Nerves of enriched categories via necklaces

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Problem 1

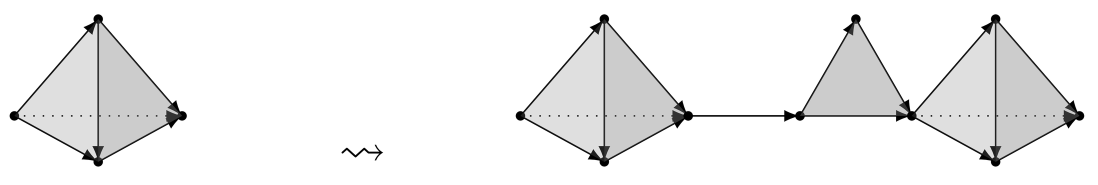
Fix a monoidally cocomplete category $\mathcal{W}$.

$$\mathbb{D} : \mathbf{\Delta} \to \mathcal{W}\,\mathrm{Cat} \qquad \longmapsto \qquad \mathrm{SSet} \underset{N^{\mathbb{D}}}{\overset{L^{\mathbb{D}}}{\rightleftarrows}} \mathcal{W}\,\mathrm{Cat}$$

The left-adjoint $L^{\mathbb{D}}$ is easy to construct but hard to describe explicitly.

Solution: Necklaces

Replace the simplex category $\mathbf{\Delta}$ by the category $\mathcal{N}ec$ of necklaces.



$\Delta^3 \quad \rightsquigarrow \quad \Delta^3 \vee \Delta^1 \vee \Delta^2 \vee \Delta^3$

For $T \in \mathcal{N}ec$, denote: $V_T = \{\text{vertices of } T\}$ $\quad J_T = \{\text{joints of } T\}$ $\quad \dim(T) = |V_T \setminus J_T|$

Necklaces allow to explicitly describe the left-adjoints of familiar nerves.
Let X be a simplicial set with $a, b \in X_0$.

Proposition (Dugger-Spivak [DS11b])**.** *Consider* $\mathfrak{C} : \mathrm{SSet} \to S\,\mathrm{Cat}$*. For all* $n \geq 0$*:*

$$n\text{-simplices of } \mathfrak{C}(X)(a,b) \quad \overset{1-1}{\longleftrightarrow} \quad \text{triples } (T, \vec{T}, \sigma)$$

with $T \in \mathcal{N}ec$, $\vec{T} = (J_T \subseteq T_1 \subseteq \ldots \subseteq T_{n-1} \subseteq V_T)$, $\sigma : T \to X_{a,b}$ *totally non-deg.*

Proposition. *Consider* $L^{dg} : \mathrm{SSet} \to dg\,\mathrm{Cat}_k$*. For any* $n \in \mathbb{Z}$*:*

$$\text{basis elements of } L^{dg}(X)_n(a,b) \quad \overset{1-1}{\longleftrightarrow} \quad \text{pairs } (T, \sigma)$$

with $T \in \mathcal{N}ec$ *such that* $\dim(T) = n$ *and* $\sigma : T \to X_{a,b}$ *totally non-degenerate.*

Problem 2

Fix a Bénabou cosmos $(\mathcal{V}, \otimes, I)$.
When $\mathcal{V}$ is non-cartesian (i.e. $\otimes \neq \times$), there is no enriched nerve

$$N_{\mathcal{V}} : \mathcal{V}\,\mathrm{Cat} \to S\mathcal{V}$$

which lifts the ordinary nerve $N : \mathrm{Cat} \to \mathrm{SSet}$.

Solution: Templicial objects

Restrict the simplex category $\mathbf{\Delta}$ to the category $\mathbf{\Delta}_f$ of finite intervals.

Definition. A templicial object in $\mathcal{V}$ is a *colax monoidal* functor

$$X : \mathbf{\Delta}_f^{op} \to \mathcal{V}$$

such that X_0 is discrete. Let $S_{\otimes}\mathcal{V}$ denote the category of templicial objects in $\mathcal{V}$.

$$\underset{(0 \leq i \leq n)}{s_i : X_n \to X_{n+1}} \qquad \underset{(0 < j < n)}{d_j : X_n \to X_{n-1}} \qquad \underset{(k,l>0)}{\mu_{k,l} : X_{k+l} \to X_k \otimes_{X_0} X_l}$$

For $\mathcal{V}$ cartesian (Leinster [Lei00]):

templicial object X in $\mathcal{V}$ $\quad = \quad$ simplicial object X in $\mathcal{V}$ with X_0 discrete.

E.g. for $\mathcal{V} = \mathrm{Set}$: $S_{\times}\,\mathrm{Set} \simeq \mathrm{SSet}$

Proposition. *There is a fully faithful right-adjoint*

$$N_{\mathcal{V}} : \mathcal{V}\,\mathrm{Cat} \hookrightarrow S_{\otimes}\mathcal{V}$$

which lifts the ordinary nerve $N : \mathrm{Cat} \to \mathrm{SSet}$.

A general procedure

Necklace maps parametrize the templicial structure:

Theorem. *There is a fully faithful left-adjoint*

$$S_{\otimes}\mathcal{V} \hookrightarrow \mathcal{V}^{\mathcal{N}ec^{op}}\text{-}\mathrm{Cat}$$

Assume that $\mathcal{W}$ is tensored and enriched over $\mathcal{V}$.

$$\underset{\text{colax/strong monoidal}}{D : \mathcal{N}ec \to \mathcal{W}} \quad \longmapsto \quad \underset{\text{colax/strong monoidal left-adjoint}}{\mathcal{V}^{\mathcal{N}ec^{op}} \to \mathcal{W}} \quad \longmapsto \quad \underset{\text{functor/right-adjoint}}{N_{\mathcal{V}}^{D} : \mathcal{W}\,\mathrm{Cat} \to \mathcal{V}^{\mathcal{N}ec^{op}}\text{-}\mathrm{Cat} \to S_{\otimes}\mathcal{V}}$$

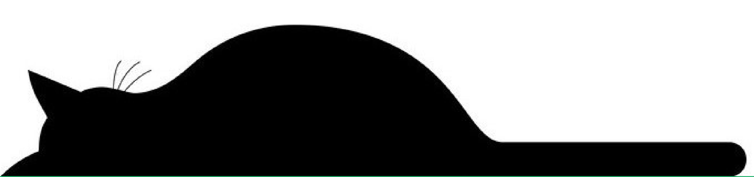
Simplifying the left-adjoint

Fix $D : \mathcal{N}ec \to \mathcal{W}$ strong monoidal $\rightsquigarrow$ $L_{\mathcal{V}}^{D} : S_{\otimes}\mathcal{V} \to \mathcal{W}\,\mathrm{Cat}$.
Let $X \in S_{\otimes}\mathcal{V}$ with vertices a and b.

Proposition. *Let* $\pi : \mathcal{W} \to \mathcal{V}$ *be cocontinuous and assume that* $\pi D(T) \simeq \coprod_{U \hookrightarrow T \text{ inj.}} \tilde{D}(U)$*. Then*

$$\pi(L_{\mathcal{V}}^{D}(X)(a,b)) \simeq \coprod_{T \in \mathcal{N}ec} X_T^{nd}(a,b) \otimes \tilde{D}(T)$$

if $X_T^{nd}(a,b)$ *exists.*



Examples

$\mathcal{V}$	$\mathcal{W}$	$D(T)$	$N_{\mathcal{V}}^{D}$
Set	Set	$\{*\}$	$N : \mathrm{Cat} \to \mathrm{SSet}$
Set	Cat	$\mathcal{P}_T \doteqdot \{U \mid V_T \subseteq U \subseteq J_T\}$	$N^{Dusk} : 2\,\mathrm{Cat} \to \mathrm{SSet}$ [Dus02]
Set	SSet	$hc(T) \doteqdot N(\mathcal{P}_T)$	$N^{hc} : S\,\mathrm{Cat} \to \mathrm{SSet}$ [Cor82]
Set	$\mathrm{Ch}(k)$	$dg(T)_{\bullet} \doteqdot k \cdot \{U \hookrightarrow T \text{ inj.} \mid \dim(U) = \bullet\}$	$N^{dg} : dg\,\mathrm{Cat}_k \to \mathrm{SSet}$ [Lur16]
Set	$c\,\mathrm{Set}$	$cub(T) \doteqdot \square^{\dim(T)}$	$N^{cub} : c\,\mathrm{Cat} \to \mathrm{SSet}$ [Le 20]
$\mathrm{SSet}^{\Theta_{n-1}^{op}}$	$\mathrm{SSet}^{\Theta_{n-1}^{op}}$	$hc(T)$	$\mathrm{SSet}^{\Theta_{n-1}^{op}}\text{-}\mathrm{Cat} \to \mathcal{PC}at(\mathrm{SSet}^{\Theta_{n-1}^{op}})$ [MRR23]
$\mathcal{V}$	$\mathcal{V}$	I	$N_{\mathcal{V}} : \mathcal{V}\,\mathrm{Cat} \to S_{\otimes}\mathcal{V}$ [LM23a]
$\mathcal{V}$	$S\mathcal{V}$	$hc(T)$	$N_{\mathcal{V}}^{hc} : S\mathcal{V}\text{-}\mathrm{Cat} \to S_{\otimes}\mathcal{V}$ [LM23a]
$\mathrm{Mod}(k)$	$\mathrm{Ch}(k)$	$dg(T)_{\bullet}$	$N_k^{dg} : dg\,\mathrm{Cat}_k \to S_{\otimes}\mathrm{Mod}(k)$ [LM23b]

Enriched quasi-categories

Definition. $X \in S_{\otimes}\mathcal{V}$ is quasi-category in $\mathcal{V}$ if

$$\begin{array}{ccc} \Sigma\Lambda_j^n & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Sigma\Delta^n & & \end{array}$$

for all $0 < j < n$ in $\mathcal{V}^{\mathcal{N}ec^{op}}$-Cat.

We have conditions on D to determine when:

- $N_{\mathcal{V}}^{D}(\mathcal{C})$ is a quasi-category in $\mathcal{V}$.
- $N_{\mathcal{V}}^{D}(\mathcal{C})$ has a *Frobenius structure*.
- $N_{\mathcal{V}}^{D_1}(\mathcal{C}) \to N_{\mathcal{V}}^{D_2}(\mathcal{C})$ is a trivial fibration.

Work in progress

- Study the homotopical properties of the nerves $N_{\mathcal{V}}^{D}$. What conditions on D and $\mathcal{W}$ make $N^{D} : \mathcal{W}\,\mathrm{Cat} \to \mathrm{SSet}_{Joyal}$ right Quillen?
- Joint with Wendy Lowen: Develop a homotopy theory for quasi-categories in $\mathcal{V}$ as models for $S\mathcal{V}$-enriched ∞-categories.
- Joint with Violeta Borges Marques: Develop a homotopy theory for Segal dg-categories based on templicial objects in $\mathrm{Ch}(k)$.

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