A ‘Basis Theorem’ for 2-rigs and Rig Geometry

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1. Rig Geometry

2. The Basis Theorem

A more-or-less standard way to build toposes 'of spaces'

$$\text{small extensive } \mathcal{C} \longmapsto \text{Gaeta topos } \mathcal{G}(\mathcal{C}) \longmapsto \mathcal{Z}(\mathcal{C}) \subseteq \mathcal{G}(\mathcal{C})$$

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Corollary (of the Comparison Lemma)

If every object of $\mathcal{C}$ is a finite coproduct of connected objects then $\mathcal{G}(\mathcal{C}) \to \mathbf{Set}$ is essential.

Example:

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Example: finite posets, affine schemes (over a fixed K).

Non-example: countable non-empyt sets (Bornological topos), affine C^∞-schemes.

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Algebraic categories $\mathcal{A}$ such that $\mathcal{A}_{fp}$ (or $\mathcal{A}_{fg}$ or ...) is coextensive.

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Example

By Hilbert's Basis Theorem, the answer is YES for $(K/\mathbf{Ring})_{fp}$, K a field.

Rig Geometry

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Given an algebraic functor $\mathcal{A} \to$ **Rig**, when is $\mathcal{A}$ coextensive?

The paper

Lawvere, F. W.
Core varieties, extensivity, and rig geometry.
Theory Appl. Categ. 20, 497-503 (2008).

gives partial answers and several examples
(with emphasis on varieties of the coextensive 2/**Rig**).

Saturated ideals

Lemma

If I is an ideal in a rig A then the kernel of $A \to A/I$ is

$$\overline{I} = \{x \in A \mid (\exists s \in I)(x + s \in I)\} \subseteq A$$

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A saturated ideal is essentially f.g. if it is the saturation of a finitely generated ideal.

Note:a f.g. saturated ideal is essentially f.g.
In **Ring**, the converse holds because ideals of rings are saturated.

Noetherian rigs

Lemma

For a rig A t.f.a.e.:

1. *Every sequence $I_0 \subseteq I_1 \subseteq \dots$ of saturated ideals of A is stationary. (I.e. there is an $m \in \mathbb{N}$ s.t. $I_m = I_n$ for every $n \geq m$.)*
2. *Every saturated ideal $I \subseteq A$ is essentially f.g..*

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2. *Every saturated ideal $I \subseteq A$ is essentially f.g..*

So, we say that A is Noetherian if the above hold.
Also, A is strongly Noetherian of every saturated ideal of A is f.g..

Lemma

Is Noetherian rig then is a finite product of directly indecomposable (Noetherian) rigs.

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Codiscrete in rings. The underlying poset in a lattice.

An ideal $I \subseteq A$ is lower-closed if $x \leq y \in I$ implies $x \in I$.
(Notice that lower-closed implies saturated.)

Theorem (The ‘lower’ Basis Theorem)

If K is s.t. every lower-closed ideal is f.g. then so is every lower-closed ideal of $K[x]$.

The 2-basis Theorem

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Corollary (The 2-Basis Theorem)

If K is a strongly Noetherian 2-rig then so is $K[x]$.

Proof.

Recall: A rig A is strongly Noetherian if every saturated ideal is f.g..
In a 2-rig, saturated iff lower-closed. □

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A rig is coconnected if

$$x + y = 1 \quad \wedge \quad xy = 0 \quad \vdash_{x,y} \quad x = 0 \quad \vee \quad y = 0$$

i.e. if it lacks complemented elements.

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Corollary

The classifier of coconnected 2-rigs is pre-cohesive.

Proof.

The lemma implies that every object in the extensive site is a finite copro of connected objects so the topos is essential. □

'integral' rigs

Example (L'2008)

Let $i\mathbf{Rig} \to 2/\mathbf{Rig}$ be the subcat of those s.t. $1 + x = 1$.
It is (coreflective and) coextensive.

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Let $i\mathbf{Rig} \to 2/\mathbf{Rig}$ be the subcat of those s.t. $1 + x = 1$.
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Proposition

The classifier of coconnected irigs is pre-cohesive.

Proof.

Again, we concentrate on essentiality.
Use the good properties of the reflection $2/\mathbf{Rig} \to i\mathbf{Rig}$ to show that every f.p. irig is a finite direct product of d.i. and f.p. irigs. □

Positive rigs

A rig is positive if $1 + x$ is invertible for every x.

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So, if we let $A \to LA$ be the (local) integral reflection of the positive A then $PA \to P(LA)$ is monic in **Bool**. (It has trivial kernel by the Lemma.)

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The left adjoint $L : p\mathbf{Rig} \to i\mathbf{Rig}$ preserves finite presentability and so, for positive f.p. A, finite $P(LA)$ implies finite PA. □

Bibliography I

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