Double Fibrations

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Pseudo Category Objects

Definition (MF, 2006)

For a 2-category K, a *pseudo category object* $\mathbb{C}$ in K consists of a diagram:

$$C_1 \times_{C_0} C_1 \xrightarrow{\ \otimes\ } C_1 \underset{\text{tgt}}{\overset{\text{src}}{\rightrightarrows}} C_0 \qquad (y \colon C_0 \to C_1)$$

with invertible 2-cells

$$\begin{array}{ccc} C_1 \times_{C_0} \times C_1 \times_{C_0} C_1 & \xrightarrow{\ 1\times\otimes\ } & C_1 \times_{C_0} C_1 \\ {\scriptstyle \otimes\times 1}\downarrow & \overset{\mathfrak{a}}{\cong} & \downarrow{\scriptstyle \otimes} \\ C_1 \times_{C_0} C_1 & \xrightarrow[\ \otimes\ ]{} & C_1 \end{array}$$

$$\begin{array}{ccccc} C_1 & \xrightarrow{\ \langle y,1\rangle\ } & C_1 \times_{C_0} C_1 & \xleftarrow{\ \langle 1,y\rangle\ } & C_1 \\ & \overset{\mathfrak{l}}{\cong} & \downarrow{\scriptstyle \otimes} & \overset{\mathfrak{r}}{\cong} & \\ & = & C_1 & = & \end{array}$$

Double Categories

- A **double category** (Ehresmann,'63) and (G-P,'99) $\mathbb{C}$ is a pseudo category object in **Cat**,

$$\mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \xrightarrow{\otimes} \mathsf{C}_1 \underset{\mathsf{tgt}}{\overset{\mathsf{src}}{\rightrightarrows}} \!\!\xleftarrow{y}\, \mathsf{C}_0 \, .$$

- It has then
  - **objects** (objects of C_0),
  - **(vertical) arrows** (arrows of C_0), denoted $\mathrm{dom}(f) \xrightarrow{f} \mathrm{cod}(f)$,
  - **(horizontal) pro-arrows** (objects of C_1), denoted $\mathrm{src}(m) \overset{m}{\nrightarrow} \mathrm{tgt}(m)$,
  - **double cells** (arrows of C_1), denoted

$$\begin{array}{ccc} A & \overset{m}{\nrightarrow} & B \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ C & \underset{n}{\nrightarrow} & D \end{array}$$

where $\mathrm{dom}(\alpha) = m$, $\mathrm{cod}(\alpha) = n$, $\mathrm{src}(\alpha) = f$, and $\mathrm{tgt}(\alpha) = g$.

Example: Monoidal Categories as Double Categories

Just like a category with one object is a monoid, a double category with $C_0 = 1$ is

$$\mathsf{C}_1 \times \mathsf{C}_1 \xrightarrow{\otimes} \mathsf{C}_1 \underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{y}}} 1$$

a monoidal category.

Examples

- For any 2-category K, the double category $\mathbb{Q}(\mathsf{K})$ with cells

$$\begin{array}{ccc} A & \overset{m}{\relbar\joinrel\mapstochar\joinrel\rightarrow} & B \\ {\scriptstyle f}\big\downarrow & \alpha & \big\downarrow{\scriptstyle g} \\ C & \underset{n}{\relbar\joinrel\mapstochar\joinrel\rightarrow} & D \end{array} \qquad \text{for each } \alpha\colon gm \Rightarrow nf \text{ in } \mathcal{C}.$$

- For a subcategory $\Sigma \subset \mathsf{K}$, $\mathbb{Q}^{\Sigma}(\mathcal{C}) \subseteq \mathbb{Q}(\mathcal{C})$ has cells:

$$\begin{array}{ccc} A & \overset{m}{\relbar\joinrel\mapstochar\joinrel\rightarrow} & B \\ {\scriptstyle f}\big\downarrow & \alpha & \big\downarrow{\scriptstyle g} \\ C & \underset{n}{\relbar\joinrel\mapstochar\joinrel\rightarrow} & D \end{array} \qquad \text{for each } \alpha\colon gm \Rightarrow nf \text{ in } \mathcal{C};\ f, g \in \Sigma.$$

- $\mathbb{R}\mathrm{el}$: sets with functions as arrows and relations as pro-arrows;
- $\mathbb{P}\mathrm{rof}$: categories with functors and profunctors;
- $\mathbb{S}\mathrm{pan}(\mathcal{A})$: objects of $\mathcal{A}$ with arrows of $\mathcal{A}$ and spans in $\mathcal{A}$.

(Pseudo) Double Functors and Transformations

There is a 2-category **DblCat** of pseudo (double=internal) categories, pseudo (double=internal) functors to be defined on the next slide, and (vertical=internal) transformations.

Pseudo Double Functors as Internal Pseudo Functors

A **pseudo functor** $F\colon \mathbb{C} \to \mathbb{D}$ consists of two arrows $F_0\colon C_0 \to D_0$ and $F_1\colon C_1 \to D_1$ and comparison invertible 2-cells (+ axioms)

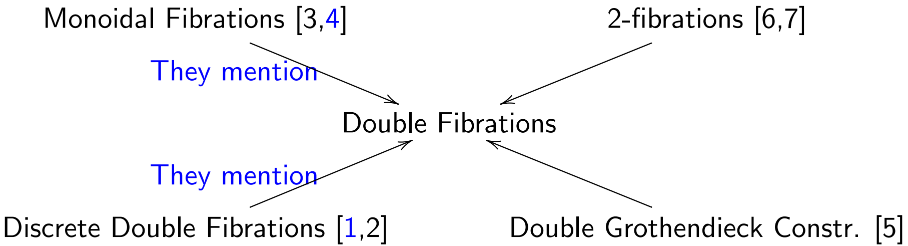
$$
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \xrightarrow{\otimes} & C_1 \\
{\scriptstyle F_1 \times_{F_0} F_1}\downarrow & \overset{\phi}{\Rightarrow} & \downarrow{\scriptstyle F_1} \\
D_1 \times_{D_0} D_1 & \xrightarrow[\otimes]{} & D_1
\end{array}
\qquad
\begin{array}{ccc}
C_0 & \xrightarrow{y} & C_1 \\
{\scriptstyle F_0}\downarrow & \overset{\iota}{\Rightarrow} & \downarrow{\scriptstyle F_1} \\
D_0 & \xrightarrow[y]{} & D_1
\end{array}
\qquad
\begin{array}{ccc}
C_1 & \overset{\mathrm{src}}{\underset{\mathrm{tgt}}{\rightrightarrows}} & C_0 \\
{\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_0} \\
C_1 & \overset{\mathrm{src}}{\underset{\mathrm{tgt}}{\rightrightarrows}} & C_0
\end{array}
$$

Note that the interaction with src and tgt is required to be **stricter** than that with y and $\otimes$.

The Topic for Today

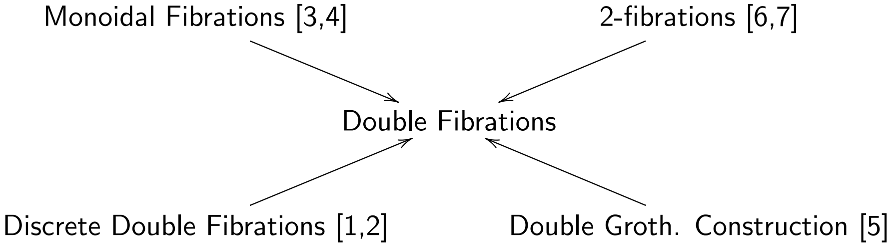
Define Double Fibrations

Goal: Generalize Various Known Concepts



[1] Discrete Double Fibrations, Lambert (2021).
[2] Yoneda Theory for Double Categories, Paré (2011).
[3] Framed Bicategories and Monoidal Fibrations, Shulman (2008).
[4] Monoidal Grothendieck Construction, Moeller and Vasilakopoulou (2020).
[5] Double Categories of Open Dynamical Systems, Myers (2021).
[6] Some Properties of FIB as a Fibred 2-Category, Hermida (1999).
[7] Fibred 2-Categories and Bicategories, Buckley (2014).

Known Properties



In each case, there is:

- A notion of fibration (between 2-categories, monoidal categories, double categories,...)
- A notion of *indexing* (indexed 2-category, indexed span, indexed double category, monoidal indexed category...)
- a Grothendieck/Elements Construction providing an equivalence of categories {Indexed} $\rightarrow$ {Fibrations}

Double Fibrations

Rather than taking double functors and determining when they are fibrations, we will start with fibrations and make them 'double' by considering pseudo category objects in a suitable 2-category of fibrations:

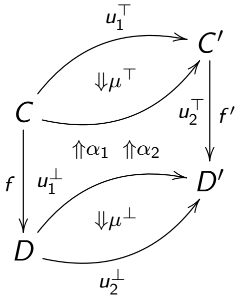
$$\mathbf{Cat}(\mathbf{Fib})$$

Arrow Categories

For any 2-category K (with 2-pullbacks), there is a 2-category $\mathbf{Arr}^p(\mathsf{K})$:

- objects are the arrows in K,
- arrows are the squares filled by an *invertible* 2-cell α,
- 2-cells are given by a pair of 2-cells satisfying an equation:

$$(C, D, f) \underset{(u_2^\top, u_2^\perp, \alpha_2)}{\overset{(u_1^\top, u_1^\perp, \alpha_1)}{\underset{\Downarrow(\mu^\top, \mu_\perp)}{\rightrightarrows}}} (C', D', f')$$



pasting α_1 with $\mu^\top$ and pasting $\mu^\perp$ with α_2 are required to be equal.

In $\mathbf{Arr}^s(\mathsf{K}) \subseteq \mathbf{Arr}^p(\mathsf{K})$ we require the invertible 2-cells α to be identities.

Internal pseudo categories in **Arr**^{p}(K) and pseudo functors

A pseudo category object $\mathbb{F} = (F_1, F_0, \mathsf{src}, \mathsf{tgt}, \iota, \phi, \mathfrak{a}, \mathfrak{r}, \mathfrak{l})$ in **Arr**^{p}(K) such that

$$\mathsf{src} : F_1 \to F_0 \leftarrow F_1 : \mathsf{tgt} \text{ are in } \mathbf{Arr}^s(\mathsf{K})$$

$$
\begin{array}{ccccc}
C_1 \times_{C_0} C_1 & \xrightarrow{\otimes^\top} & C_1 & \begin{array}{c}\xrightarrow{\mathsf{src}^\top} \\ \xleftarrow{y^\top} \\ \xrightarrow[\mathsf{tgt}^\top]{} \end{array} & C_0 \\
{\scriptstyle F_1 \times_{F_0} F_1}\big\downarrow & \overset{\phi}{\Rightarrow} & \big\downarrow{\scriptstyle F_1} & \overset{\iota}{\Leftarrow} & \big\downarrow{\scriptstyle F_0} \\
D_1 \times_{D_0} D_1 & \xrightarrow{\otimes^\perp} & D_1 & \begin{array}{c}\xrightarrow{\mathsf{src}^\perp} \\ \xleftarrow{y^\perp} \\ \xrightarrow[\mathsf{tgt}^\perp]{} \end{array} & D_0
\end{array}
\tag{4.1}
$$

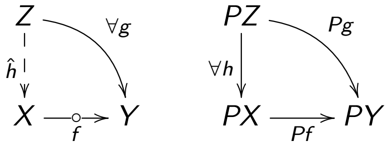
is the same as a pseudo functor F between two pseudo categories (internal in K):

This restricts: $\mathbb{F}$ pseudo category in **Arr**^{s}(K) $\iff$ F strict functor.

Fibrations

Let $P : \mathscr{E} \longrightarrow \mathscr{B}$ be a functor between categories.

- An arrow f of $\mathscr{E}$ is Cartesian if:



- P is a fibration when: $B^* \xrightarrow{u^*E} E \;\leftsquigarrow\; B \xrightarrow{u} PE$

  (Cartesian lift)

- A cleavage is a choice of a Cartesian lift for each arrow of $\mathscr{B}$.
  A cloven fibration is a fibration and a chosen cleavage.

-Any cloven fibration gives rise to an Indexed category $F : \mathscr{B}^{op} \to \mathbf{Cat}$.
-Any indexed category gives rise to a cloven fibration by its Grothendieck construction/category of elements.

Morphisms of Fibrations

Given two cloven fibrations $P : \mathscr{E} \longrightarrow \mathscr{B}$ and $P' : \mathscr{E}' \longrightarrow \mathscr{B}'$,

- A morphism f between them is:

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{f^{\top}} & \mathscr{E}' \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P'} \\ \mathscr{B} & \xrightarrow[f^{\perp}]{} & \mathscr{B}' \end{array}$$

  where $f^{\top}$ preserves the Cartesian arrows.
- f is said to be cleavage-preserving when $f^{\top}$ maps the arrows of the cleavage of P to arrows in the cleavage of P'.
- This defines 2-categories $\mathbf{cFib} \subseteq \mathbf{Fib} \subseteq \mathbf{Arr}^s(\mathbf{Cat})$ (full on 2-cells, with objects the cloven fibrations).

The classical equivalence $\mathbf{Fib} \simeq \mathbf{ICat}$ (with pseudo transformations) restricts to $\mathrm{c}\mathbf{Fib} \simeq \mathbf{ICat}_t$ (with strict natural transformations.)

What is a Double Fibration?

First idea – worked for discrete double fibrations (Lambert, 2021):

A double category is a pseudo category in **Cat** $\rightsquigarrow$
A double fibration is a pseudo category in **Fib**.

Since **Fib** $\subseteq$ **Arr**^{s}(**Cat**), the equation (4.1) in this case looks like this:

$$\begin{array}{ccccc}
\mathscr{E}_1 \times_{\mathscr{E}_0} \mathscr{E}_1 & \xrightarrow{\otimes^{\top}} & \mathscr{E}_1 & \begin{array}{c} \xrightarrow{\text{src}^{\top}} \\ \xleftarrow{y^{\top}} \\ \xrightarrow[\text{tgt}^{\top}]{} \end{array} & \mathscr{E}_0 \\
\Big\downarrow {\scriptstyle P_1 \times_{P_0} P_1} & & \Big\downarrow {\scriptstyle P_1} & & \Big\downarrow {\scriptstyle P_0} \\
\mathscr{B}_1 \times_{\mathscr{B}_0} \mathscr{B}_1 & \xrightarrow{\otimes^{\perp}} & \mathscr{B}_1 & \begin{array}{c} \xrightarrow{\text{src}^{\perp}} \\ \xleftarrow{y^{\perp}} \\ \xrightarrow[\text{tgt}^{\perp}]{} \end{array} & \mathscr{B}_0
\end{array}$$

So a double fibration could be seen, as expected, as a (strict) double functor between double categories, with extra properties.

What is a Double Fibration?

First problem

Fib doesn't have all the 2-pullbacks on the previous slide!

Also, the *fibrational strictness* of src and tgt is the same as that of y and $\otimes$, which is not in line with the exponential example we saw before.

The solution

A double fibration is a pseudo category in **Fib** such that src and tgt are in c**Fib** (that is, they preserve the chosen cleavages).

This translates into:

Definition of a Double Fibration

A **double fibration** as defined on the previous slide is the same as a (strict) double functor $P : \mathbb{E} \to \mathbb{B}$ between (pseudo) double categories

$$\begin{array}{ccccc}
\mathscr{E}_1 \times_{\mathscr{E}_0} \mathscr{E}_1 & \xrightarrow{\otimes_{\mathbb{E}}} & \mathscr{E}_1 & \underset{\mathrm{tgt}_{\mathbb{E}}}{\overset{\mathrm{src}_{\mathbb{E}}}{\rightrightarrows}} \ \xleftarrow{y_{\mathbb{E}}} & \mathscr{E}_0 \\
\downarrow {\scriptstyle P_1 \times_{P_0} P_1} & & \downarrow {\scriptstyle P_1} & & \downarrow {\scriptstyle P_0} \\
\mathscr{B}_1 \times_{\mathscr{B}_0} \mathscr{B}_1 & \xrightarrow{\otimes_{\mathbb{B}}} & \mathscr{B}_1 & \underset{\mathrm{tgt}_{\mathbb{B}}}{\overset{\mathrm{src}_{\mathbb{B}}}{\rightrightarrows}} \ \xleftarrow{y_{\mathbb{B}}} & \mathscr{B}_0
\end{array}$$

such that

1. P_0 and P_1 are fibrations,
2. they admit a cleavage such that $\mathrm{src}_{\mathbb{E}}$ and $\mathrm{tgt}_{\mathbb{E}}$ are cleavage-preserving, and
3. $y_{\mathbb{E}}$ and $\otimes_{\mathbb{E}}$ are Cartesian-morphism preserving.

Some Examples

- When $\mathscr{E}_0 = \mathscr{B}_0 = 1$, we recover monoidal fibrations [3];
- For any 2-functor $P : \mathsf{E} \to \mathsf{B}$, we have that P is a 2-fibration [7] if and only if $\mathbb{Q}P : \mathbb{Q}\mathsf{E} \to \mathbb{Q}\mathsf{B}$ is a double fibration;
- When P_0 and P_1 are discrete fibrations, we recover discrete double fibrations [1];
- The double Grothendieck construction in Definition 5.3 of [5] is also a double fibration.

[1] Discrete Double Fibrations, Lambert (2021).
[3] Framed Bicategories and Monoidal Fibrations, Shulman (2008).
[5] Double Categories of Open Dynamical Systems, Myers (2021).
[7] Fibred 2-Categories and Bicategories, Buckley (2014).

More Examples

- The domain fibration: $\text{dom}\colon \mathbb{D}^2 \to \mathbb{D}$,

$$\begin{array}{ccc} \mathbb{D}^2_1 & \overset{\text{tgt}}{\underset{\text{src}}{\rightrightarrows}} & \mathbb{D}^2_0 \\ {\scriptstyle \text{dom}}\downarrow & & \downarrow{\scriptstyle \text{dom}} \\ \mathbb{D}_1 & \overset{\text{tgt}}{\underset{\text{src}}{\rightrightarrows}} & \mathbb{D}_0 \end{array}$$

- $Im\colon \mathbb{S}\text{pan} \to \mathbb{R}\text{el}$ is a double opfibration.
- There is a codomain fibration $\text{cod}\colon \mathbb{D}^2 \to \mathbb{D}$ if
  - $\mathbb{D}_1$ and $\mathbb{D}_0$ have chosen finite limits,
  - these limits are preserved on the nose by src and tgt
  - and up to iso by y and $\otimes$.

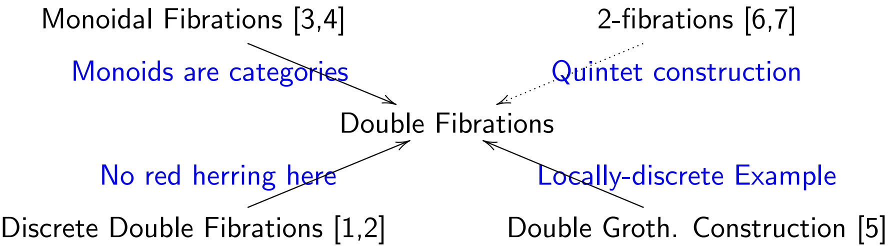
Double Fibrations are Internal Fibrations

The notion of *internal fibration* for a 2-category was given by Street in 1974. Let **DblCat** be the 2-category of pseudo double categories, pseudo functors and vertical/internal transformations.

Theorem [Cruttwell, Lambert, P., Szyld]

A *strict* double functor $P : \mathbb{E} \to \mathbb{B}$ is an internal fibration in **DblCat** if and only if it is a double fibration

Connection with other known constructions.



In each case, there is:

- A notion of fibration (between 2-categories, monoidal categories, double categories,...)
- an *indexed* notion (indexed 2-category, indexed span, indexed double category, monoidal indexed category...)
- a Grothendieck Construction (a.k.a. Elements Construction) providing an equivalence of categories $\{\text{Indexed}\} \to \{\text{Fibrations}\}$

Indexing double functors?

In order to obtain a representation theorem for double fibrations over a given double category, we need to decide on the codomain for the indexing functors.

Categories are Monoids in $\mathbb{S}\text{pan}$

- Categories: $C_1 \times_{C_0} C_1 \xrightarrow{\otimes} C_1 \begin{array}{c} \xrightarrow{\text{src}} \\ \xleftarrow{y} \\ \xrightarrow[\text{tgt}]{} \end{array} C_0$

- Monoids: $m \colon A \longrightarrow\!\!\!\!\!\!\!+\;\; A$,
$$\begin{array}{ccccc} A & \xrightarrow{m} & A & \xrightarrow{m} & A \\ \| & & \mu & & \| \\ A & & \xrightarrow[m]{} & & A \end{array} \quad , \quad \begin{array}{ccc} A & \xrightarrow{id_A} & A \\ \| & \iota & \| \\ A & \xrightarrow[m]{} & A \end{array}$$

- $\mathbb{S}\text{pan}$ has $A \leftarrow S \rightarrow B$ as horizontal arrows, and double cells

$$\begin{array}{ccc} A & \xrightarrow{S} & B \\ f \downarrow & v & \downarrow g \\ X & \xrightarrow[T]{} & Y \end{array} \quad = \quad \begin{array}{ccccc} A & \longleftarrow & S & \longrightarrow & B \\ f \downarrow & & \downarrow v & & \downarrow g \\ X & \longleftarrow & T & \longrightarrow & Y \end{array}$$

$m \colon A \longrightarrow\!\!\!\!\!\!\!+\;\; A$ corresponds to $\text{src} : C_0 \leftarrow C_1 \rightarrow C_0 : \text{tgt}$
μ corresponds to $\otimes$, ι corresponds to y.

Lift this to double categories...

In [4] the monoidal indexed categories are obtained by lifting $\mathbf{Fib} \simeq \mathbf{ICat}$ to $\mathbf{PsMon}(\mathbf{Fib}, \times) \simeq \mathbf{PsMon}(\mathbf{ICat}, \times)$.

We use the same strategy, but it requires some tweaking...

We use double 2-categories (i.e. pseudo categories internal to **2Cat**), and pseudo monoids in them. We then show:

Pseudo categories in $\mathsf{K} \cong$ Pseudo monoids in a double 2-category $\mathbb{S}\mathrm{pan}(\mathsf{K})$.

Furthermore, most importantly for us:

Pseudo categories in K with source and target in a family of arrows Σ are pseudo monoids in a double 2-category $\mathbb{S}\mathrm{pan}_\Sigma(\mathsf{K})$ (where the spans are formed by arrows in Σ).

The {Fibrations} $\overset{\simeq}{\leftarrow}$ {Indexed} Theorem

Let $\mathbf{I}\mathbb{S}\mathrm{pan}(\mathbf{Cat})$ be the category of contravariant lax pseudo double functors valued in the double 2-category $\mathbb{S}\mathrm{pan}(\mathbf{Cat})$.

The {Fibrations} $\overset{\simeq}{\leftarrow}$ {Indexed} Theorem

Let $\mathbf{I}\mathbb{S}\mathrm{pan}(\mathbf{Cat})$ be the category of contravariant lax pseudo double functors valued in the double 2-category $\mathbb{S}\mathrm{pan}(\mathbf{Cat})$.

Theorem [Cruttwell, Lambert, P., Szyld]

There is an equivalence of categories $\mathbf{DblFib} \simeq \mathbf{I}\mathbb{S}\mathrm{pan}(\mathbf{Cat})$

Idea for the proof: use pseudo monoids in double 2-categories.

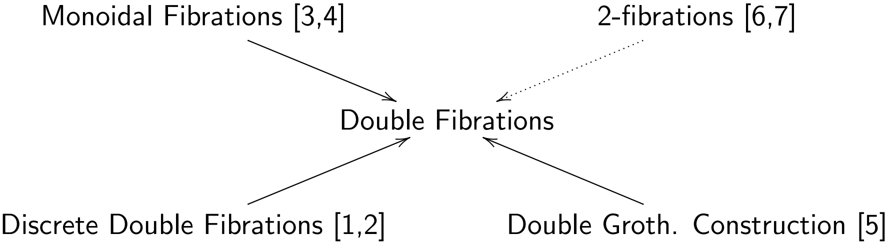
$$\mathbf{Fib} \simeq \mathbf{ICat} \text{ restricts to } \mathrm{c}\mathbf{Fib} \simeq \mathbf{ICat}_t, \text{ so } \mathbb{S}\mathrm{pan}_c(\mathbf{Fib}) \simeq \mathbb{S}\mathrm{pan}_t(\mathbf{ICat}).$$

Now we lift:

$$\mathbf{DblFib} := \mathbf{PsMon}(\mathbb{S}\mathrm{pan}_c(\mathbf{Fib})) \simeq \mathbf{PsMon}(\mathbb{S}\mathrm{pan}_t(\mathbf{ICat})) \simeq \mathbf{I}\mathbb{S}\mathrm{pan}(\mathbf{Cat}) \quad \square$$

Restricting (to monoidal or to discrete fibrations) we recover the theorems in [1,4]. The right-to-left functor restricts to the constructions spelled out in [2,5] (double category of elements, double Grothendieck construction).

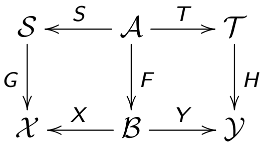
Connection with known constructions.



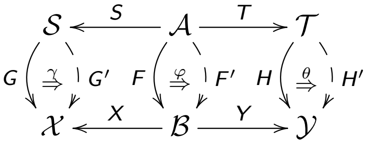
We establish an $\{\text{Indexed}\} \overset{\simeq}{\to} \{\text{Fibrations}\}$ theorem for double fibrations. It *restricts* to the theorems that were shown for monoidal fibrations [4] and for discrete double fibrations [1].
The functor is given by a more general elements construction for an object of $\{\text{Indexed}\}$, that *restricts* to the known elements constructions [2,5].

The Double 2-Category $\mathbb{S}\mathrm{pan}(\mathbf{Cat})$

- **Objects**: Categories $\mathcal{A}$, $\mathcal{B}$, ...
- **Arrows**: Functors $F : \mathcal{A} \to \mathcal{B}$
- **Pro-Arrows**: Spans $\mathcal{S} \xleftarrow{S} \mathcal{A} \xrightarrow{T} \mathcal{T}$
- **Double 2-Cells**: Commutative diagrans



- **Double 3-Cells**:



The Double Category of Elements

- Let $F\colon \mathbb{D}^{\mathrm{op}} \to \mathbb{S}\mathrm{pan}(\mathbf{Cat})$ be a lax double pseudo functor.
- Its components are pseudofunctors

$$F_0\colon D_0 \to \mathbb{S}\mathrm{pan}(\mathbf{Cat})_0 = \mathbf{Cat} \text{ and } F_1\colon D_1 \to \mathbb{S}\mathrm{pan}(\mathbf{Cat})_1.$$

- There is a further induced functor $D_1^{\mathrm{op}} \xrightarrow{F_1} \mathbb{S}\mathrm{pan}(\mathbf{Cat})_1 \xrightarrow{\mathrm{apx}} \mathbf{Cat}$.
- Now apply the ordinary elements construction to F_0 and $\mathrm{apx} \circ F_1$.
- This gives us cloven fibrations

$$\mathbb{E}\mathrm{l}(F)_0 \to D_0 \text{ and } \mathbb{E}\mathrm{l}(F)_1 \to D_1$$

that are part of a double fibration $\mathbb{E}\mathrm{l}(F) \to \mathbb{D}$.

Example - Decorated Cospans (Patterson, 2023)

Set-Up:

- $\mathcal{A}$ a category with pushouts and $\mathbb{C}\mathrm{sp}(\mathcal{A})$ the double category of cospans.
- The decorations are given by a lax double pseudo functor

$$F\colon \mathbb{C}\mathrm{sp}(\mathcal{A}) \to \mathbb{S}\mathrm{pan}(\mathbf{Cat})$$

Definition (Patterson, 2023)

The double category F-$\mathbb{C}\mathrm{sp}$ is the double category of elements $\mathbb{E}\mathrm{l}(F)$.

- This slightly generalizes the definition given by (Baez-Courser-Vasilakopoulou), allowing for decorations on both spans and objects.
- This would allow for a new way to model open systems in classical mechanics (Baez, Weisbart, Yassine, 2021).

Example - Open Dynamical Systems (D.J. Myers, 2020)

- D.J. Myers introduced a double Grothendieck construction $\int\int F$ for a double functor $F\colon \mathbb{D} \to \mathbb{P}\mathrm{rof}$.
- There is a double functor $J\colon \mathbb{P}\mathrm{rof} \to \mathbb{S}\mathrm{pan}(\mathbf{Cat})$ which is the identity on objects and arrows and sends each pro-arrow (profunctor) $\mathcal{A} \overset{P}{\nrightarrow} \mathcal{B}$ to the span of projections

$$\mathcal{A} \longleftarrow \mathrm{disc}\left(\coprod_{a\in\mathcal{A}, b\in\mathcal{B}} P(a,b)\right) \longrightarrow \mathcal{B}$$

- Then $\mathbb{E}\mathrm{l}(JF) = \int\int F$.

Oplax Colimit? (Work in Progress)

- The classical category of elements is an oplax colimit for the diagram defined by the indexing functor.
- This result cannot work for the current set-up because the objects of $\mathbb{S}\mathrm{pan}(\mathbf{Cat})$ are categories rather than double categories.
- There is a one-to-one correspondence of indexing functors between lax double pseudo functors

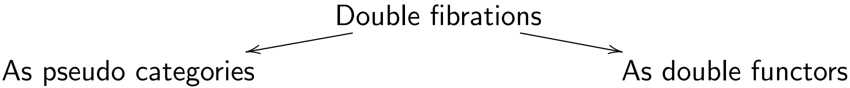
$$F\colon \mathbb{D} \to \mathbb{S}\mathrm{pan}(\mathbf{Cat})$$

  and normal lax double pseudo functors

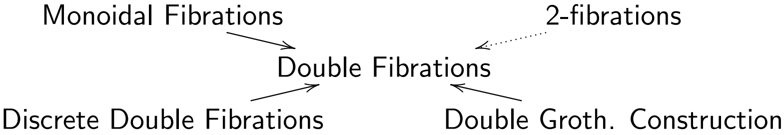
$$\tilde{F}\colon \mathbb{D} \to \mathbb{P}\mathrm{rof}(\mathbf{DblCat})$$

- Conjecture: $\mathbb{E}\mathrm{l}(F)$ is the oplax double colimit (in the arrow direction) of $\tilde{F}$.

Wrapping up...



Thm: Double fibration as pseudo category = Internal fibration in DblCat



Thm: There is an equivalence $\mathbf{I}\mathbb{S}\text{pan}(\mathbf{Cat}) \to \mathbf{DblFib}$ given by the elements/Grothendieck construction for double categories.

Thank you!