExtS}(B)$ is a right adjoint of K.
As we have seen previously, this right adjoint exists exactly when the category $\mathbb{X}$ is (LACC) and, when it is the case, η_S is necessarily monic. In conclusion, we obtain the following result:

Theorem

A semi-abelian category $\mathbb{X}$ has a **universal Kaluzhnin–Krasner embedding theorem for split extensions** if and only if $\mathbb{X}$ is (LACC).

We already know that a Kaluzhnin–Krasner embedding theorem for **arbitrary** extensions exists for groups and Lie algebras and this last result tells us that we cannot hope for a lot of other examples since such an example must be (LACC). In particular, this excludes **all** types of algebras over an infinite field except Lie algebras [GMVdL19]. On the other hand, examples of categories where such an embedding has a chance to exist include crossed modules and certain categories of internal Lie algebras [GMG21].

A crude embedding for arbitrary extensions

The forgetful functor $P \colon \mathrm{Ext}(B) \to \mathrm{ExtS}(B)$ has a left adjoint L so we have the composite adjunction

$$\mathrm{Ext}(B) \underset{P}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathrm{ExtS}(B) \underset{R}{\overset{K}{\underset{\perp}{\rightleftarrows}}} \mathbb{X},$$

which gives us the following universal embedding.

Theorem

In a (LACC) semi-abelian category, for every object X, there exists a universal extension $W(X) \coloneqq PR(X)$ in which each extension E with $KL(E) = X$ embeds.

However, KL is **not** the forgetful functor $U \colon \mathrm{Ext}(B) \to \mathbb{X}$ which sends each extension to its kernel object so the result above is not a generalisation of the classical Kaluzhnin–Krasner embedding theorem.

Embedding into the wreath product $A \wr B$

Instead of embedding each extension E in the wreath product $WKL(E) = WUPL(E)$ as above, we would like to have an embedding $\phi \colon E \to WU(E)$. If such a ϕ exists, we have

$$\begin{array}{ccc} E & \longrightarrow & WUPL(E) \\ & {\scriptstyle \phi} \searrow & \downarrow{\scriptstyle W(\overline{\phi})} \\ & & WU(E) \end{array} \qquad \begin{array}{c} UPL(E) \\ \downarrow{\scriptstyle \exists!\overline{\phi}} \\ U(E) \end{array}$$

and we can ask ourself if this $\overline{\phi}$ comes from a morphism of extensions $PL(E) \to E$. Unfortunately, this is the case precisely when E is **split** so, for arbitrary extensions, we cannot use just the adjunctions we have considered above and we must work on a case-by-case basis, by studying the structure of the extension

$$PL(E) = (0 \longrightarrow KL(E) \longrightarrow G + B \xrightarrow{\langle f, 1_B \rangle} B \longrightarrow 0)$$

for any extension $E = (0 \dashrightarrow A \dashrightarrow G \overset{f}{\dashrightarrow} B \dashrightarrow 0)$.

Conclusion

In conclusion, we can hope to have a **universal Kaluzhnin–Krasner embedding theorem**—already known to exist in groups and Lie algebras—only if our category $\mathbb{X}$ is **locally algebraically cartesian closed** (LACC) since this condition is necessary to obtain such an embedding for **split** extensions. In particular, this excludes many examples of semi-abelian categories such as rings, loops and any variety of algebras over an infinite field other than the one of Lie algebras. For the (LACC) categories where such a result **may** be possible, we can cite crossed modules or certain categories of internal Lie algebras.

Bibliography

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