Resource-sensitive model theory

A categorical view

Luca Reggio

University College London

International Category Theory Conference

Louvain-la-Neuve, July 6, 2023

Outline

1. Logic, categories, and resources
2. Games: unravelling and covering
3. Counting, definability, and beyond

1. Logic, categories, and resources

2. Games: unravelling and covering

3. Counting, definability, and beyond

Categorical logic (after Lawvere)

Algebraization of first-order logic:

- Lindenbaum-Tarski algebras are replaced by syntactic categories $\mathbf{Syn}(T)$
- Quantifiers as adjoints (thus a property, rather than extra structure)
- Models of T as structure-preserving functors defined on $\mathbf{Syn}(T)$

Categorical logic (after Lawvere)

Algebraization of first-order logic:

- Lindenbaum-Tarski algebras are replaced by syntactic categories $\mathbf{Syn}(T)$
- Quantifiers as adjoints (thus a property, rather than extra structure)
- Models of T as structure-preserving functors defined on $\mathbf{Syn}(T)$

(Boolean) hyperdoctrine associated to T: fiberwise Lindenbaum-Tarski algebras—stratify in terms of free variables. Studying first-order logic is like studying Boolean algebras in $\widehat{\mathbf{Fin}}$.

Dual perspective: Joyal's polyadic spaces, which encode the spaces of n-types of T. (This viewpoint has been recently exploited and considerably extended by Jérémie Marquès.)

These ideas lead to a syntax-independent view of logic and model theory.

Categorical logic

Various approaches to categorical model theory, including:

- accessible categories (Adámek, Lair, Makkai, Paré, Rosický,...)
- μ-abstract elementary classes (Boney, Grossberg, Lieberman, Rosický, Vasey)

where compactness is replaced by λ-accessibility.

These are powerful tools to investigate infinitary extensions of first-order logic and non-elementary classes of mathematical structures in a syntax-free way.

Categorical logic

Various approaches to categorical model theory, including:

- accessible categories (Adámek, Lair, Makkai, Paré, Rosický,...)
- μ-abstract elementary classes (Boney, Grossberg, Lieberman, Rosický, Vasey)

where compactness is replaced by λ-accessibility.

These are powerful tools to investigate infinitary extensions of first-order logic and non-elementary classes of mathematical structures in a syntax-free way.

We adopt a different perspective: motivated by the needs of finite model theory and descriptive complexity, we are interested in capturing fine structure "down below", typically in fragments of first-order logic.

Desideratum: A structure theory of logical resources in (finite) model theory

This suggests an orthogonal direction: drop compactness altogether.

Life without compactness

Finite model theory $=$ Model theory $-$ Compactness

A proof that does not use compactness (ultraproducts, saturated extensions, etc.)

- is more likely to admit a relativisation to finite models
- typically carries quantitative information about the complexity of the objects
- often results in stronger conclusions (both for finite and infinite models).

Life without compactness

Finite model theory $=$ Model theory $-$ Compactness

A proof that does not use compactness (ultraproducts, saturated extensions, etc.)

- is more likely to admit a relativisation to finite models
- typically carries quantitative information about the complexity of the objects
- often results in stronger conclusions (both for finite and infinite models).

In the absence of compactness, combinatorial and game-theoretic arguments are key tools.

Challenge: Develop an “axiomatic model theory”, based on structural methods, which is resource-sensitive and well adapted for finite and algorithmic model theory.

Three examples (I)

Homomorphism preservation theorems relate the syntactic shape of sentences with the semantic property of being preserved under a certain class of homomorphisms.

Three examples (I)

Homomorphism preservation theorems relate the syntactic shape of sentences with the semantic property of being preserved under a certain class of homomorphisms.

Homomorphism Preservation Theorem (Łoś, Lyndon, Tarski, 1950s)

A first-order sentence is preserved under homomorphisms if, and only if, it is equivalent to an existential positive one.

Using $\wedge, \vee, \exists$ but not $\neg, \forall$

Three examples (I)

Homomorphism preservation theorems relate the syntactic shape of sentences with the semantic property of being preserved under a certain class of homomorphisms.

Homomorphism Preservation Theorem (Łoś, Lyndon, Tarski, 1950s)

A first-order sentence is preserved under homomorphisms if, and only if, it is equivalent to an existential positive one.

Equirank HPT (Rossman, 2005)

A first-order sentence of quantifier rank $\leq k$ is preserved under homomorphisms if, and only if, it is equivalent to an existential positive sentence of quantifier rank $\leq k$.

Maximum nesting of quantifiers

Three examples (I)

Homomorphism preservation theorems relate the syntactic shape of sentences with the semantic property of being preserved under a certain class of homomorphisms.

Homomorphism Preservation Theorem (Łoś, Lyndon, Tarski, 1950s)

A first-order sentence is preserved under homomorphisms if, and only if, it is equivalent to an existential positive one.

Equirank HPT (Rossman, 2005)

A first-order sentence of quantifier rank $\leq k$ is preserved under homomorphisms if, and only if, it is equivalent to an existential positive sentence of quantifier rank $\leq k$.

Finite HPT (Rossman, 2005)

A first-order sentence is preserved under homomorphisms between finite structures if, and only if, it is equivalent over finite structures to an existential positive one.

Three examples (II)

Modal logic: $p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \neg\varphi \mid \Box\varphi \mid \Diamond\varphi$

$$\mathrm{ML} \xrightarrow{\text{standard translation}} \mathrm{FO}[x]$$

$$[\![p]\!]_x := P(x)$$
$$[\![\Box\varphi]\!]_x := \forall y.\, R(x,y) \rightarrow [\![\varphi]\!]_y$$
$$[\![\Diamond\varphi]\!]_x := \exists y.\, R(x,y) \wedge [\![\varphi]\!]_y$$

Three examples (II)

Modal logic: $p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \neg\varphi \mid \Box\varphi \mid \Diamond\varphi$

$$\mathrm{ML} \xrightarrow{\text{standard translation}} \mathrm{FO}[x]$$

$$[\![p]\!]_x := P(x)$$
$$[\![\Box\varphi]\!]_x := \forall y.\, R(x,y) \rightarrow [\![\varphi]\!]_y$$
$$[\![\Diamond\varphi]\!]_x := \exists y.\, R(x,y) \wedge [\![\varphi]\!]_y$$

Van Benthem's theorem characterises the image of the standard translation:

Theorem (van Benthem, 1976)

A first-order formula $\varphi(x)$ is invariant under bisimulations between Kripke models if, and only if, it is equivalent to (the standard translation of) a modal formula.

Three examples (II)

Modal logic: $p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \neg\varphi \mid \Box\varphi \mid \Diamond\varphi$

$$\mathrm{ML} \xrightarrow{\text{standard translation}} \mathrm{FO}[x]$$

$$[\![p]\!]_x := P(x)$$
$$[\![\Box\varphi]\!]_x := \forall y.\, R(x,y) \to [\![\varphi]\!]_y$$
$$[\![\Diamond\varphi]\!]_x := \exists y.\, R(x,y) \wedge [\![\varphi]\!]_y$$

Van Benthem's theorem characterises the image of the standard translation:

Theorem (van Benthem, 1976)

A first-order formula $\varphi(x)$ is invariant under bisimulations between Kripke models if, and only if, it is equivalent to (the standard translation of) a modal formula.

Theorem (Rosen, 1997)

A first-order formula $\varphi(x)$ of quantifier rank $\leq k$ is invariant under bisimulations between Kripke models if, and only if, it is equivalent to a modal formula of modal depth $\leq 2^k$.

Maximum nesting of modalities $\Box, \Diamond$

Three examples (II)

Modal logic: $p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \neg\varphi \mid \Box\varphi \mid \Diamond\varphi$

$$\mathrm{ML} \xrightarrow{\text{standard translation}} \mathrm{FO}[x]$$

$$[\![p]\!]_x := P(x)$$
$$[\![\Box\varphi]\!]_x := \forall y.\, R(x,y) \rightarrow [\![\varphi]\!]_y$$
$$[\![\Diamond\varphi]\!]_x := \exists y.\, R(x,y) \wedge [\![\varphi]\!]_y$$

Van Benthem's theorem characterises the image of the standard translation:

Theorem (van Benthem, 1976)

A first-order formula $\varphi(x)$ is invariant under bisimulations between Kripke models if, and only if, it is equivalent to (the standard translation of) a modal formula.

Theorem (Rosen, 1997)

A first-order formula $\varphi(x)$ of quantifier rank $\leq k$ is invariant under bisimulations between Kripke models if, and only if, it is equivalent to a modal formula of modal depth $\leq 2^k$.

The same result holds relative to finite Kripke models.

Three examples (III)

Homomorphism counting results in finite model theory: Two finite relational structures (e.g., two graphs) satisfy the same properties—expressible in a given logic—exactly when they are indistinguishable in terms of homomorphism counts.

Three examples (III)

Homomorphism counting results in finite model theory: Two finite relational structures (e.g., two graphs) satisfy the same properties—expressible in a given logic—exactly when they are indistinguishable in terms of homomorphism counts.

Gr	:	category of graphs and graph homomorphisms
$\mathrm{FO}^n(\#)$	:	n-variable first-order logic with counting quantifiers

For each $i \in \mathbb{N}$, add $\exists_{\geq i}$

Three examples (III)

Homomorphism counting results in finite model theory: Two finite relational structures (e.g., two graphs) satisfy the same properties—expressible in a given logic—exactly when they are indistinguishable in terms of homomorphism counts.

Gr	:	category of graphs and graph homomorphisms
$\mathrm{FO}^n(\#)$	:	*n*-variable first-order logic with counting quantifiers

Theorem (Dvořák, 2010)

The following statements are equivalent for all finite graphs G and H:

1. $G \equiv^{\mathrm{FO}^n(\#)} H$
2. $\mathbf{Gr}(F, G) \cong \mathbf{Gr}(F, H)$ for all finite graphs F of tree-width $< n$.

Prototypical hom-counting result: Lovász (1967), generalised in Pultr (1973) and Isbell (1991).

1. Logic, categories, and resources

2. Games: unravelling and covering

3. Counting, definability, and beyond

Games: the case of modal logic

In (finite) model theory, one is typically not interested in objects up to isomorphism, but only up to definable properties.

Games are a useful tool to establish whether two structures are equivalent with respect to a certain logic fragment; they are also well-suited for a finer analysis of logical resources.

Games: the case of modal logic

In (finite) model theory, one is typically not interested in objects up to isomorphism, but only up to definable properties.

Games are a useful tool to establish whether two structures are equivalent with respect to a certain logic fragment; they are also well-suited for a finer analysis of logical resources.

Bisimulation game for modal logic

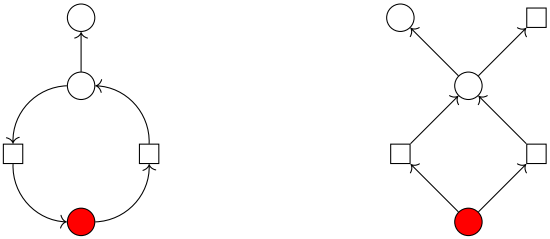
Played by Spoiler (S) and Duplicator (D) on two pointed Kripke models (A, a) and (B, b).

Initial position: $(a_0, b_0) := (a, b)$. At round i, with current position (a_i, b_i), S picks one of the models, e.g. A, and $a_{i+1} \in A$ such that $a_i \, R^A \, a_{i+1}$. D responds with $b_{i+1} \in B$ such that $b_i \, R^B \, b_{i+1}$.

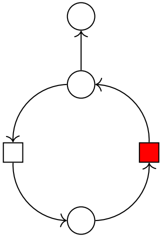
If D has no such response available, they lose. D wins after k rounds if a_i and b_i satisfy the same unary predicates, for all i with $0 \leq i \leq k$.

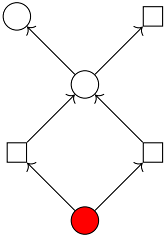
Theorem (Hennessy–Milner, 1980)

D has a winning strategy in the k-round bisimulation game if, and only if, (A, a) and (B, b) satisfy the same modal formulas of depth $\leq k$.

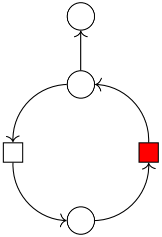


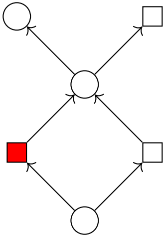
Initial position





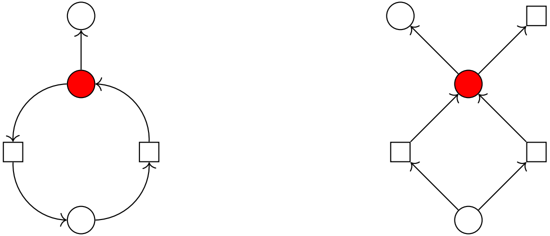
Round 1



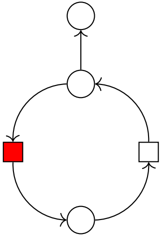


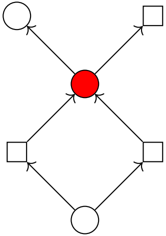
Round 1

Round 2

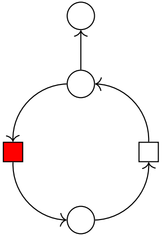


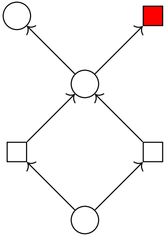
Round 2





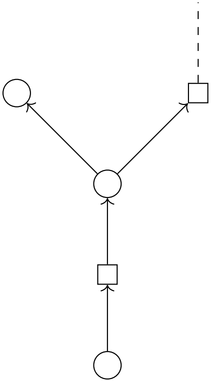
Round 3

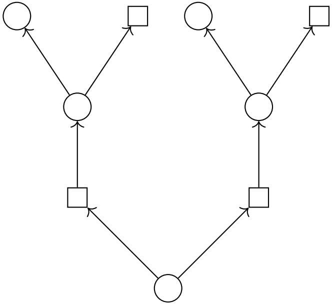


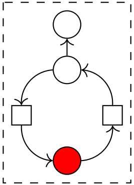


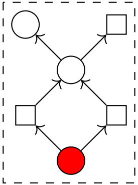
Round 3

Tree unravelling









Tree unravelling

The tree unravelling $R(A, a)$ of a Kripke model (A, a) is again a Kripke model, with distinguished element (a)—the one-element sequence. Moreover, it satisfies:

$(\star)$ $\forall x \in R(A, a)$, there is a unique path from the distinguished element of $R(A, a)$ to x.

Tree unravelling

The tree unravelling $R(A, a)$ of a Kripke model (A, a) is again a Kripke model, with distinguished element (a)—the one-element sequence. Moreover, it satisfies:

$(\star)$ $\forall x \in R(A, a)$, there is a unique path from the distinguished element of $R(A, a)$ to x.

Let $\mathbf{K}$ be the category of Kripke models, and $\mathbf{S}$ its full subcategory defined by the objects satisfying $(\star)$. The tree unravelling exhibits $\mathbf{S}$ as a coreflective subcategory of $\mathbf{K}$:

$$\mathbf{S} \underset{\hookrightarrow}{\overset{R}{\underset{\top}{\longleftarrow}}} \mathbf{K}$$

The objects of $\mathbf{S}$ are called synchronization trees, as they carry a “definable” tree order, namely the reflexive transitive closure of their Kripke relation.

Similarly, for each positive integer n, the tree unravelling to depth n defines a coreflection $R_n \colon \mathbf{K} \to \mathbf{S}_n$ of $\mathbf{K}$ onto the full subcategory $\mathbf{S}_n$ of synchronization trees of height at most n.

From unravellings to coverings

The usefulness of unravelling in modal logic has been long recognised.

However, only recently it has emerged that various notions of games can be encoded by means of **comonads** on the category of structures [Abramsky, Dawar & Wang, 2017], but the latter are not idempotent in general.

Given such a game comonad G, we can think of GA as a “covering” of the structure A.

From unravellings to coverings

The usefulness of unravelling in modal logic has been long recognised.

However, only recently it has emerged that various notions of games can be encoded by means of **comonads** on the category of structures [Abramsky, Dawar & Wang, 2017], but the latter are not idempotent in general.

Given such a game comonad G, we can think of GA as a "covering" of the structure A.

In concrete cases, the universe of GA is the set of plays in A, and the counit

$$\begin{array}{c} GA \\ \downarrow \\ A \end{array}$$

sends a play in A, i.e. a finite list $(a_1, \ldots, a_m)$, to its last element.

The Ehrenfeucht-Fraïssé comonad

Ehrenfeucht-Fraïssé (EF) game

Played by S and D on two structures A and B. Adjust the bisimulation game as follows:
No initial position, and S and D are not required to move along any "accessible relation".
D wins after k rounds if the relation $\{(a_i, b_i) \mid 1 \leq i \leq k\}$ is a partial isomorphism.

Theorem (Ehrenfeucht & Fraïssé, 1950s)

D has a winning strategy in the k-round Ehrenfeucht-Fraïssé game if, and only if, A and B satisfy the same sentences of quantifier rank $\leq k$.

The Ehrenfeucht-Fraïssé comonad

Ehrenfeucht-Fraïssé (EF) game

Played by S and D on two structures A and B. Adjust the bisimulation game as follows: No initial position, and S and D are not required to move along any "accessible relation". D wins after k rounds if the relation $\{(a_i, b_i) \mid 1 \leq i \leq k\}$ is a partial isomorphism.

Theorem (Ehrenfeucht & Fraïssé, 1950s)

D has a winning strategy in the k-round Ehrenfeucht-Fraïssé game if, and only if, A and B satisfy the same sentences of quantifier rank $\leq k$.

Let **R** be the category of (relational) structures and their homomorphisms. For each $k \geq 1$, we define an EF comonad $\mathbb{E}_k$ on **R**.

- The universe of $\mathbb{E}_k A$ is $\bigcup_{i=1}^{k} A^i$, the set of plays in A of length at most k.
  If R is a (say, binary) relation, $R^{\mathbb{E}_k A}$ consists of the pairs of sequences (s, t) such that: s and t are comparable in the prefix order, and $(\texttt{last}(s), \texttt{last}(t)) \in R^A$.
- The counit is $\texttt{last}_A \colon \mathbb{E}_k A \to A$, $[a_1, \ldots, a_m] \mapsto a_m$.
- The comultiplication $\mathbb{E}_k A \to \mathbb{E}_k^2 A$ is $[a_1, \ldots, a_m] \mapsto [[a_1], [a_1, a_2], \ldots, [a_1, \ldots, a_m]]$.

The Ehrenfeucht-Fraïssé comonad

The **coalgebras** for the EF comonads capture the combinatorial parameter of tree-depth:

Theorem (Abramsky & Shah, 2018)

$A \in \mathbf{R}$ has tree-depth $\leq k$ just when it admits a coalgebra structure $\alpha\colon A \to \mathbb{E}_k A$.

Coalgebras for $\mathbb{E}_k$ can be described as objects of $\mathbf{R}$ equipped with a compatible forest order of height $\leq k$. Coalgebra morphisms are homomorphisms that preserve the forest order.

The Ehrenfeucht-Fraïssé comonad

The **coalgebras** for the EF comonads capture the combinatorial parameter of tree-depth:

Theorem (Abramsky & Shah, 2018)

$A \in \mathbf{R}$ has tree-depth $\leq k$ just when it admits a coalgebra structure $\alpha\colon A \to \mathbb{E}_k A$.

Coalgebras for $\mathbb{E}_k$ can be described as objects of $\mathbf{R}$ equipped with a compatible forest order of height $\leq k$. Coalgebra morphisms are homomorphisms that preserve the forest order.

Preservation of logic fragments. Denote by FO_k first-order logic with quantifier rank $\leq k$. In the Kleisli category of $\mathbb{E}_k$,

- the homomorphism preorder captures preservation of existential positive FO_k-sentences
- the isomorphism relation captures equivalence in $\mathrm{FO}_k(\#)$

Other fragments of FO_k can be captured in the Eilenberg–Moore category of $\mathbb{E}_k$. E.g., equivalence in FO_k corresponds to the existence of a span of open maps.

Categories from games

The ideas just outlined are not specific to EF or bisimulation games and have been extended to a number of other model comparison games, yielding:

- pebbling comonads
- MSO comonads
- guarded quantifier comonads
- generalized quantifier comonads
- hybrid comonads
- ...

Categories from games

The ideas just outlined are not specific to EF or bisimulation games and have been extended to a number of other model comparison games, yielding:

- pebbling comonads
- MSO comonads
- guarded quantifier comonads
- generalized quantifier comonads
- hybrid comonads
- ...

In each case, we have tight connections, via the categories of coalgebras, with combinatorial invariants of structures and preservation of corresponding logic fragments.

Starting from concrete notions of games, we build a (resource-indexed) family of comonads and consider the associated categories of coalgebras.

Can we recognise the comonads arising from games, and their categories of coalgebras? We can attempt to isolate the fundamental properties of these categories; this leads to an **axiomatic perspective** on game comonads.

Games from categories

Let $\mathbf{C}$ be a (well-powered) category equipped with a proper factorisation system $(\mathcal{Q}, \mathcal{M})$.

Games from categories

Let **C** be a (well-powered) category equipped with a proper factorisation system $(\mathcal{Q}, \mathcal{M})$.

Definition

1. An object P of **C** is a path provided its poset $\mathbb{S}\,P$ of $\mathcal{M}$-subobjects is a finite chain.
2. A path embedding in **C** is an $\mathcal{M}$-morphism $P \rightarrowtail X$ whose domain is a path.

For any $X \in$ **C**, the sub-poset $\mathbb{P}\,X$ of $\mathbb{S}\,X$ consisting of the path embeddings is a tree.
Further, $\mathbb{P}\,X$ is non-empty if the factorisation system is stable and **C** has an initial object.

Games from categories

Let $\mathbf{C}$ be a (well-powered) category equipped with a proper factorisation system $(\mathcal{Q}, \mathcal{M})$.

Definition

1. An object P of $\mathbf{C}$ is a path provided its poset $\mathbb{S}\,P$ of $\mathcal{M}$-subobjects is a finite chain.
2. A path embedding in $\mathbf{C}$ is an $\mathcal{M}$-morphism $P \rightarrowtail X$ whose domain is a path.

For any $X \in \mathbf{C}$, the sub-poset $\mathbb{P}\,X$ of $\mathbb{S}\,X$ consisting of the path embeddings is a tree. Further, $\mathbb{P}\,X$ is non-empty if the factorisation system is stable and $\mathbf{C}$ has an initial object.

This minimal amount of structure on $\mathbf{C}$ allows us to define various abstract notions of games between objects X, Y by playing on the associated trees $\mathbb{P}\,X, \mathbb{P}\,Y$.

Back-and-forth game in C. Played by S and D on objects X and Y. This is essentially the bisimulation game played on the trees $\mathbb{P}\,X$ and $\mathbb{P}\,Y$ with initial position given by the roots. The accessibility relation is the immediate-successor relation, and at each round D must ensure that the selected path embeddings have isomorphic domains.

Games from categories

The ensuing notion of back-and-forth equivalence on objects of **C** can be transferred to other categories **E** via adjunctions:

$$\mathbf{C} \underset{L}{\overset{R}{\underset{\top}{\leftrightarrows}}} \mathbf{E} \tag{$\heartsuit$}$$

For all $a, b \in \mathbf{E}$, define $a \leftrightarrow^{R} b$ just when Ra and Rb are back-and-forth equivalent in **C**.

Games from categories

The ensuing notion of back-and-forth equivalence on objects of **C** can be transferred to other categories **E** via adjunctions:

$$\mathbf{C} \underset{L}{\overset{R}{\rightleftarrows}} \mathbf{E} \qquad (L \dashv R) \tag{$\heartsuit$}$$

For all $a, b \in \mathbf{E}$, define $a \leftrightarrow^R b$ just when Ra and Rb are back-and-forth equivalent in **C**.

All concrete examples of game comonads fit in this framework: $\mathbf{E} = \mathbf{R}$ is the category of relational structures, and **C** is the category of coalgebras for the comonad. The abstract game in **C** coincides with the corresponding concrete game (e.g., EF or bisimulation games).

Games from categories

The ensuing notion of back-and-forth equivalence on objects of $\mathbf{C}$ can be transferred to other categories $\mathbf{E}$ via adjunctions:

$$\mathbf{C} \underset{L}{\overset{R}{\rightleftarrows}} \mathbf{E}, \quad L \dashv R \qquad (\heartsuit)$$

For all $a, b \in \mathbf{E}$, define $a \leftrightarrow^R b$ just when Ra and Rb are back-and-forth equivalent in $\mathbf{C}$.

All concrete examples of game comonads fit in this framework: $\mathbf{E} = \mathbf{R}$ is the category of relational structures, and $\mathbf{C}$ is the category of coalgebras for the comonad. The abstract game in $\mathbf{C}$ coincides with the corresponding concrete game (e.g., EF or bisimulation games).

Theorem (LR & Riba)

Suppose that $\mathbf{C}$ and $\mathbf{E}$ are lfp, paths in $\mathbf{C}$ are fp, and the adjunction ($\heartsuit$) is finitely accessible. If L detects path embeddings, then $a \leftrightarrow^R b$ whenever a and b satisfy the same $\mathcal{L}_{\infty,\omega}$-sentences.

The proof combines Gabriel–Ulmer duality with Hodges' word-constructions.

Arboreal categories

In the examples, the categories of coalgebras for game comonads satisfy additional properties. Most important of all, the full subcategory of paths $\mathbf{C}_p \hookrightarrow \mathbf{C}$ is dense.

These extra properties lead to the notion of arboreal category [Abramsky & LR, 2021].

Arboreal categories

In the examples, the categories of coalgebras for game comonads satisfy additional properties. Most important of all, the full subcategory of paths $\mathbf{C}_p \hookrightarrow \mathbf{C}$ is dense.

These extra properties lead to the notion of arboreal category [Abramsky & LR, 2021].

In arboreal categories, back-and-forth equivalence coincides with bisimilarity in the sense of [Joyal, Nielsen & Winskel, 1993]. Let us say that a morphism $X \to Y$ in an arboreal category $\mathbf{C}$ is open if it has the right lifting property wrt morphisms between paths.

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ Q & \longrightarrow & Y \end{array}$$

(Tightly related to the concept of open maps in presheaf toposes [Joyal & Moerdijk, 1994])

Two objects of $\mathbf{C}$ are bisimilar if they are connected by a span of open morphisms.

Arboreal categories

In the examples, the categories of coalgebras for game comonads satisfy additional properties. Most important of all, the full subcategory of paths $\mathbf{C}_p \hookrightarrow \mathbf{C}$ is dense.

These extra properties lead to the notion of arboreal category [Abramsky & LR, 2021].

In arboreal categories, back-and-forth equivalence coincides with bisimilarity in the sense of [Joyal, Nielsen & Winskel, 1993]. Let us say that a morphism $X \to Y$ in an arboreal category $\mathbf{C}$ is open if it has the right lifting property wrt morphisms between paths.

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ Q & \longrightarrow & Y \end{array}$$

(Tightly related to the concept of open maps in presheaf toposes [Joyal & Moerdijk, 1994])

Two objects of $\mathbf{C}$ are bisimilar if they are connected by a span of open morphisms.

Equivalences wrt resource-indexed logic fragments are obtained by transferring bisimilarity (homomorphism preorder, isomorphism, ...) along an arboreal adjunction $\mathbf{C} \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathbf{E}$.

E.g., preservation of existential (resp., existential positive) fragments corresponds to constructing a compatible cocone of $\mathcal{M}$-morphisms (resp., of morphisms) in $\mathbf{C}$.

1. Logic, categories, and resources

2. Games: unravelling and covering

3. Counting, definability, and beyond

Homomorphism counting revisited

Theorem (Lovász, 1967)

Finite relational structures $A, B \in \mathbf{R}$ *are isomorphic iff* $\mathbf{R}(C, A) \cong \mathbf{R}(C, B)$ *for all finite* $C \in \mathbf{R}$.

Generalised to locally finite categories with appropriate factorisation systems in [Pultr, 1973] and [Isbell, 1991]. The categorical perspective, combined with game comonads, yields a uniform approach to homomorphism counting results [Dawar, Jakl & LR, 2021].

Homomorphism counting revisited

Theorem (Lovász, 1967)

Finite relational structures $A, B \in \mathbf{R}$ are isomorphic iff $\mathbf{R}(C, A) \cong \mathbf{R}(C, B)$ for all finite $C \in \mathbf{R}$.

Generalised to locally finite categories with appropriate factorisation systems in [Pultr, 1973] and [Isbell, 1991]. The categorical perspective, combined with game comonads, yields a uniform approach to homomorphism counting results [Dawar, Jakl & LR, 2021]. E.g.,

Theorem (Grohe, 2020)

The following statements are equivalent for all finite $A, B \in \mathbf{R}$ and all $k > 0$:

1. $A \equiv^{\mathrm{FO}_k(\#)} B$
2. $\mathbf{R}(C, A) \cong \mathbf{R}(C, B)$ *for all finite $C \in \mathbf{R}$ of tree-depth $\leq k$.*

follows by considering the EF comonads $\mathbb{E}_k$; similarly, the modal comonads yield a (new) homomorphism counting result for (graded) modal logic.

Homomorphism counting revisited

Theorem (Lovász, 1967)

Finite relational structures $A, B \in \mathbf{R}$ are isomorphic iff $\mathbf{R}(C, A) \cong \mathbf{R}(C, B)$ for all finite $C \in \mathbf{R}$.

Generalised to locally finite categories with appropriate factorisation systems in [Pultr, 1973] and [Isbell, 1991]. The categorical perspective, combined with game comonads, yields a uniform approach to homomorphism counting results [Dawar, Jakl & LR, 2021]. E.g.,

Theorem (Grohe, 2020)

The following statements are equivalent for all finite $A, B \in \mathbf{R}$ and all $k > 0$:

1. $A \equiv^{\mathrm{FO}_k(\#)} B$
2. $\mathbf{R}(C, A) \cong \mathbf{R}(C, B)$ *for all finite $C \in \mathbf{R}$ of tree-depth $\leq k$.*

follows by considering the EF comonads $\mathbb{E}_k$; similarly, the modal comonads yield a (new) homomorphism counting result for (graded) modal logic.

When (co)free coalgebras are infinite, a more general approach is available [LR, 2022], which applies e.g. to lfp categories and is related to [Fiore & Menni, 2005].

Homomorphism preservation theorems revisited

Using arboreal adjunctions, we get simple proofs of new homomorphism preservation theorems for modal and guarded logics [Abramsky & LR], along with relativisations to subclasses of structures, e.g. the finite ones. We also get an axiomatic proof of

Equirank HPT (Rossman, 2005)

A first-order sentence of quantifier rank $\leq k$ is preserved under homomorphisms if, and only if, it is equivalent to an existential positive sentence of quantifier rank $\leq k$.

Homomorphism preservation theorems revisited

Using arboreal adjunctions, we get simple proofs of new homomorphism preservation theorems for modal and guarded logics [Abramsky & LR], along with relativisations to subclasses of structures, e.g. the finite ones. We also get an axiomatic proof of

Equirank HPT (Rossman, 2005)

A first-order sentence of quantifier rank $\leq k$ is preserved under homomorphisms if, and only if, it is equivalent to an existential positive sentence of quantifier rank $\leq k$.

The key idea is that of upgrading: given $a, b \in \mathbf{E}$, construct extensions $a^*, b^* \in \mathbf{E}$ such that $R(a^*)$ and $R(b^*)$ are bisimilar whenever Ra and Rb are hom-equivalent.

$$\begin{array}{ccc} a^* & \leftrightarrow^R & b^* \\ \uparrow & & \uparrow \\ a & \rightleftarrows^R & b \end{array}$$

Assuming further properties of the arboreal adjunction, the extensions a^*, b^* can be constructed via a small object argument.

Outlook

What I haven't mentioned:

- More work on the concrete level: composition methods in finite model theory (Jakl, Marsden, Shah); other game comonads (Abramsky, Ó Conghaile, Dawar, Marsden, Montacute, Shah)
- Combinatorial parameters and density comonads (Abramsky, Jakl, Paine)
- ...

Outlook

What I haven't mentioned:

- More work on the concrete level: composition methods in finite model theory (Jakl, Marsden, Shah); other game comonads (Abramsky, Ó Conghaile, Dawar, Marsden, Montacute, Shah)
- Combinatorial parameters and density comonads (Abramsky, Jakl, Paine)
- ...

Future directions:

- New examples: applications of arboreal adjunctions beyond (finite) model theory
- Translations between logic fragments: the category of arboreal categories
- Homotopy theory of logical resources

Upgrading arguments, pervasive in (finite) model theory, as instances of fibrant replacement for an appropriate model category structure on the presheaf category $\widehat{\mathbf{C}_p}$.

Resource-sensitive model theory

A categorical view

Luca Reggio

University College London

International Category Theory Conference

Louvain-la-Neuve, July 6, 2023

References I

Homomorphism preservation theorems

B. Rossman (2005)
Existential positive types and preservation under homomorphisms
Proceedings of LiCS 2005.

Modal logic

J. van Benthem (1976)
Modal correspondence theory
PhD thesis, University of Amsterdam.

M. Hennessy & R. Milner (1976)
On observing nondeterminism and concurrency
Automata, Languages and Programming.

E. Rosen (1997)
Modal logic over finite structures
Journal of Logic, Language and Information.

Homomorphism counting & Factorization monads

A. Dawar, T. Jakl & L. Reggio (2021)
Lovász-type theorems and game comonads
Proceedings of LiCS 2021.

Z. Dvořák (2010)
On recognizing graphs by numbers of homomorphisms
Journal of Graph Theory.

M. Fiore & M. Menni (2005)
Reflective Kleisli subcategories of the category of Eilenberg–Moore algebras for factorization monads
Theory and Applications of Categories.

M. Grohe (2020)
Counting bounded tree depth homomorphisms
Proceedings of LiCS 2020.

References II

J. Isbell (1991)
Some inequalities in hom sets
Journal of Pure and Applied Algebra.

L. Lovász (1967)
Operations with structures
Acta Mathematica Academiae Scientiarum Hungaricae.

A. Pultr (1973)
Isomorphism types of objects in categories determined by numbers of morphisms
Acta Scientiarum Mathematicarum.

L. Reggio (2022)
Polyadic sets and homomorphism counting
Advances in Mathematics.

Game comonads

S. Abramsky, A. Dawar & P. Wang (2017)
The pebbling comonad in finite model theory
Proceedings of LiCS 2017.

S. Abramsky & N. Shah (2018)
Relating structure and power: Comonadic semantics for computational resources
Proceedings of CSL 2018 (Extended version: Journal of Logic and Computation, 2021).

Arboreal categories

S. Abramsky & L. Reggio (2021)
Arboreal categories and resources
Proceedings of ICALP 2021 (Extended version to appear in Logical Methods in Computer Science).

References III

S. Abramsky & L. Reggio

Arboreal categories and homomorphism preservation theorem

Preprint available on arXiv.

L. Reggio & C. Riba

Finitely accessible arboreal adjunctions and Hintikka formulae

Preprint available on arXiv.

Open maps

A. Joyal & I. Moerdijk (1994)

A completeness theorem for open maps

Annals of Pure and Applied Logic.

A. Joyal, M. Nielsen & G. Winskel (1993)

Bisimulation and open maps

Proceedings of LiCS 1993.