

# Factorization systems as double categories

Miloslav Štěpán

Masaryk University  
*miloslav.stepan@mail.muni.cz*

Category Theory 2023  
June 6, 2023

# Plan of the presentation

- 1 Some double category theory,
- 2 Strict factorization systems  $\leftrightarrow$  (certain) double categories,
- 3 Orthogonal factorization systems  $\leftrightarrow$  (certain) double categories.

# Plan of the presentation

- 1 Some double category theory,
- 2 Strict factorization systems  $\leftrightarrow$  (certain) double categories,
- 3 Orthogonal factorization systems  $\leftrightarrow$  (certain) double categories.

# Plan of the presentation

- 1 Some double category theory,
- 2 Strict factorization systems  $\longleftrightarrow$  (certain) double categories,
- 3 Orthogonal factorization systems  $\longleftrightarrow$  (certain) double categories.

## Definition

A *double category*  $X$  consists of objects, horizontal morphisms, vertical morphisms, and squares:

$$\begin{array}{ccc}
 a & \xrightarrow{g} & b \\
 u \downarrow & \Downarrow \alpha & \downarrow v \\
 c & \xrightarrow{h} & d
 \end{array}$$

The squares can be composed horizontally and vertically and both compositions are associative and unital.

It can be equivalently described as a category object in  $\text{Cat}$ , i.e. a diagram in  $\text{Cat}$  satisfying some properties:

$$\begin{array}{ccccc}
 & \xrightarrow{d_2} & & \xleftarrow{d_1} & \\
 X_2 & \xrightarrow{d_1} & X_1 & \xrightarrow{s} & X_0 \\
 & \xrightarrow{d_0} & & \xleftarrow{d_0} &
 \end{array}$$

## Definition

A *double category*  $X$  consists of objects, horizontal morphisms, vertical morphisms, and squares:

$$\begin{array}{ccc}
 a & \xrightarrow{g} & b \\
 u \downarrow & \Downarrow \alpha & \downarrow v \\
 c & \xrightarrow{h} & d
 \end{array}$$

The squares can be composed horizontally and vertically and both compositions are associative and unital.

It can be equivalently described as a category object in  $\text{Cat}$ , i.e. a diagram in  $\text{Cat}$  satisfying some properties:

$$\begin{array}{ccccc}
 & \xrightarrow{d_2} & & \xleftarrow{d_1} & \\
 X_2 & \xrightarrow{d_1} & X_1 & \xrightarrow{s} & X_0 \\
 & \xrightarrow{d_0} & & \xleftarrow{d_0} &
 \end{array}$$

# Duals

A double category  $X$  admits 8 duals: the *vertical opposite*  $X^v$ , *horizontal opposite*  $X^h$ , *transpose*  $X^T$  ...

For example:

$$\begin{array}{ccc}
 a & \xrightarrow{g} & b \\
 u \downarrow & \Downarrow \alpha & \downarrow v \\
 c & \xrightarrow{h} & d
 \end{array}
 \quad \text{in } X
 \quad \iff \quad
 \begin{array}{ccc}
 a & \xrightarrow{u} & c \\
 g \downarrow & \Downarrow \alpha & \downarrow h \\
 b & \xrightarrow{v} & d
 \end{array}
 \quad \text{in } X^T$$

# Duals

A double category  $X$  admits 8 duals: the *vertical opposite*  $X^v$ , *horizontal opposite*  $X^h$ , *transpose*  $X^T$  ...

For example:

$$\begin{array}{ccc}
 a & \xrightarrow{g} & b \\
 u \downarrow & \Downarrow \alpha & \downarrow v \\
 c & \xrightarrow{h} & d
 \end{array}
 \quad \text{in } X
 \quad \iff \quad
 \begin{array}{ccc}
 a & \xrightarrow{u} & c \\
 g \downarrow & \Downarrow \alpha & \downarrow h \\
 b & \xrightarrow{v} & d
 \end{array}
 \quad \text{in } X^T$$



# Basic examples

## Example

$\mathcal{C}$  a category, there is double category  $\text{Sq}(\mathcal{C})$  such that:

- objects are the objects of  $\mathcal{C}$ ,
- vertical and horizontal morphisms are morphisms of  $\mathcal{C}$ ,
- squares are commutative squares in  $\mathcal{C}$

## Example

We will encounter these two of its sub-double categories:

$\text{PbSq}(\mathcal{C}) \supseteq \text{MonoPbSq}(\mathcal{C})$

## Example

There is double category  $\text{BOFib}$  of (small) categories, bijections on objects, discrete opfibrations, pullback squares.

# Basic examples

## Example

$\mathcal{C}$  a category, there is double category  $\text{Sq}(\mathcal{C})$  such that:

- objects are the objects of  $\mathcal{C}$ ,
- vertical and horizontal morphisms are morphisms of  $\mathcal{C}$ ,
- squares are commutative squares in  $\mathcal{C}$

## Example

We will encounter these two of its sub-double categories:

$\text{PbSq}(\mathcal{C}) \supseteq \text{MonoPbSq}(\mathcal{C})$

## Example

There is double category  $\text{BOFib}$  of (small) categories, bijections on objects, discrete opfibrations, pullback squares.

## Basic examples

### Example

$\mathcal{C}$  a category, there is double category  $\text{Sq}(\mathcal{C})$  such that:

- objects are the objects of  $\mathcal{C}$ ,
- vertical and horizontal morphisms are morphisms of  $\mathcal{C}$ ,
- squares are commutative squares in  $\mathcal{C}$

### Example

We will encounter these two of its sub-double categories:

$\text{PbSq}(\mathcal{C}) \supseteq \text{MonoPbSq}(\mathcal{C})$

### Example

There is double category  $\text{BOFib}$  of (small) categories, bijections on objects, discrete opfibrations, pullback squares.

# Strict factorization systems $\leftrightarrow$ (certain) double categories

# Strict factorization systems

## Definition

A *strict factorization system* on a category  $\mathcal{C}$  consists of two wide subcategories  $\mathcal{E}, \mathcal{M} \subseteq \mathcal{C}$  with the property that:

For every morphism  $f \in \mathcal{C}$  there exist unique  $e \in \mathcal{E}, m \in \mathcal{M}$  with:

$$f = m \circ e.$$

## Definition

Denote by  $\mathcal{SFS}$  the category whose:

- objects are strict factorization systems  $\mathcal{E} \subseteq \mathcal{C} \supseteq \mathcal{M}$ ,
- a morphism  $(\mathcal{E} \subseteq \mathcal{C} \supseteq \mathcal{M}) \rightarrow (\mathcal{E}' \subseteq \mathcal{C}' \supseteq \mathcal{M}')$  is a functor  $F : \mathcal{C} \rightarrow \mathcal{C}'$  satisfying  $F(\mathcal{E}) \subseteq \mathcal{E}'$  and  $F(\mathcal{M}) \subseteq \mathcal{M}'$ .

# Strict factorization systems

## Definition

A *strict factorization system* on a category  $\mathcal{C}$  consists of two wide subcategories  $\mathcal{E}, \mathcal{M} \subseteq \mathcal{C}$  with the property that:

For every morphism  $f \in \mathcal{C}$  there exist unique  $e \in \mathcal{E}, m \in \mathcal{M}$  with:

$$f = m \circ e.$$

## Definition

Denote by  $\mathcal{SFS}$  the category whose:

- objects are strict factorization systems  $\mathcal{E} \subseteq \mathcal{C} \supseteq \mathcal{M}$ ,
- a morphism  $(\mathcal{E} \subseteq \mathcal{C} \supseteq \mathcal{M}) \rightarrow (\mathcal{E}' \subseteq \mathcal{C}' \supseteq \mathcal{M}')$  is a functor  $F : \mathcal{C} \rightarrow \mathcal{C}'$  satisfying  $F(\mathcal{E}) \subseteq \mathcal{E}'$  and  $F(\mathcal{M}) \subseteq \mathcal{M}'$ .

## Example

Given categories  $\mathcal{A}, \mathcal{B}$ , consider  $\mathcal{A} \times \mathcal{B}$  and denote:

$$\begin{aligned} \mathcal{E} &:= \{(f, 1_b) \mid f \in \text{mor } \mathcal{A}, b \in \mathcal{B}\}, \\ \mathcal{M} &:= \{(1_a, g) \mid g \in \text{mor } \mathcal{B}, a \in \mathcal{A}\}, \end{aligned}$$

Every morphism  $(f, g) \in \mathcal{A} \times \mathcal{B}$  admits a unique  $(\mathcal{E}, \mathcal{M})$ -factorization:

$$(f, g) = (1, g) \circ (f, 1).$$

## Example

Given categories  $\mathcal{A}, \mathcal{B}$ , consider  $\mathcal{A} \times \mathcal{B}$  and denote:

$$\begin{aligned} \mathcal{E} &:= \{(f, 1_b) \mid f \in \text{mor } \mathcal{A}, b \in \mathcal{B}\}, \\ \mathcal{M} &:= \{(1_a, g) \mid g \in \text{mor } \mathcal{B}, a \in \mathcal{A}\}, \end{aligned}$$

Every morphism  $(f, g) \in \mathcal{A} \times \mathcal{B}$  admits a unique  $(\mathcal{E}, \mathcal{M})$ -factorization:

$$(f, g) = (1, g) \circ (f, 1).$$



# Codomain-discrete double categories

## Definition

A double category  $X$  will be called *codomain-discrete* if every top-right corner can be uniquely filled into a square:

$$\begin{array}{ccc}
 a & \xrightarrow{g} & b \\
 \vdots & & \downarrow \exists! \\
 \bullet & \dashrightarrow & c
 \end{array}$$

The diagram illustrates a square in a double category. The top-left corner is labeled  $a$ , the top-right corner is labeled  $b$ , and the bottom-right corner is labeled  $c$ . A solid arrow labeled  $g$  points from  $a$  to  $b$ . A solid arrow labeled  $u$  points from  $b$  to  $c$ . A dashed arrow points from  $a$  down to a small downward-pointing arrowhead, and another dashed arrow points from  $\bullet$  to  $c$ . A central arrow points from the top-right corner  $b$  down to the bottom-right corner  $c$ , labeled with  $\exists!$ , indicating a unique filling of the square.

## Remark

This amounts to requiring that the codomain functor  $d_0 : X_1 \rightarrow X_0$  is a discrete opfibration.

# Codomain-discrete double categories

## Definition

A double category  $X$  will be called *codomain-discrete* if every top-right corner can be uniquely filled into a square:

$$\begin{array}{ccc}
 a & \xrightarrow{g} & b \\
 \vdots & & \downarrow \exists! \\
 \bullet & \dashrightarrow & c
 \end{array}$$

The diagram shows a square with vertices  $a$  (top-left),  $b$  (top-right),  $\bullet$  (bottom-left), and  $c$  (bottom-right). A solid arrow  $g$  points from  $a$  to  $b$ . A solid arrow  $u$  points from  $b$  to  $c$ . A dashed arrow points from  $\bullet$  to  $c$ . A vertical arrow points from  $a$  to  $\bullet$ . A vertical arrow points from  $b$  to  $c$ , with the label  $\exists!$  placed to its left.

## Remark

This amounts to requiring that the codomain functor  $d_0 : X_1 \rightarrow X_0$  is a discrete opfibration.

## Example

If  $T$  is a very nice 2-monad on  $\text{Cat}$ , for any  $T$ -algebra  $(A, a)$ , its *resolution*:

$$\begin{array}{ccccc}
 & & \longrightarrow & m_A & \longrightarrow & & & \\
 & & & & & & & \\
 T^2A & \longleftarrow & Tl_A & \longrightarrow & TA & & & \\
 & & & & & & & \\
 & & \longrightarrow & Ta & \longrightarrow & & & 
 \end{array}$$

Is a double category and its transpose is codomain-discrete.

## C.d. double categories $\rightsquigarrow$ SFS' (1/2)

### Construction

Let  $X$  be codomain-discrete. By the *category of corners* associated to  $X$  we mean a category  $\text{Cnr}(X)$  such that:

- objects are the objects of  $X$ ,
- a morphism  $a \rightarrow b$  is a tuple  $(u, g)$  of a vertical and a horizontal morphism in  $X$  (below left):

$$\begin{array}{ccc}
 a & & a \\
 u \downarrow & & \parallel \\
 a' & \xrightarrow{g} & b \\
 & & a \quad \longleftarrow \quad a
 \end{array}$$

- the identity on an object  $a$  is the corner  $(1_a, 1_a)$  (above right).

## C.d. double categories $\rightsquigarrow$ SFS' (1/2)

### Construction

Let  $X$  be codomain-discrete. By the *category of corners* associated to  $X$  we mean a category  $\text{Cnr}(X)$  such that:

- objects are the objects of  $X$ ,
- a morphism  $a \rightarrow b$  is a tuple  $(u, g)$  of a vertical and a horizontal morphism in  $X$  (below left):

$$\begin{array}{ccc}
 a & & a \\
 u \downarrow & & \parallel \\
 a' & \xrightarrow{g} & b \\
 & & a \quad \longleftarrow \quad a
 \end{array}$$

- the identity on an object  $a$  is the corner  $(1_a, 1_a)$  (above right).

## C.d. double categories $\rightsquigarrow$ SFS' (1/2)

### Construction

Let  $X$  be codomain-discrete. By the *category of corners* associated to  $X$  we mean a category  $\text{Cnr}(X)$  such that:

- objects are the objects of  $X$ ,
- a morphism  $a \rightarrow b$  is a tuple  $(u, g)$  of a vertical and a horizontal morphism in  $X$  (below left):

$$\begin{array}{ccc}
 a & & a \\
 u \downarrow & & \parallel \\
 a' & \xrightarrow{g} & b \\
 & & a \longleftarrow a
 \end{array}$$

- the identity on an object  $a$  is the corner  $(1_a, 1_a)$  (above right).

## C.d. double categories $\rightsquigarrow$ SFS' (1/2)

### Construction

Let  $X$  be codomain-discrete. By the *category of corners* associated to  $X$  we mean a category  $\text{Cnr}(X)$  such that:

- objects are the objects of  $X$ ,
- a morphism  $a \rightarrow b$  is a tuple  $(u, g)$  of a vertical and a horizontal morphism in  $X$  (below left):

$$\begin{array}{ccc}
 a & & a \\
 u \downarrow & & \parallel \\
 a' \xrightarrow{g} b & & a = a
 \end{array}$$

- the identity on an object  $a$  is the corner  $(1_a, 1_a)$  (above right).

## C.d. double categories $\rightsquigarrow$ SFS' (1/2)

### Construction

Let  $X$  be codomain-discrete. By the *category of corners* associated to  $X$  we mean a category  $\text{Cnr}(X)$  such that:

- objects are the objects of  $X$ ,
- a morphism  $a \rightarrow b$  is a tuple  $(u, g)$  of a vertical and a horizontal morphism in  $X$  (below left):

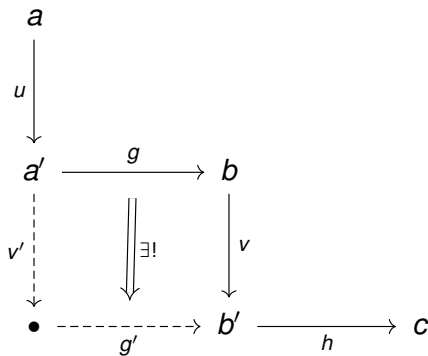
$$\begin{array}{ccc}
 a & & a \\
 u \downarrow & & \parallel \\
 a' \xrightarrow{g} b & & a \text{ --- } a
 \end{array}$$

- the identity on an object  $a$  is the corner  $(1_a, 1_a)$  (above right).



## C.d. double categories $\rightsquigarrow$ SFS' (2/2)

The composite of  $(u, g) : a \rightarrow b$  and  $(v, h) : b \rightarrow c$  is defined using the unique filler square, in this case it is the corner  $(v' \circ u, h \circ g') : a \rightarrow c$ :



The category of corners  $\text{Cnr}(X)$  has two canonical wide subcategories consisting of “vertical” and “horizontal” corners:

$$\mathcal{E}_X := \{(u, 1) \mid u \in \text{vmor } X\} \quad \mathcal{M}_X := \{(1, g) \mid g \in \text{hmor } X\}.$$

### Lemma

Let  $X$  be codomain-discrete. Then  $(\mathcal{E}_X, \mathcal{M}_X)$  is a strict factorization system on the category  $\text{Cnr}(X)$ .

### Proof

Every corner  $(u, g)$  factors uniquely as  $(1, g) \circ (u, 1)$ :

$$\begin{array}{ccccc}
 a & & & & \\
 u \downarrow & & & & \\
 a' & \xlongequal{\quad} & a' & & \\
 \parallel & & \Downarrow & & \parallel \\
 a' & \xlongequal{\quad} & a' & \xrightarrow{g} & b
 \end{array}$$

The category of corners  $\text{Cnr}(X)$  has two canonical wide subcategories consisting of “vertical” and “horizontal” corners:

$$\mathcal{E}_X := \{(u, 1) \mid u \in \text{vmor } X\} \quad \mathcal{M}_X := \{(1, g) \mid g \in \text{hmor } X\}.$$

### Lemma

Let  $X$  be codomain-discrete. Then  $(\mathcal{E}_X, \mathcal{M}_X)$  is a strict factorization system on the category  $\text{Cnr}(X)$ .

### Proof

Every corner  $(u, g)$  factors uniquely as  $(1, g) \circ (u, 1)$ :

$$\begin{array}{ccccc}
 a & & & & \\
 u \downarrow & & & & \\
 a' & \xlongequal{\quad} & a' & & \\
 \parallel & & \Downarrow & & \parallel \\
 a' & \xlongequal{\quad} & a' & \xrightarrow{g} & b
 \end{array}$$

The category of corners  $\text{Cnr}(X)$  has two canonical wide subcategories consisting of “vertical” and “horizontal” corners:

$$\mathcal{E}_X := \{(u, 1) \mid u \in \text{vmor } X\} \quad \mathcal{M}_X := \{(1, g) \mid g \in \text{hmor } X\}.$$

### Lemma

Let  $X$  be codomain-discrete. Then  $(\mathcal{E}_X, \mathcal{M}_X)$  is a strict factorization system on the category  $\text{Cnr}(X)$ .

### Proof

Every corner  $(u, g)$  factors uniquely as  $(1, g) \circ (u, 1)$ :

$$\begin{array}{ccccc}
 a & & & & \\
 u \downarrow & & & & \\
 a' & \xlongequal{\quad} & a' & & \\
 \parallel & & \Downarrow & & \parallel \\
 a' & \xlongequal{\quad} & a' & \xrightarrow{g} & b
 \end{array}$$

# SFS' $\rightsquigarrow$ c.d. double categories (1/2)

## Construction

Let  $(\mathcal{E}, \mathcal{M})$  be two classes of morphisms in a category  $\mathcal{C}$ , both closed under composition and containing all identities. Define a double category  $D_{\mathcal{E}, \mathcal{M}}$  as follows:

- The objects are the objects of  $\mathcal{C}$ ,
- vertical morphisms are those of  $\mathcal{E}$ ,
- horizontal morphisms are those of  $\mathcal{M}$ ,
- the squares are commutative squares in  $\mathcal{C}$ .

# SFS' $\rightsquigarrow$ c.d. double categories (1/2)

## Construction

Let  $(\mathcal{E}, \mathcal{M})$  be two classes of morphisms in a category  $\mathcal{C}$ , both closed under composition and containing all identities. Define a double category  $D_{\mathcal{E}, \mathcal{M}}$  as follows:

- The objects are the objects of  $\mathcal{C}$ ,
- vertical morphisms are those of  $\mathcal{E}$ ,
- horizontal morphisms are those of  $\mathcal{M}$ ,
- the squares are commutative squares in  $\mathcal{C}$ .

# SFS' $\rightsquigarrow$ c.d. double categories (1/2)

## Construction

Let  $(\mathcal{E}, \mathcal{M})$  be two classes of morphisms in a category  $\mathcal{C}$ , both closed under composition and containing all identities. Define a double category  $D_{\mathcal{E}, \mathcal{M}}$  as follows:

- The objects are the objects of  $\mathcal{C}$ ,
- vertical morphisms are those of  $\mathcal{E}$ ,
- horizontal morphisms are those of  $\mathcal{M}$ ,
- the squares are commutative squares in  $\mathcal{C}$ .

# SFS' $\rightsquigarrow$ c.d. double categories (1/2)

## Construction

Let  $(\mathcal{E}, \mathcal{M})$  be two classes of morphisms in a category  $\mathcal{C}$ , both closed under composition and containing all identities. Define a double category  $D_{\mathcal{E}, \mathcal{M}}$  as follows:

- The objects are the objects of  $\mathcal{C}$ ,
- vertical morphisms are those of  $\mathcal{E}$ ,
- horizontal morphisms are those of  $\mathcal{M}$ ,
- the squares are commutative squares in  $\mathcal{C}$ .



# SFS' $\rightsquigarrow$ c.d. double categories (1/2)

## Construction

Let  $(\mathcal{E}, \mathcal{M})$  be two classes of morphisms in a category  $\mathcal{C}$ , both closed under composition and containing all identities. Define a double category  $D_{\mathcal{E}, \mathcal{M}}$  as follows:

- The objects are the objects of  $\mathcal{C}$ ,
- vertical morphisms are those of  $\mathcal{E}$ ,
- horizontal morphisms are those of  $\mathcal{M}$ ,
- the squares are commutative squares in  $\mathcal{C}$ .

# SFS' $\rightsquigarrow$ c.d. double categories (1/2)

## Construction

Let  $(\mathcal{E}, \mathcal{M})$  be two classes of morphisms in a category  $\mathcal{C}$ , both closed under composition and containing all identities. Define a double category  $D_{\mathcal{E}, \mathcal{M}}$  as follows:

- The objects are the objects of  $\mathcal{C}$ ,
- vertical morphisms are those of  $\mathcal{E}$ ,
- horizontal morphisms are those of  $\mathcal{M}$ ,
- the squares are commutative squares in  $\mathcal{C}$ .

# SFS' $\rightsquigarrow$ c.d. double categories (2/2)

## Lemma

Let  $(\mathcal{E}, \mathcal{M})$  be a strict factorization system on a category  $\mathcal{C}$ . Then  $D_{\mathcal{E}, \mathcal{M}}$  is codomain-discrete.

## Proof

The unique filler square is given by the unique  $(\mathcal{E}, \mathcal{M})$ -factorization of the morphism  $m \circ e$  in  $\mathcal{C}$ :

$$\begin{array}{ccc}
 a & \xrightarrow{e} & b \\
 \downarrow e' & & \downarrow m \\
 \bullet & \xrightarrow{m'} & c
 \end{array}$$

The diagram shows a commutative square with a filler. The top row is a solid arrow from  $a$  to  $b$  labeled  $e$ . The right side is a solid arrow from  $b$  to  $c$  labeled  $m$ . The bottom row is a dashed arrow from a dot to  $c$  labeled  $m'$ . The left side is a dashed arrow from  $a$  to the dot labeled  $e'$ . A double-lined arrow points from  $b$  down to the dot, representing the unique factorization of  $m \circ e$ .

# SFS' $\rightsquigarrow$ c.d. double categories (2/2)

## Lemma

Let  $(\mathcal{E}, \mathcal{M})$  be a strict factorization system on a category  $\mathcal{C}$ . Then  $\mathcal{D}_{\mathcal{E}, \mathcal{M}}$  is codomain-discrete.

## Proof

The unique filler square is given by the unique  $(\mathcal{E}, \mathcal{M})$ -factorization of the morphism  $m \circ e$  in  $\mathcal{C}$ :

$$\begin{array}{ccc}
 a & \xrightarrow{e} & b \\
 \downarrow e' & & \downarrow m \\
 \bullet & \xrightarrow{m'} & c
 \end{array}$$

The diagram shows a commutative square with a double arrow in the center. The top horizontal arrow is labeled  $e$ , the right vertical arrow is labeled  $m$ , the bottom horizontal arrow is labeled  $m'$ , and the left vertical arrow is labeled  $e'$ . The bottom-left corner contains a solid black dot. The center of the square has a double arrow pointing downwards.

# SFS' $\longleftrightarrow$ cod. discr. double categories

## Theorem

The assignments:

$$\begin{aligned} (\mathcal{E}, \mathcal{M}) &\mapsto D_{\mathcal{E}, \mathcal{M}}, \\ X &\mapsto (\mathcal{E}_X, \mathcal{M}_X), \end{aligned}$$

Are equivalence inverse to each other and thus induce an equivalence between strict factorization systems and codomain-discrete double categories.

$$\begin{array}{ccc} & \text{Cnr}(-) & \\ & \longleftarrow & \\ SFS & \xleftarrow{\quad \simeq \quad} & \text{CodDiscr} \\ & \xrightarrow{D} & \end{array}$$

# Orthogonal factorization systems $\leftrightarrow$ (certain) double categories

# OFS $\longleftrightarrow$ (certain) double categories

The goal is to prove an analogue of the above result for orthogonal factorization systems. To do this, we need three ingredients:

- 1 bicartesian squares,
- 2 invariance,
- 3 the notion of a “joint monicity” of a pair of a vertical and a horizontal morphism in a double category.

# OFS $\longleftrightarrow$ (certain) double categories

The goal is to prove an analogue of the above result for orthogonal factorization systems. To do this, we need three ingredients:

- 1 bicartesian squares,
- 2 invariance,
- 3 the notion of a “joint monicity” of a pair of a vertical and a horizontal morphism in a double category.



# OFS $\longleftrightarrow$ (certain) double categories

The goal is to prove an analogue of the above result for orthogonal factorization systems. To do this, we need three ingredients:

- 1 bicartesian squares,
- 2 invariance,
- 3 the notion of a “joint monicity” of a pair of a vertical and a horizontal morphism in a double category.

# OFS $\longleftrightarrow$ (certain) double categories

The goal is to prove an analogue of the above result for orthogonal factorization systems. To do this, we need three ingredients:

- 1 bicartesian squares,
- 2 invariance,
- 3 the notion of a “joint monicity” of a pair of a vertical and a horizontal morphism in a double category.

# Orthogonal factorization systems

## Definition

An *orthogonal factorization system*  $(\mathcal{E}, \mathcal{M})$  on a category  $\mathcal{C}$  consists of two wide sub-categories  $\mathcal{E}, \mathcal{M} \subseteq \mathcal{C}$  satisfying:

- For every morphism  $f \in \mathcal{C}$  there exist  $e \in \mathcal{E}, m \in \mathcal{M}$  such that  $f = m \circ e$ , and if  $f = m' \circ e'$  is a second factorization with  $e' \in \mathcal{E}, m' \in \mathcal{M}$ , there exists a unique morphism  $\theta$  so that this commutes:

$$\begin{array}{ccccc}
 a & \xrightarrow{e} & a' & \xrightarrow{m} & b \\
 \parallel & & \downarrow \exists! \theta & & \parallel \\
 a & \xrightarrow{e'} & a'' & \xrightarrow{m'} & b
 \end{array}$$

- we have that  $\mathcal{E} \cap \mathcal{M} = \{\text{isomorphisms in } \mathcal{C}\}$ .

# Bicrossed double categories (1/2)

## Definition

A square  $\lambda$  in a double category  $X$  will be called *opcartesian* if it's an opcartesian morphism with respect to the codomain functor  $d_0 : X_1 \rightarrow X_0$ . In elementary terms:

$$\begin{array}{ccc}
 a & \xrightarrow{g} & b \\
 \downarrow u & \Downarrow \lambda & \downarrow v \\
 c & \xrightarrow{h} & d \\
 \downarrow \theta & \Downarrow \exists! \epsilon & \parallel \\
 e & \xrightarrow{k} & d
 \end{array}
 =
 \begin{array}{ccc}
 a & \xrightarrow{g} & b \\
 \downarrow l & \Downarrow \forall \alpha & \downarrow v \\
 e & \xrightarrow{k} & d
 \end{array}$$

# Bicrossed double categories (1/2)

## Definition

A square  $\lambda$  in a double category  $X$  will be called *opcartesian* if it's an opcartesian morphism with respect to the codomain functor  $d_0 : X_1 \rightarrow X_0$ . In elementary terms:

$$\begin{array}{ccc}
 a & \xrightarrow{g} & b \\
 \downarrow u & \Downarrow \lambda & \downarrow v \\
 c & \xrightarrow{h} & d \\
 \downarrow \theta & \Downarrow \exists! \epsilon & \parallel \\
 e & \xrightarrow{k} & d
 \end{array}
 =
 \begin{array}{ccc}
 a & \xrightarrow{g} & b \\
 \downarrow l & \Downarrow \forall \alpha & \downarrow v \\
 e & \xrightarrow{k} & d
 \end{array}$$

## Bicrossed double categories (2/2)

Given a double category  $X$ , denote  $X^* := ((X^v)^h)^T$ .

### Definition - Ingredient 1

A square  $\lambda$  in a double category  $X$  will be called *bicartesian* if it is opcartesian in both  $X$  and  $X^*$ .

### Definition

A double category  $X$  will be called *bicrossed* if every top-right corner can be filled to a (not necessarily unique) bicartesian square. Moreover, bicartesian squares are closed under horizontal and vertical compositions and identities.

## Bicrossed double categories (2/2)

Given a double category  $X$ , denote  $X^* := ((X^v)^h)^T$ .

### Definition - Ingredient 1

A square  $\lambda$  in a double category  $X$  will be called *bicartesian* if it is opcartesian in both  $X$  and  $X^*$ .

### Definition

A double category  $X$  will be called *bicrossed* if every top-right corner can be filled to a (not necessarily unique) bicartesian square. Moreover, bicartesian squares are closed under horizontal and vertical compositions and identities.

## Bicrossed double categories (2/2)

Given a double category  $X$ , denote  $X^* := ((X^v)^h)^T$ .

### Definition - Ingredient 1

A square  $\lambda$  in a double category  $X$  will be called *bicartesian* if it is opcartesian in both  $X$  and  $X^*$ .

### Definition

A double category  $X$  will be called *bicrossed* if every top-right corner can be filled to a (not necessarily unique) bicartesian square. Moreover, bicartesian squares are closed under horizontal and vertical compositions and identities.



# Bicrossed double categories - Examples

## Example

$\mathcal{C}$  a category with pullbacks,  $\text{Sq}(\mathcal{C})^\vee$ ,  $\text{PbSq}(\mathcal{C})^\vee$ ,  $\text{MonoPbSq}(\mathcal{C})^\vee$ . In each of these the filler is given by a pullback square:

$$\begin{array}{ccc}
 a & \xrightarrow{g} & b \\
 \uparrow & & \uparrow \\
 \bullet & \dashrightarrow & c
 \end{array}$$

The diagram shows a pullback square. The top-left object is  $a$ , the top-right is  $b$ , and the bottom-right is  $c$ . A solid arrow  $g$  points from  $a$  to  $b$ . A solid arrow  $u$  points from  $c$  to  $b$ . A dashed arrow points from  $\bullet$  to  $a$ , and another dashed arrow points from  $\bullet$  to  $c$ . A small comma is located in the center of the square.

## Example

$\text{BOFib}^\vee$ . This is because both bijections on objects and discrete opfibrations are stable under pullbacks.

# Bicrossed double categories - Examples

## Example

$\mathcal{C}$  a category with pullbacks,  $\text{Sq}(\mathcal{C})^\vee$ ,  $\text{PbSq}(\mathcal{C})^\vee$ ,  $\text{MonoPbSq}(\mathcal{C})^\vee$ . In each of these the filler is given by a pullback square:

$$\begin{array}{ccc}
 a & \xrightarrow{g} & b \\
 \uparrow & & \uparrow \\
 \bullet & \dashrightarrow & c
 \end{array}$$

(Note: The left vertical arrow is dashed, and the bottom horizontal arrow is dashed. The top horizontal arrow is labeled  $g$ , the right vertical arrow is labeled  $u$ , and the bottom horizontal arrow is labeled  $c$ .)

## Example

$\text{BOFib}^\vee$ . This is because both bijections on objects and discrete opfibrations are stable under pullbacks.

# The category of corners

## Construction

Let  $X$  be bicrossed. Assume every square is bicartesian. By the *category of corners* associated to  $X$  we mean a category  $\text{Cnr}(X)$  such that:

- objects are the objects of  $X$ ,
- a morphism  $a \rightarrow b$  is an equivalence class  $[e, m]$  of tuples of a vertical morphism followed by a horizontal one in  $X$ ,
- the identity on an object  $a$  is the equivalence class  $[1_a, 1_a]$  (above right).

# The category of corners

## Construction

Let  $X$  be bicrossed. Assume every square is bicartesian. By the *category of corners* associated to  $X$  we mean a category  $\text{Cnr}(X)$  such that:

- objects are the objects of  $X$ ,
- a morphism  $a \rightarrow b$  is an equivalence class  $[e, m]$  of tuples of a vertical morphism followed by a horizontal one in  $X$ ,
- the identity on an object  $a$  is the equivalence class  $[1_a, 1_a]$  (above right).

# The category of corners

## Construction

Let  $X$  be bicrossed. Assume every square is bicartesian. By the *category of corners* associated to  $X$  we mean a category  $\text{Cnr}(X)$  such that:

- objects are the objects of  $X$ ,
- a morphism  $a \rightarrow b$  is an equivalence class  $[e, m]$  of tuples of a vertical morphism followed by a horizontal one in  $X$ ,
- the identity on an object  $a$  is the equivalence class  $[1_a, 1_a]$  (above right).

# The category of corners

## Construction

Let  $X$  be bicrossed. Assume every square is bicartesian. By the *category of corners* associated to  $X$  we mean a category  $\text{Cnr}(X)$  such that:

- objects are the objects of  $X$ ,
- a morphism  $a \rightarrow b$  is an equivalence class  $[e, m]$  of tuples of a vertical morphism followed by a horizontal one in  $X$ ,
- the identity on an object  $a$  is the equivalence class  $[1_a, 1_a]$  (above right).

# The category of corners

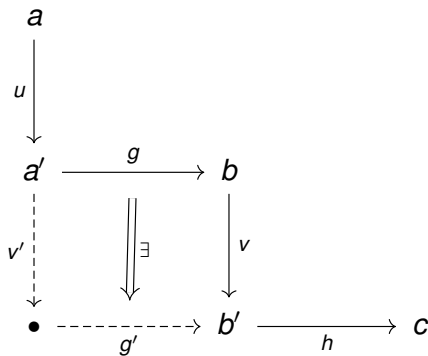
We consider two corners  $(e, m)$ ,  $(e', m')$  with the same domain and codomain equivalent if and only if there exists a square  $\beta$  like this:

$$\begin{array}{ccccc}
 a & \xlongequal{\quad} & a & & \\
 e \downarrow & & \downarrow e & & \\
 a' & \xlongequal{\quad} & a' & \xrightarrow{m} & b \\
 \theta \downarrow & \Downarrow \beta & \parallel & & \parallel \\
 a'' & \xrightarrow{\psi} & a' & \xrightarrow{m} & b \\
 & \searrow m' & & & 
 \end{array}$$

The diagram illustrates a commutative square  $\beta$  between two corners. The top row shows the identity  $a \xlongequal{\quad} a$ . The middle row shows the corner  $(e, m)$  with  $a' \xlongequal{\quad} a'$  and  $a' \xrightarrow{m} b$ . The bottom row shows the corner  $(e', m')$  with  $a'' \xrightarrow{\psi} a'$  and  $a' \xrightarrow{m} b$ . A curved arrow  $e'$  on the left indicates the equivalence between the two corners. A curved arrow  $m'$  at the bottom indicates the equivalence between the two morphisms  $\psi$  and  $m$ . The square  $\beta$  is represented by a double arrow  $\Downarrow \beta$  between the two  $a'$  nodes, with a vertical double arrow  $\parallel$  between the two  $a'$  nodes and a vertical double arrow  $\parallel$  between the two  $b$  nodes.

# The category of corners

The composite of  $[u, g] : a \rightarrow b$  and  $[v, h] : b \rightarrow c$  is defined using a **choice** of **some** bicartesian filler square, in this case it is the equivalence class  $[v' \circ u, h \circ g'] : a \rightarrow c$ :





# The category of corners - Examples

## Example

Consider  $\text{PbSq}(\mathcal{C})^\vee$  for  $\mathcal{C}$  with pullbacks.  $\text{Cnr}(\text{PbSq}(\mathcal{C})^\vee)$  has objects the objects of  $\mathcal{C}$ , while a morphism is an equivalence class of corners:

$$\begin{array}{ccc} & a & \\ & \uparrow u & \\ a' & \xrightarrow{g} & b \end{array}$$

In fact,  $\text{Cnr}(\text{PbSq}(\mathcal{C})^\vee) \cong \text{Span}(\mathcal{C})$ .

Similarly,

$$\text{Cnr}(\text{MonoPbSq}(\mathcal{C})^\vee) \cong \text{Par}(\mathcal{C}),$$

$$\text{Cnr}(\text{BOFib}^\vee) \cong \text{Cof}.$$

# The category of corners - Examples

## Example

Consider  $\text{PbSq}(\mathcal{C})^\vee$  for  $\mathcal{C}$  with pullbacks.  $\text{Cnr}(\text{PbSq}(\mathcal{C})^\vee)$  has objects the objects of  $\mathcal{C}$ , while a morphism is an equivalence class of corners:

$$\begin{array}{ccc} & a & \\ & \uparrow u & \\ a' & \xrightarrow{g} & b \end{array}$$

In fact,  $\text{Cnr}(\text{PbSq}(\mathcal{C})^\vee) \cong \text{Span}(\mathcal{C})$ .

Similarly,

$$\text{Cnr}(\text{MonoPbSq}(\mathcal{C})^\vee) \cong \text{Par}(\mathcal{C}),$$

$$\text{Cnr}(\text{BOFib}^\vee) \cong \text{Cof}.$$

# The category of corners - Examples

## Example

Consider  $\text{PbSq}(\mathcal{C})^\vee$  for  $\mathcal{C}$  with pullbacks.  $\text{Cnr}(\text{PbSq}(\mathcal{C})^\vee)$  has objects the objects of  $\mathcal{C}$ , while a morphism is an equivalence class of corners:

$$\begin{array}{ccc} & a & \\ & \uparrow u & \\ a' & \xrightarrow{g} & b \end{array}$$

In fact,  $\text{Cnr}(\text{PbSq}(\mathcal{C})^\vee) \cong \text{Span}(\mathcal{C})$ .

Similarly,

$$\text{Cnr}(\text{MonoPbSq}(\mathcal{C})^\vee) \cong \text{Par}(\mathcal{C}),$$

$$\text{Cnr}(\text{BOFib}^\vee) \cong \text{Cof}.$$

# The category of corners - Examples

## Example

Consider  $\text{PbSq}(\mathcal{C})^\vee$  for  $\mathcal{C}$  with pullbacks.  $\text{Cnr}(\text{PbSq}(\mathcal{C})^\vee)$  has objects the objects of  $\mathcal{C}$ , while a morphism is an equivalence class of corners:

$$\begin{array}{ccc} & a & \\ & \uparrow u & \\ a' & \xrightarrow{g} & b \end{array}$$

In fact,  $\text{Cnr}(\text{PbSq}(\mathcal{C})^\vee) \cong \text{Span}(\mathcal{C})$ .

Similarly,

$$\text{Cnr}(\text{MonoPbSq}(\mathcal{C})^\vee) \cong \text{Par}(\mathcal{C}),$$

$$\text{Cnr}(\text{BOFib}^\vee) \cong \text{Cof}.$$

# The category of corners - Examples

## Example

Consider  $\text{PbSq}(\mathcal{C})^\vee$  for  $\mathcal{C}$  with pullbacks.  $\text{Cnr}(\text{PbSq}(\mathcal{C})^\vee)$  has objects the objects of  $\mathcal{C}$ , while a morphism is an equivalence class of corners:

$$\begin{array}{ccc} & a & \\ & \uparrow u & \\ a' & \xrightarrow{g} & b \end{array}$$

In fact,  $\text{Cnr}(\text{PbSq}(\mathcal{C})^\vee) \cong \text{Span}(\mathcal{C})$ .

Similarly,

$$\begin{aligned} \text{Cnr}(\text{MonoPbSq}(\mathcal{C})^\vee) &\cong \text{Par}(\mathcal{C}), \\ \text{Cnr}(\text{BOFib}^\vee) &\cong \text{Cof}. \end{aligned}$$

# Ingredient 2

## Definition - Ingredient 2

A double category  $X$  is *invariant* if the following boundaries admit a unique filler:

$$\begin{array}{ccc}
 a & \overset{\exists!}{\dashrightarrow} & b \\
 \Downarrow \cong & & \Downarrow \exists! \\
 d & \longrightarrow & c
 \end{array}$$

$$\begin{array}{ccc}
 a & \xrightarrow{\cong} & b \\
 \Downarrow \exists! & & \Downarrow \exists! \\
 d & \xrightarrow{\cong} & c
 \end{array}$$

## Example

All of our previous guests:  $Sq(\mathcal{C})$ ,  $PbSq(\mathcal{C})$ ,  $MonoPbSq(\mathcal{C})$ ,  $BOFib$ .

## Ingredient 2

### Definition - Ingredient 2

A double category  $X$  is *invariant* if the following boundaries admit a unique filler:

$$\begin{array}{ccc}
 a & \overset{\exists!}{\dashrightarrow} & b \\
 \cong \downarrow & \Downarrow \exists! & \downarrow \cong \\
 d & \longrightarrow & c
 \end{array}$$

$$\begin{array}{ccc}
 a & \xrightarrow{\cong} & b \\
 \exists! \downarrow & \Downarrow \exists! & \downarrow \\
 d & \xrightarrow{\cong} & c
 \end{array}$$

### Example

All of our previous guests:  $\text{Sq}(\mathcal{C})$ ,  $\text{PbSq}(\mathcal{C})$ ,  $\text{MonoPbSq}(\mathcal{C})$ ,  $\text{BOFib}$ .

# Ingredient 3

## Definition - ingredient 3

A top-left corner  $(\pi_1, \pi_2)$  in a double category  $X$  is said to be *jointly monic* if, given squares  $\kappa_1, \kappa_2$  pictured below:

$$\begin{array}{ccc} a' & \xrightarrow{\pi_2} & b \\ \pi_1 \downarrow & & \\ a & & \end{array}$$

$$\begin{array}{ccc} a'' & \xrightarrow{\psi} & a' \\ \theta \downarrow & \Downarrow \kappa_1 & \parallel \\ a' & \xlongequal{\quad} & a' \end{array}$$

$$\begin{array}{ccc} a'' & \xrightarrow{\psi'} & a' \\ \theta' \downarrow & \Downarrow \kappa_2 & \parallel \\ a' & \xlongequal{\quad} & a' \end{array}$$

We have the following implication:

$$(\pi_1\theta = \pi_1\theta' \wedge \pi_2\psi = \pi_2\psi') \Rightarrow (\theta = \theta', \psi = \psi').$$



## Ingredient 3 - Example

### Example

In  $\text{Sq}(\mathcal{C})$  a pair  $(\pi_1, \pi_2)$  of pullback projections is jointly monic, as this condition reduces to:

$$(\pi_1\theta = \pi_1\theta' \wedge \pi_2\theta = \pi_2\theta') \Rightarrow (\theta = \theta').$$

### Example

In  $\text{MonoPbSq}(\mathcal{C})$  any pair  $(\pi_1, \pi_2)$  is jointly monic because  $\pi_1$  is a monomorphism.

### Example

In  $\text{BOFib}$  any pair is jointly monic. It can be proven.

## Ingredient 3 - Example

### Example

In  $\text{Sq}(\mathcal{C})$  a pair  $(\pi_1, \pi_2)$  of pullback projections is jointly monic, as this condition reduces to:

$$(\pi_1\theta = \pi_1\theta' \wedge \pi_2\theta = \pi_2\theta') \Rightarrow (\theta = \theta').$$

### Example

In  $\text{MonoPbSq}(\mathcal{C})$  any pair  $(\pi_1, \pi_2)$  is jointly monic because  $\pi_1$  is a monomorphism.

### Example

In  $\text{BOFib}$  any pair is jointly monic. It can be proven.

## Ingredient 3 - Example

### Example

In  $\text{Sq}(\mathcal{C})$  a pair  $(\pi_1, \pi_2)$  of pullback projections is jointly monic, as this condition reduces to:

$$(\pi_1\theta = \pi_1\theta' \wedge \pi_2\theta = \pi_2\theta') \Rightarrow (\theta = \theta').$$

### Example

In  $\text{MonoPbSq}(\mathcal{C})$  any pair  $(\pi_1, \pi_2)$  is jointly monic because  $\pi_1$  is a monomorphism.

### Example

In  $\text{BOFib}$  any pair is jointly monic. It can be proven.

# Fact. double categories $\rightsquigarrow$ OFS'

## Definition

A double category  $X$  is said to be a *factorization double category* if:

- every square is bicartesian and every top-right corner can be filled to a square,
- $X$  is invariant,
- every top-left corner in  $X^\vee$  is jointly monic.

Let  $X$  be a factorization double category. Define the classes of “vertical” and “horizontal” corners  $\mathcal{E}_X, \mathcal{M}_X$  on the category  $\text{Cnr}(X)$  as before. We have:

## Proposition

Let  $X$  be a factorization double category. Then  $(\mathcal{E}_X, \mathcal{M}_X)$  is an orthogonal factorization system on the category  $\text{Cnr}(X)$ .

# Fact. double categories $\rightsquigarrow$ OFS'

## Definition

A double category  $X$  is said to be a *factorization double category* if:

- every square is bicartesian and every top-right corner can be filled to a square,
- $X$  is invariant,
- every top-left corner in  $X^\vee$  is jointly monic.

Let  $X$  be a factorization double category. Define the classes of “vertical” and “horizontal” corners  $\mathcal{E}_X, \mathcal{M}_X$  on the category  $\text{Cnr}(X)$  as before. We have:

## Proposition

Let  $X$  be a factorization double category. Then  $(\mathcal{E}_X, \mathcal{M}_X)$  is an orthogonal factorization system on the category  $\text{Cnr}(X)$ .

# Fact. double categories $\rightsquigarrow$ OFS'

## Definition

A double category  $X$  is said to be a *factorization double category* if:

- every square is bicartesian and every top-right corner can be filled to a square,
- $X$  is invariant,
- every top-left corner in  $X^\vee$  is jointly monic.

Let  $X$  be a factorization double category. Define the classes of “vertical” and “horizontal” corners  $\mathcal{E}_X, \mathcal{M}_X$  on the category  $\text{Cnr}(X)$  as before. We have:

## Proposition

Let  $X$  be a factorization double category. Then  $(\mathcal{E}_X, \mathcal{M}_X)$  is an orthogonal factorization system on the category  $\text{Cnr}(X)$ .

Fact. double categories  $\longleftrightarrow$  OFS'

## Proposition

Let  $(\mathcal{E}, \mathcal{M})$  be an orthogonal factorization system on a category  $\mathcal{C}$ . Then  $D_{\mathcal{E}, \mathcal{M}}$  is a factorization double category.

## Theorem

The assignments are again equivalence inverse to each other and induce an equivalence:

$$\begin{array}{ccc}
 & \text{Cnr}(-) & \\
 \text{OFS} & \xleftarrow{\quad} & \text{FactDbL} \\
 & \underset{D}{\xrightarrow{\quad}} & \\
 & \simeq & 
 \end{array}$$

Fact. double categories  $\longleftrightarrow$  OFS'

## Proposition

Let  $(\mathcal{E}, \mathcal{M})$  be an orthogonal factorization system on a category  $\mathcal{C}$ . Then  $D_{\mathcal{E}, \mathcal{M}}$  is a factorization double category.

## Theorem

The assignments are again equivalence inverse to each other and induce an equivalence:

$$\begin{array}{ccc}
 & \text{Cnr}(-) & \\
 & \longleftarrow & \\
 \text{OFS} & \xrightarrow{\simeq} & \text{FactDbI} \\
 & \longleftarrow & \\
 & D & 
 \end{array}$$



# Examples (1/2)

## Example

$\mathcal{C}$  a category with pullbacks,  $\text{MonoPbSq}(\mathcal{C})^\vee$  is a factorization double category. Thus  $\text{Cnr}(\text{MonoPbSq}(\mathcal{C})^\vee) = \text{Par}(\mathcal{C})$  admits an orthogonal factorization system given by “restricted identity maps” and *total maps*:

$$\begin{array}{ccc}
 a & & a \\
 \uparrow \iota & & \parallel \\
 a' & \xlongequal{\quad} & a' & & a & \xrightarrow{g} & b
 \end{array}$$

## Examples (2/2)

### Example

$\text{BOFib}^V$  is a factorization double category and  $\text{Cnr}(\text{BOFib}^V) = \text{Cof}$  comes equipped with an orthogonal factorization system given by (the opposites of) bijections on objects followed by discrete opfibrations.

### Example

If  $P : \mathcal{E} \rightarrow \mathcal{B}$  is a fibration, there is a double category  $X_P$  such that:

- objects are the objects of  $\mathcal{E}$ ,
- vertical morphisms are  $P$ -vertical morphisms,
- horizontal morphisms are  $P$ -cartesian morphisms,
- squares are commutative squares.

$X_P$  is a factorization double category and  $\text{Cnr}(X_P) = \mathcal{E}$  admits an orthogonal factorization system given by  $P$ -vertical morphisms followed by  $P$ -cartesian morphisms.

## Examples (2/2)

### Example

$\text{BOFib}^V$  is a factorization double category and  $\text{Cnr}(\text{BOFib}^V) = \text{Cof}$  comes equipped with an orthogonal factorization system given by (the opposites of) bijections on objects followed by discrete opfibrations.

### Example

If  $P : \mathcal{E} \rightarrow \mathcal{B}$  is a fibration, there is a double category  $X_P$  such that:

- objects are the objects of  $\mathcal{E}$ ,
- vertical morphisms are  $P$ -vertical morphisms,
- horizontal morphisms are  $P$ -cartesian morphisms,
- squares are commutative squares.

$X_P$  is a factorization double category and  $\text{Cnr}(X_P) = \mathcal{E}$  admits an orthogonal factorization system given by  $P$ -vertical morphisms followed by  $P$ -cartesian morphisms.

# References



Mark Weber (2015)

Internal algebra classifiers as codescent objects of crossed internal categories

*Theory and Applications of Categories* 30.50 (2015): 1713-1792.



Miloslav Štěpán (2023)

Factorization systems and double categories

*arXiv preprint arXiv:2305.06714* 12(3)

Thank you.