Bifunctor Thm. and strictification tensor product for double categories with lax double functors

What is a Gray tensor product?

Gray $\otimes$ gives a closed monoidal structure on a category. For 2Cat, the cat. of 2-categories:

$$2\mathrm{Cat}(\mathcal{A}\otimes\mathcal{B},\mathcal{C})\cong 2\mathrm{Cat}(\mathcal{B},\mathrm{Fun}(\mathcal{A},\mathcal{C})).$$

- Writing out what a 2-functor $F:\mathcal{B}\longrightarrow\mathrm{Fun}(\mathcal{A},\mathcal{C})$ is,
- one obtains "quasi-functor of two variables" $H:\mathcal{A}\times\mathcal{B}\longrightarrow\mathcal{C}$ defined by relations among $F(B)(A)$, $A\in\mathcal{A}, B\in\mathcal{B}$, and
- concludes which relations should hold in $\mathcal{A}\otimes\mathcal{B}$.

The relation that differs from what holds in $-\times-$ is:

$$(f\otimes B')(A\otimes g)\overset{\neq}{\Rightarrow}(A'\otimes g)(f\otimes B).$$

Gray proved that $\mathcal{A}\otimes\mathcal{B}$ yields a monoidal product on 2Cat.

Candidate for Gray $\otimes$ for double cats and lax double functors

- In [1] the existence of a Gray monoidal structure is proved for strict double categories, and it was fully described by generators and relations in [3].
- In [4] we defined $[\![\mathbb{A},\mathbb{B}]\!]$ to consist of:
  - 0: lax double functors
  - 1v: vert. lax transf.
  - 1h: horiz. oplax transf.
  - modifications

We characterized a lax d. functor $F:\mathbb{A}\longrightarrow[\![\mathbb{B},\mathbb{C}]\!]$, got to the notion of *lax double quasi-functor* $H:\mathbb{A}\times\mathbb{B}\longrightarrow\mathbb{C}$, and defined $\mathbb{A}\otimes\mathbb{B}$ (in this lax setting).

What is lost in the lax case?

- $[\![-,-]\!]$ is NOT a bifunctor
- associativity constraint is not an isomorphism
- no enrichment: the composition on $[\![\mathbb{A},\mathbb{B}]\!]$ can not be defined (*horizontal* composition and the interchange law of h. o. t. require invertibility of double functors).

$\rightsquigarrow$ no Gray monoidal structure.

References

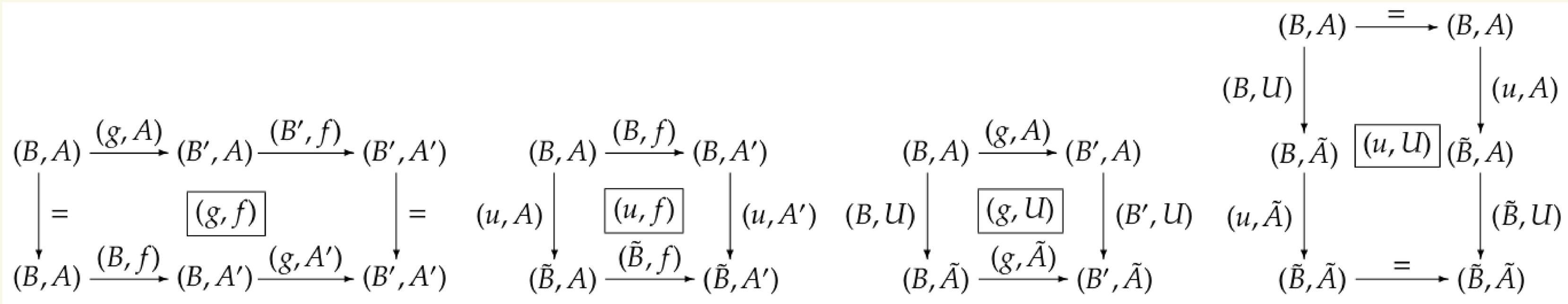
- G. Böhm: *The Gray Monoidal Product of Double Categories (2020)*
- *P.F. Faul, G. Manuell, J. Siqueira:* 2-Dimensional Bifunctor Theorems and Distributive laws (2021)
- B. Femić: *Enrichment and internalization in tricategories, the case of tensor categories and alternative notion to intercategories*
- *B. Femić:* Bifunctor Theorem and Gray monoidal structure for double categories with lax double functors

Lax double quasi-functor

...consists of lax double functors

$$(-,A):\mathbb{B}\longrightarrow\mathbb{C}\quad\text{and}\quad(B,-):\mathbb{A}\longrightarrow\mathbb{C}$$

coinciding on 0-cells, and 2-cells



in $\mathbb{C}$ which satisfy 20 axioms.

Double category isomorphisms

We introduce double cats and construct isos:

$$q\text{-}\mathrm{Lax}_{hop}(\mathbb{A}\times\mathbb{B},\mathbb{C})\overset{*}{\cong}\mathrm{Lax}_{hop}(\mathbb{A},[\![\mathbb{B},\mathbb{C}]\!])$$

$$q\text{-}\mathrm{Lax}_{hop}(\mathbb{A}\times\mathbb{B},\mathbb{C})\cong\mathrm{Dbl}_{hop}(\mathbb{A}\otimes\mathbb{B},\mathbb{C})$$

$\rightsquigarrow$ $\mathrm{Dbl}_{hop}(\mathbb{A}\otimes\mathbb{B},\mathbb{C})\cong\mathrm{Lax}_{hop}(\mathbb{A},[\![\mathbb{B},\mathbb{C}]\!])$

- Hence, there is a natural isomorphism of sets:

$$Dbl_{st}(\mathbb{A}\otimes\mathbb{B},\mathbb{C})\cong Dbl_{lx}(\mathbb{A},[\![\mathbb{B},\mathbb{C}]\!]).$$

Double categorical Bifunctor Theorem

Passing to strict vert. trans. we get a double functor

$$\mathcal{F}:q\text{-}\mathrm{Lax}^{st}_{hop}(\mathbb{A}\times\mathbb{B},\mathbb{C})\longrightarrow\mathrm{Lax}_{hop}(\mathbb{A}\times\mathbb{B},\mathbb{C}).$$

It restricts to double equivalences:

$$\mathcal{F}':q\text{-}\mathrm{Lax}^{st-u}_{hop}(\mathbb{A}\times\mathbb{B},\mathbb{C})\overset{\simeq}{\longrightarrow}\mathrm{Lax}^{u-d}_{hop}(\mathbb{A}\times\mathbb{B},\mathbb{C})$$

$$\mathcal{F}'':q\text{-}\mathrm{Ps}^{st}_{hop}(\mathbb{A}\times\mathbb{B},\mathbb{C})\overset{\simeq}{\longrightarrow}\mathrm{Ps}_{hop}(\mathbb{A}\times\mathbb{B},\mathbb{C}).$$

"(Un)currying" double functor

Accordingly we get **uncurrying** double functor:

$$\mathrm{Lax}_{hop}(\mathbb{A},[\![\mathbb{B},\mathbb{C}]\!]^{st})\longrightarrow\mathrm{Lax}_{hop}(\mathbb{A}\times\mathbb{B},\mathbb{C}).$$

It restricts to a double equivalence - **currying** d.f.

$$\mathrm{Lax}^{ud}_{hop}(\mathbb{A}\times\mathbb{B},\mathbb{C})\simeq\mathrm{Lax}^{u}_{hop}(\mathbb{A},[\![\mathbb{B},\mathbb{C}]\!]^{st-u}).$$

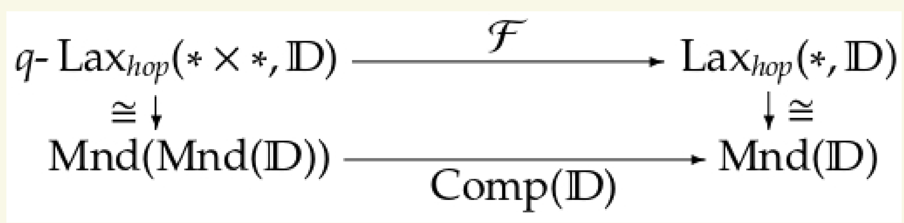
Connection to double monads

There are double category isomorphisms:

$$\mathrm{Lax}_{hop}(*,\mathbb{D})\cong\mathrm{Mnd}(\mathbb{D})$$

$$q\text{-}\mathrm{Lax}_{hop}(*\times*,\mathbb{D})\cong\mathrm{Mnd}(\mathrm{Mnd}(\mathbb{D})).$$

Bifunctor Theorem as a generalization of Beck's result on the composition of monads:



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