Strong pseudomonads and premonoidal bicategories

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Summary

Question: how to bicategorify the following structures?

A **left strength** for a monad (T, μ, η) on a monoidal category $(\mathbb{C}, \otimes, I)$ is a natural transformation $A \otimes TB \to T(A \otimes B)$ subject to coherence laws.

A **bistrong monad** is a monad with both a left strength $t_{A,B} : A \otimes TB \to T(A \otimes B)$ and a right strength $s_{A,B} : T(A) \otimes B \to T(A \otimes B)$, subject to a compatibility law relating s and t.

Premonoidal categories axiomatise the structure of the Kleisli category $\mathbb{C}_T$ when T is bistrong. A strict premonoidal category is a monoid in **Cat** with the funny tensor product.

Strategy: equations on 2-cells

1. Start by bicategorifying correspondences from the 1-dimensional setting;
2. Show basic examples and theory also lift;
3. Prove (some) coherence results.

Motivation

Build a framework to capture recent bicategorical models for programming languages and linear logic (*e.g.* [11, 4, 6, 1]).

Strengths for pseudomonads

Definition

A **left strength** for a pseudomonad T on a monoidal bicategory $\mathcal{B}$ consists of a pseudonatural transformation $t_{A,B} : A \otimes TB \to T(A \otimes B)$ together with invertible modifications witnessing the four axioms of a strong monad, subject to coherence equations.

A **bistrong** pseudomonad has a left strength and a right strength, related by an invertible modification satisfying coherence axioms. Every strong pseudomonad on a symmetric monoidal bicategory is bistrong.

Results mirroring the 1-dimensional setting, and coherence

To give a left strength for a monad (T, μ, η) on $(\mathbb{C}, \otimes, I)$ is to give a left action $\mathbb{C} \times \mathbb{C}_T \to \mathbb{C}_T$, such that $\eta \circ (-) : \mathbb{C} \to \mathbb{C}_T$ is a strict morphism of actions. A bicategorical correlate:

Proposition. *For a pseudomonad T on a monoidal bicategory $(\mathcal{B}, \otimes, I)$, there is an equivalence between the category of left strengths on T and the category of 0-strict morphisms of actions with $\mathsf{J} = \eta \circ (-)$:*

$$\begin{array}{ccc} \mathcal{B} \times \mathcal{B} & \xrightarrow{\otimes} & \mathcal{B} \\ {\scriptstyle \mathrm{id} \times \mathsf{J}}\downarrow & \overset{\theta}{\cong} & \downarrow{\scriptstyle \mathsf{J}} \\ \mathcal{B} \times \mathcal{B}_T & \xrightarrow{\triangleright} & \mathcal{B}_T \end{array}$$

A strong monad on $\mathbb{V}$ is an internal monad in the 2-category $\mathbb{V}$-Act.

Proposition. *A strong pseudomonad on $\mathcal{B}$ is precisely an internal pseudomonad in the tricategory $\mathcal{B}$-*Act *(see above right). Hence, by Lack's coherence theorem [9], every diagram of structural 2-cells commutes.*

Examples

- Any pseudomonad on $(\mathbf{Cat}, \times, 1)$;
- If $\mathcal{B}$ is cartesian closed $(\times, 1, \Rightarrow)$, the state pseudomonad $S \Rightarrow (S \times -)$ and continuation pseudomonad $(- \Rightarrow R) \Rightarrow R$ are canonically strong.
- Any pseudomonad wrt coproducts $(0, +)$;
- $(-) \otimes M$ for any pseudomonoid M;
- Any strong monad on $\mathbb{C}$ lifts to a strong pseudomonad: on $\mathrm{Para}(\mathbb{C})$ (see *e.g.* [5]) if $\mathbb{C}$ is monoidal; on $\mathrm{Span}(\mathbb{C})$ if $\mathbb{C}$ is lextensive (see [2]).

The 1-dimensional story: premonoidal and Freyd structure

Premonoidal categories are weakenings of monoidal categories in which $\otimes$ is only assumed to be a functor in each argument separately, *i.e.* interchange fails.

A **binoidal category** $(\mathbb{P}, \rtimes, \ltimes)$ is a category $\mathbb{P}$ equipped with a mapping $\otimes : ob(\mathbb{P}) \times ob(\mathbb{P}) \to ob(\mathbb{P})$ and functors $A \rtimes (-), (-) \ltimes B : \mathbb{P} \to \mathbb{P}$ for every $A, B \in \mathbb{P}$, such that $A \rtimes B = A \otimes B = A \ltimes B$ on objects. A map $f : A \to A'$ in $\mathbb{P}$ is **central** if for any $g : B \to B'$ interchange holds on both sides:

$$\begin{array}{ccc} A \otimes B & \xrightarrow{A \rtimes g} & A \otimes B' \\ {\scriptstyle f \ltimes B}\downarrow & & \downarrow{\scriptstyle f \ltimes B'} \\ A' \otimes B & \xrightarrow[A' \rtimes g]{} & A' \otimes B' \end{array} \qquad \begin{array}{ccc} B \otimes A & \xrightarrow{g \ltimes A} & B' \otimes A \\ {\scriptstyle B \rtimes f}\downarrow & & \downarrow{\scriptstyle B' \rtimes f} \\ B \otimes A' & \xrightarrow[g \ltimes A']{} & B' \otimes A' \end{array}$$

A **premonoidal category** $(\mathbb{P}, \rtimes, \ltimes, I)$ is a binoidal category equipped with a unit $I \in \mathbb{P}$ and component-wise central natural isomorphisms α, λ, ρ satisfying triangle and pentagon laws as in a monoidal category.

Actions and extensions for bicategories

A **left action** of a monoidal bicategory $(\mathcal{V}, \otimes, I)$ on a bicategory $\mathcal{B}$ is defined as a degenerate 2-object tricategory (*c.f.* [7]).

$$\begin{array}{ccc} \mathcal{V} \times \mathcal{B} & \xrightarrow{\triangleright} & \mathcal{V} \\ {\scriptstyle \mathrm{id} \times \mathsf{J}}\downarrow & \overset{\theta}{\Longleftarrow} & \downarrow{\scriptstyle \mathsf{J}} \\ \mathcal{V} \times \mathcal{C} & \xrightarrow{\blacktriangleright} & \mathcal{C} \end{array}$$

Proposition. *There is a tricategory $\mathcal{V}$-Act with objects left $\mathcal{V}$-actions. 1-cells are pseudofunctors $\mathsf{J} : \mathcal{B} \to \mathcal{C}$ with a pseudonatural transformation θ as on the right and modifications similar to those of a monoidal pseudofunctor, subject to coherence axioms (*c.f. *[3, 12]).*

Proposition. *For a fixed monoidal $\mathcal{V}$, there is a biequivalence $\mathcal{V}\text{-Act}(\mathcal{B}) \simeq \mathrm{MonBicat}(\mathcal{V}, \mathrm{Hom}(\mathcal{B}, \mathcal{B}))$ between left $\mathcal{V}$-actions on $\mathcal{B}$ and monoidal pseudofunctors $\mathcal{V} \to \mathrm{Hom}(\mathcal{B}, \mathcal{B})$ (see [3]).*

A **0-strict morphism of $\mathcal{V}$-actions** is a 1-cell (J, θ) such that θ is invertible, J strictly preserves the 1-cell structural data, and J preserves the 2-cell structural data modulo θ.

Premonoidal bicategories

Binoidal structure

A **binoidal bicategory** is a bicategory $\mathcal{B}$ equipped with a mapping $\otimes : ob(\mathcal{B}) \times ob(\mathcal{B}) \to ob(\mathcal{B})$ and pseudofunctors $A \rtimes (-), (-) \ltimes B : \mathcal{B} \to \mathcal{B}$ for every $A, B \in \mathcal{B}$, such that $A \rtimes B = A \otimes B = A \ltimes B$ on objects.

Centrality as data

A **central 1-cell** is a 1-cell $f : A \to A'$ with 2-cells lc^f_g and rc^f_g for each $g : B \to B'$

$$\begin{array}{ccc} A \otimes B & \xrightarrow{A \rtimes g} & A \otimes B' \\ {\scriptstyle f \ltimes B}\downarrow & \overset{\mathsf{lc}^f_g}{\cong} & \downarrow{\scriptstyle f \ltimes B'} \\ A' \otimes B & \xrightarrow[A' \rtimes g]{} & A' \otimes B' \end{array} \qquad \begin{array}{ccc} B \otimes A & \xrightarrow{g \ltimes A} & B' \otimes A \\ {\scriptstyle B \rtimes f}\downarrow & \overset{\mathsf{rc}^f_g}{\cong} & \downarrow{\scriptstyle B' \rtimes f} \\ B \otimes A' & \xrightarrow[g \ltimes A']{} & B' \otimes A' \end{array}$$

such that we get pseudonatural transformations

$$\mathsf{lc}^f : A \rtimes (-) \Rightarrow A' \rtimes (-) \quad , \quad \mathsf{lc}^f_B := \left(A \rtimes B = A \ltimes B \xrightarrow{f \ltimes B} A' \ltimes B = A' \rtimes B\right)$$
$$\mathsf{rc}^f : (-) \ltimes A \Rightarrow (-) \ltimes A' \quad , \quad \mathsf{rc}^f_B := \left(B \ltimes A = B \rtimes A \xrightarrow{B \rtimes f} B \rtimes A' = B \ltimes A'\right)$$

A **central 2-cell** $(f, \mathsf{lc}^f, \mathsf{rc}^f) \Rightarrow (f', \mathsf{lc}^{f'}, \mathsf{rc}^{f'})$ is a 2-cell $\sigma : f \Rightarrow f'$ such that $\sigma \rtimes (-)$ and $(-) \ltimes \sigma$ define modifications between the induced pseudonatural transformations.

Premonoidal structure

A **premonoidal bicategory** is a binoidal bicategory with a unit $I \in \mathcal{B}$, component-wise central pseudonatural transformations and component-wise central modifications as in a monoidal bicategory.

A subtlety. For the structural modifications that use interchange, we need to use the transformation given by centrality:

$$\begin{array}{ccc} (I \rtimes -) \rtimes B & \xrightarrow{\lambda \ltimes B} & (- \ltimes B) \\ & {\scriptstyle \alpha_{I,-,B}}\searrow \quad \mathsf{l}_{-,B} \quad \nearrow{\scriptstyle \lambda_{- \ltimes B}} & \\ & I \rtimes (- \ltimes B) & \end{array} \qquad \begin{array}{ccc} (IA) \rtimes (-) & \xrightarrow{\mathsf{lc}^{\lambda}} & (A \rtimes -) \\ & {\scriptstyle \alpha_{I,A,-}}\searrow \quad \mathsf{l}_{A,-} \quad \nearrow{\scriptstyle \lambda_{A \rtimes -}} & \\ & I \rtimes (A \ltimes -) & \end{array}$$

Examples

- Any monoidal bicategory;
- The Kleisli bicategory $\mathcal{B}_T$ for a bistrong pseudomonad;
- The bicategory $[\mathcal{B}, \mathcal{C}]_{\mathrm{unnat}}$ of pseudofunctors, *unnatural* transformations, and *unnatural* modifications.
- For any **bistrong graded monad** $T : \mathbb{E} \to [\mathbb{C}, \mathbb{C}]_{\mathrm{bistrong}}$ (*c.f.* [13, 10, 8]), the Kleisli bicategory $\mathcal{K}_T$ with 1-cells $A \twoheadrightarrow B$ given by arrows $A \to T_e B$.

See our Arxiv preprint for extension to **Freyd bicategories**, their relationship to actions, and more!

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