Internal Parameterization of Hyperconnected Quotients

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Idea

“External” $\longleftrightarrow$ “Internal”
(bijection; “Internal Parameterization”)

Subtopos	$\longleftrightarrow$ by Lawvere and Tierney	Lawvere Tierney topology $\Omega \to \Omega$	Known
Quotient	$\longleftrightarrow$ Lawvere's open problem	?	Open
$\cup$			
Hyperconnected Quotient	$\longleftrightarrow$ Main Theorem	Main Definition $\Xi \to \Omega$	My result

Ξ: local state classifier

This talk in one slide

1. I have been studying the first problem of **Lawvere's open problems in topos theory.**

2. I construct the bijective correspondence between
   - hyperconnected geometric morphisms from $\mathcal{E}$ and
   - internal semilattice homomorphisms $\Xi \to \Omega$ in $\mathcal{E}$.

   We call this main theorem **Internal Parameterization of Hyperconnected Quotients [Hor23]**.

3. As a key notion, I define the notion of **local state classifier** Ξ, which is just a colimit of all monomorphisms! (And it's fun to calculate them!)

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Hyperconnected Quotient $\longleftrightarrow$ (Main Theorem) Main Definition $\Xi \to \Omega$ — My result

Ξ: local state classifier

Lawvere's open problem

In his webpage "open problems in topos theory"[1], Lawvere asks the following question:

Lawvere's open problems #1

[...] Is there a Grothendieck topos for which the number of these **quotients** is not small? At the other extreme, could they be **parameterized internally, as subtoposes are?**

Today, we focus on "the other extreme".

[1] https://www.acsu.buffalo.edu/~wlawvere/Openproblemstopos.htm

Lawvere's open problem

Theorem (Internal Parameterization of Subtoposes, Lawvere Tierney)

For a topos $\mathcal{E}$, there is a natural bijective correspondence between the following two data:

1. *subtoposes of $\mathcal{E}$ (i.e., embedding into $\mathcal{E}$)*
2. *LT-topology (i.e. idempotent internal semi-lattice homomorphisms $\Omega \to \Omega$)*

Lawvere's open problem

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Imaginary "dual" theorem that does not (yet) exist

For a topos $\mathcal{E}$, there exists a natural bijective correspondence between the following two concepts:

- quotients of $\mathcal{E}$ (i.e., connected geometric morphism from $\mathcal{E}$)
- internal homomorphisms (from one internal algebraic object to another).

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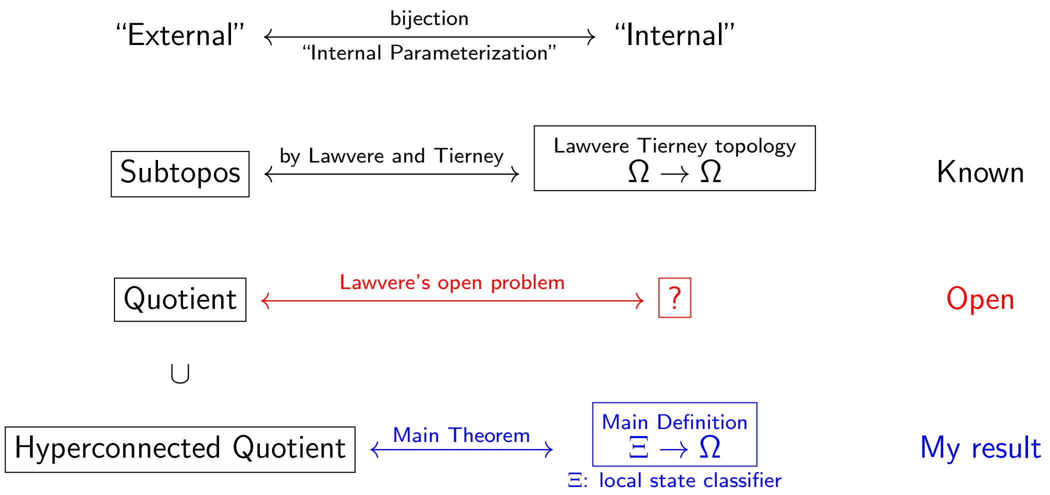
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Internal Parameterization of Hyperconnected Quotients

Definition

A hyperconnected quotient of $\mathcal{E}$ is a hyperconnected geometric morphism from $\mathcal{E}$.

Although I did not define a local state classifier until now, it suffices to know that every Grothendieck topos has it.

Main Theorem (Internal Parameterization of Hyperconnected Quotients [Hor23])

For a topos $\mathcal{E}$ with a local state classifier Ξ (e.g., any Grothendieck topos), there is a natural bijective correspondence between the following two data:

1. hyperconnected quotients of $\mathcal{E}$
2. internal semi-lattice homomorphisms $\Xi \to \Omega$

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$\cup$

Hyperconnected Quotient ⟷ Main Definition $\Xi \to \Omega$ — Main Theorem — My result

Ξ: local state classifier

Local state classifier

I defined a local state classifier (LSC) Ξ in the context of topos theory.

However, even if I forgot all topos theoretic motivations, this notion would still be my favorite!

Local state classifier

- The definition makes sense for (general) categories. There might be other applications.
- It is a variant of the notion of a terminal object, and easy to define.
- Once understood, the definition is intuitive. LSC is intended to be "the set of all local states."
- For a presheaf category, it is "dual" to the subobject classifier!
- Concrete calculations are often possible, and it's fun to discover connections with "usual" mathematics!

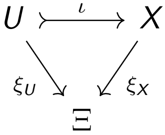
Local state classifier

Definition (local state classifier [Hor23])

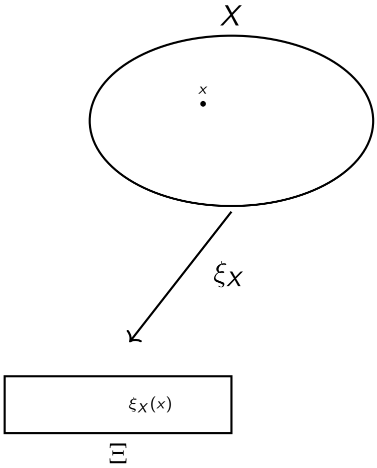
A local state classifier Ξ of a category $\mathcal{C}$ is a colimit of all monomorphisms:

$$\Xi = \mathrm{colim}(\mathcal{C}_{\mathrm{mono}} \rightarrowtail \mathcal{C}).$$

The associated cocone is referred to as $\{\xi_X \colon X \to \Xi\}_{X \in \mathrm{ob}(\mathcal{C})}$.



ξ_X is "locally determined"



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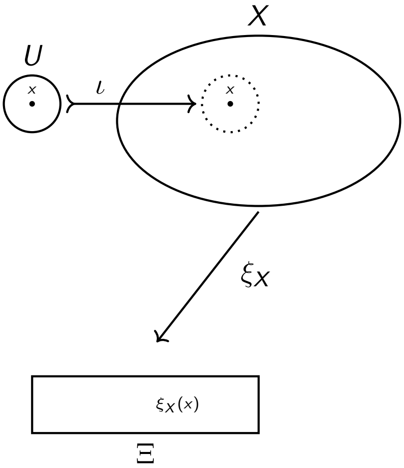

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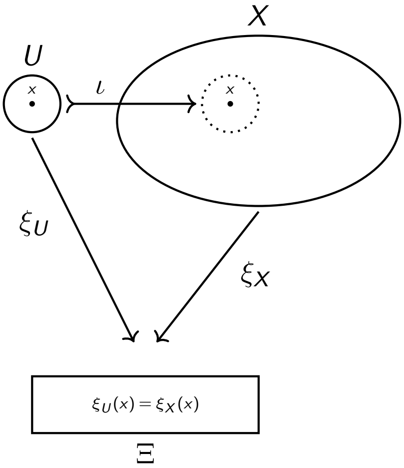
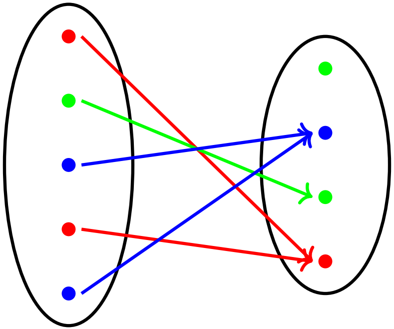


Toy Example: LSC of 3-colored sets

What is the local state classifier of the category of 3-colored sets $\mathrm{Set}/\{\mathrm{R,G,B}\}$?



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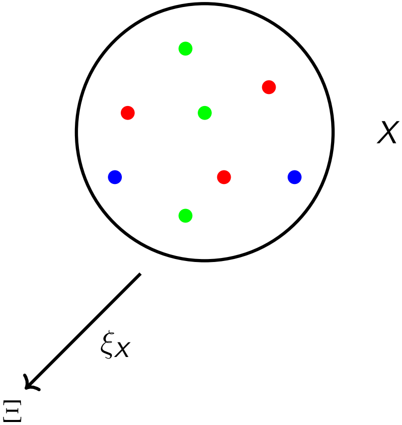

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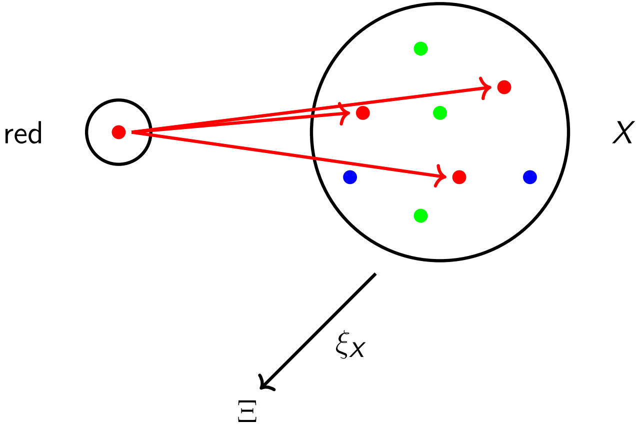

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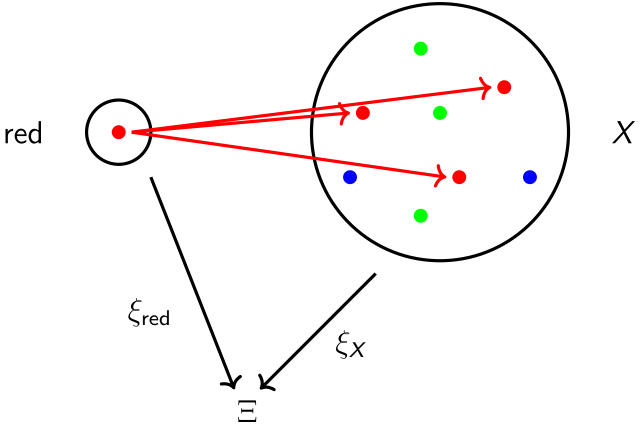

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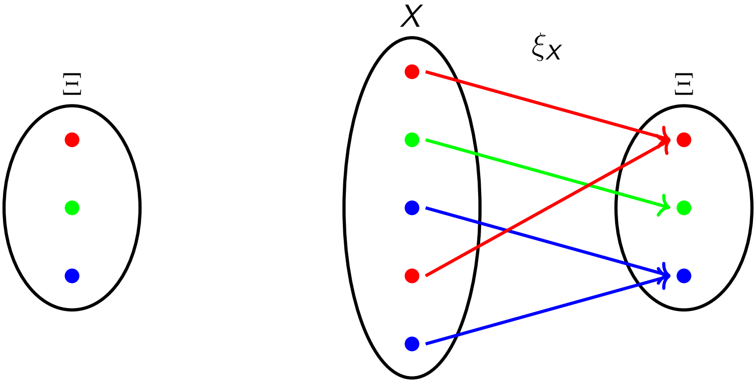

Existence of LSC

Proposition (Existence of LSC for a Grothendieck topos [Hor23])

Every Grothendieck topos has a local state classifier.

Existence of LSC

Proposition (Existence of LSC for a Grothendieck topos [Hor23])

Every Grothendieck topos has a local state classifier.

- The elementary topos $\mathrm{FinSet}^{\mathbb{Z}}$ does not have a local state classifier.
- The category of algebras (for example, groups, rings, Lie algebras) has a local state classifier, (by the existence theorem in [Hor23].)

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More Examples: presheaf topos

Let $\mathcal{C}$ be a small category. Recall that the subobject classifier of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the presheaf of subobjects of representable presheaves (sieves):

$$\Omega\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

$$c \longmapsto \{S \rightarrowtail y(c)\}$$

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Proposition (LSC of a presheaf topos [Hor23])

The local state classifier of the presheaf topos $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the presheaf of co-subobjects (quotients) of representable presheaves:

$$\Xi \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

$$c \longmapsto \{y(c) \twoheadrightarrow S\}$$

More Examples: presheaf topos

Example (Group action topos)

LSC of the topos of G-sets $\mathrm{Set}^{G^{\mathrm{op}}}$ is the set of all subgroups:

$$\Xi = \mathrm{Sub}_{\mathrm{Group}}(G).$$

ξ_X sends each element $x \in X$ to its stabilizer subgroup.

Example (Directed Graphs)

LSC of the topos of directed graphs $\mathrm{DirGraph} = \mathrm{Set}^{\rightrightarrows^{\mathrm{op}}}$ is given by the "2-bouquet":

[being a loop]  [not being a loop].

More Examples: Relation to the terminal object

A local state classifier Ξ, which is a colimit of

$$\iota \colon \mathcal{C}_{\mathrm{mono}} \to \mathcal{C},$$

is similar to the terminal object.

Fact (Characterization of a terminal object)

An object of a category $\mathcal{C}$ is terminal, if and only if it is a colimit of the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$.

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Then, it is natural to ask when LSC becomes terminal (or, "degenerate").

More Examples: Relation to the terminal object

As an immediate corollary of our main theorem (and hyperconnected localic factorization by Johnstone), we obtain a simple answer to this question.

Theorem (Characterization of Localic toposes [Hor23])

For a Grothendieck topos $\mathcal{E}$, the followings are equivalent:

- *the local state classifier Ξ is terminal*
- *$\mathcal{E}$ is localic.*

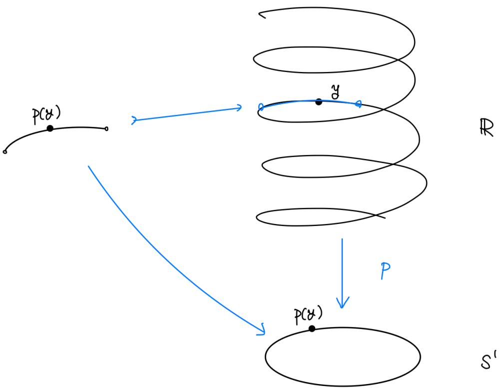
For example, the local state classifier of a sheaf topos $\mathrm{Sh}(X)$ over a topological space X is terminal.

More Examples: Relation to the terminal object

$\mathrm{Sh}(X)$ is the category of étale bundles over X (i.e., **local** homeomorphisms to X).

An element $y \in Y$ in the étale bundle $p : Y \to X$ does not have any non-trivial "local data" except the underlying point $p(y) \in X$ ("where it locates").

Roughly speaking, this is the reason why LSC of $\mathrm{Sh}(X)$ is X itself, (i.e., the terminal bundle $\mathrm{id}_X : X \to X$).



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Things I couldn't talk about

On this theorem

- Theoretical details of the bijective correspondence
- Concrete classification of hyperconnected quotients
- Complete answer to the Boolean Grothendieck toposes
- Why hyperconnected quotient?
- Why "local state"?

On future works

- Remaining Gaps
- Generalization of LSC
- interaction with LT-topology
- Quotients of $\mathrm{Set}^{\mathbb{N}}$ and $\mathrm{DirGraph}$
- Quotients of monoid action topos
- LSC of locally presentable category
- functor that preserves LSC