THE RIGHT-CONNECTED COMPLETION OF A DOUBLE CATEGORY

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MOTIVATION & OVERVIEW

$$\begin{tikzcd}
\boxed{\text{algebraic weak factorisation systems}} \arrow[d, hook, "{(L,R) \text{ on } \mathcal{C} \;\mapsto\; R\text{-}\mathbb{A}\mathrm{lg}}"] \\
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  - it is right-connected
  - satisfies a monadicity condition.

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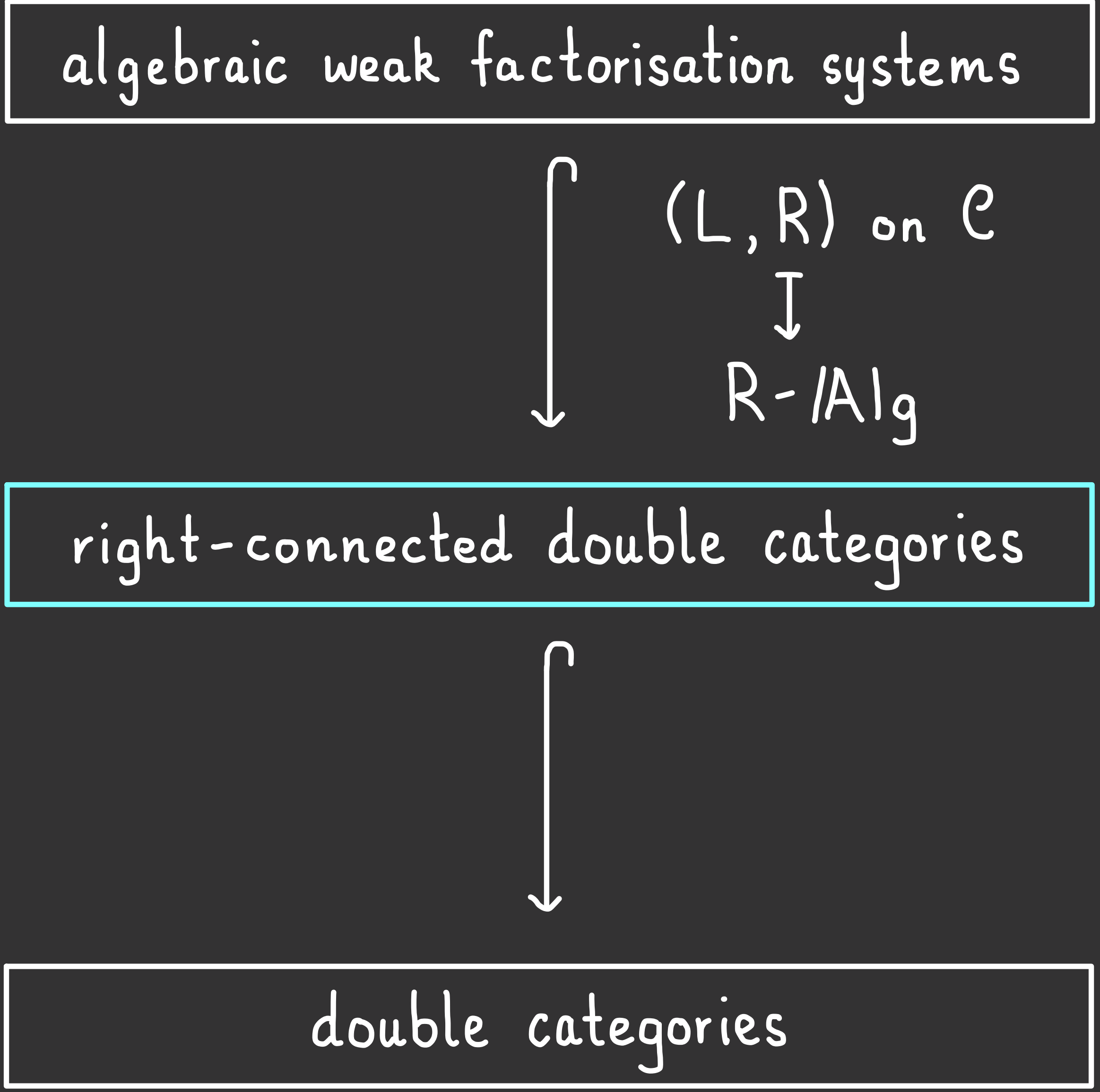

algebraic weak factorisation systems

(L, R) on $\mathcal{C}$ $\mapsto$ $\mathsf{R}\text{-}\mathbb{A}\mathrm{lg}$

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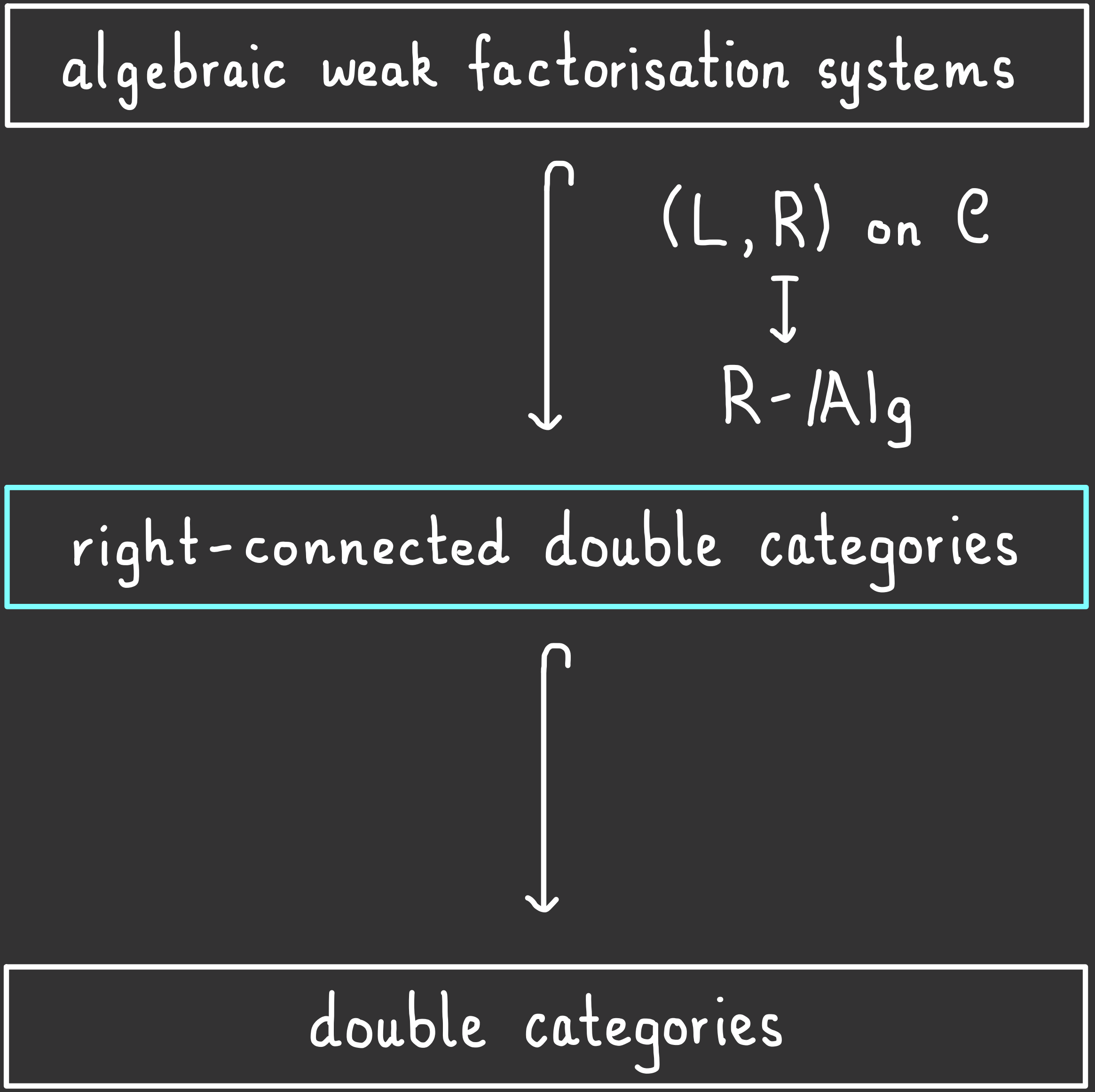

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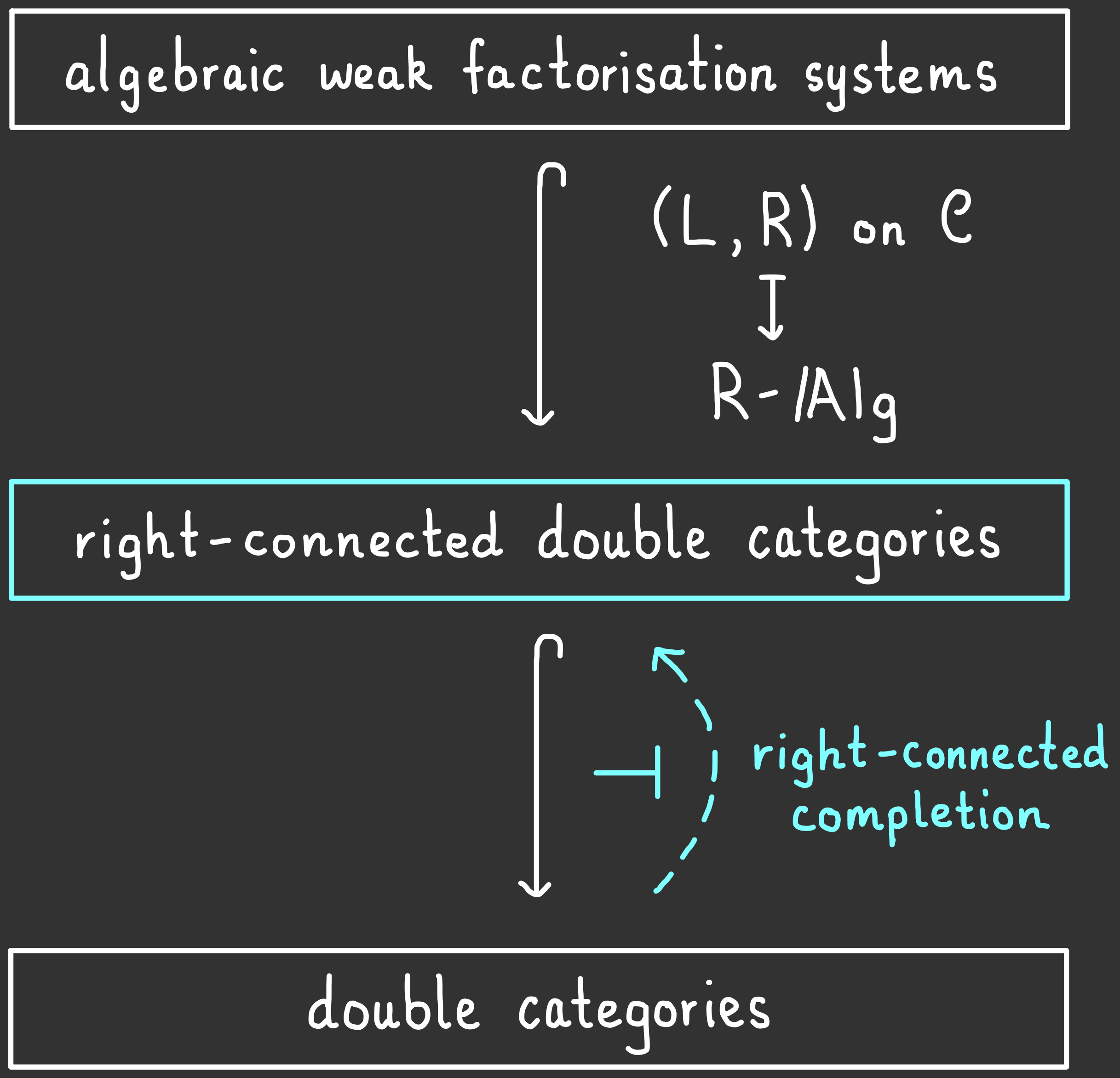

algebraic weak factorisation systems

(L, R) on $\mathcal{C}$

$\downarrow$

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$\dashv$ right-connected completion

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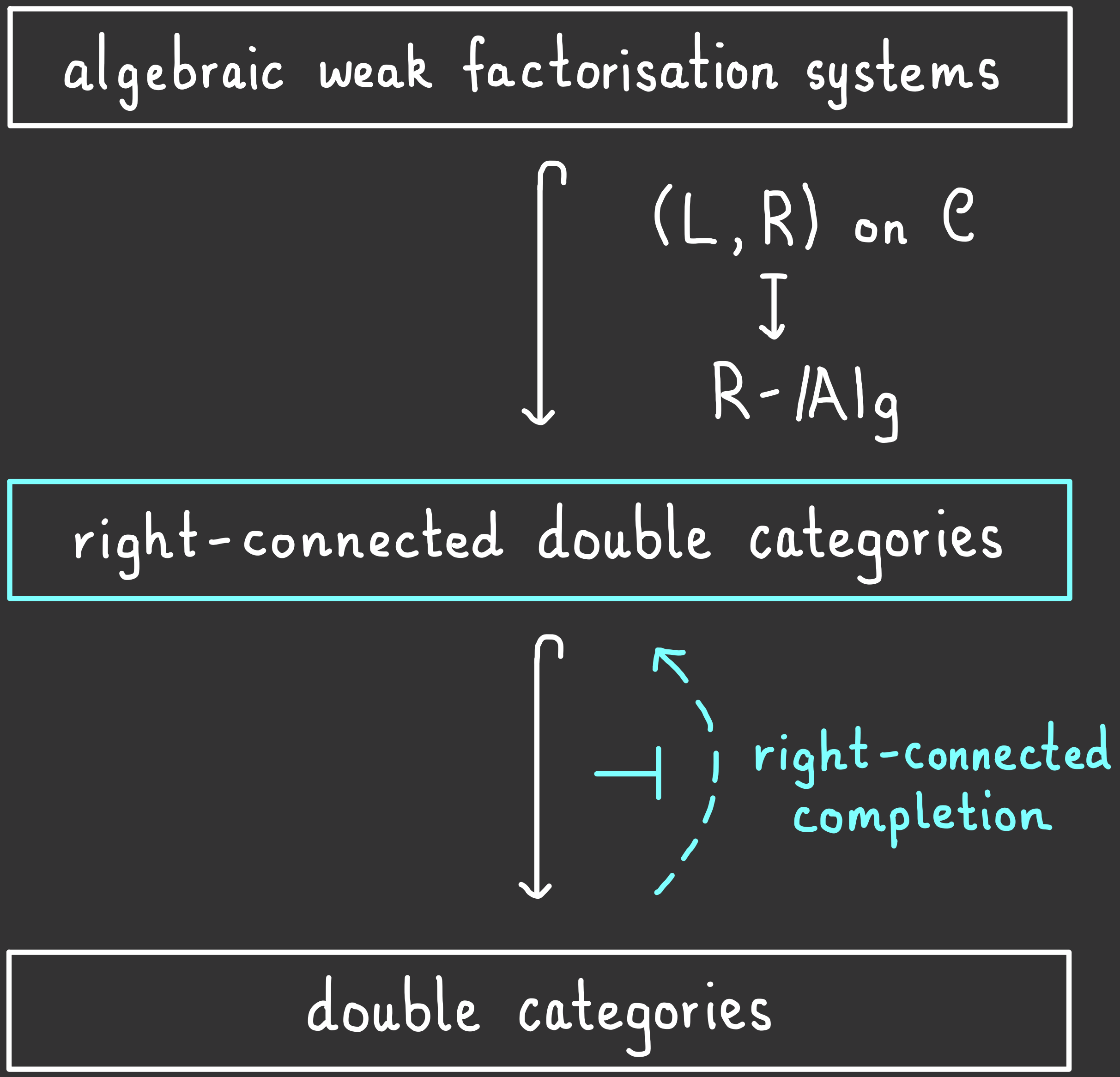

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OUTLINE OF THE TALK

1. Three approaches to the R.C.C.
2. Examples + (co)monadicity conditions

DOUBLE CATEGORIES

A double category consists of:

- objects $A, B, C, D, \ldots$
- horizontal morphisms $\bullet \longrightarrow \bullet$
- vertical morphisms $\bullet \mathrel{\mapstochar\relbar\joinrel\rightarrow} \bullet$
- cells

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"', "\shortmid" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g", "\shortmid" marking] \\
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A **double category** $\mathbb{D}$ is an internal category in the 2-category CAT.

$$\begin{tikzcd}
\mathcal{D}_0 \arrow[r, "id" description] & \mathcal{D}_1 \arrow[l, shift right=2.5ex, "dom"'] \arrow[l, shift left=2.5ex, "cod"] & \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \arrow[l, "comp"']
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The **nerve** of a double category

$$\mathrm{DBL} \xhookrightarrow{N} [\Delta^{op}, \mathrm{CAT}]$$

is 2-functor $N_{\mathbb{D}} \cong \mathrm{DBL}(\mathbb{V}(-), \mathbb{D})$ where:

$$\Delta \hookrightarrow \mathrm{CAT}_{\ell d} \xrightarrow{\mathbb{V}} \mathrm{DBL}$$

TWO RUNNING EXAMPLES

$\mathbb{K}\ell(\mathcal{C}, Q)$ for comonad Q on $\mathcal{C}$.

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A \arrow[r] & C \\
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B \arrow[r] & D
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B \arrow[ur, shift left=1, "s"] \arrow[rrr, "k"'] & & & D
\end{tikzcd}$$

We can use this to compose R-algebras, and define a double category R-$\mathbb{A}$lg:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "id" description] & R\text{-}Alg \arrow[l, shift right=3, "dom"'] \arrow[l, shift left=3, "cod"]
\end{tikzcd}$$

RIGHT-CONNECTED DOUBLE CATEGORIES

A double cat. is right-connected if:

$$\begin{tikzcd} \mathbb{D}_0 \arrow[r, bend left, "\mathrm{id}"] \arrow[r, phantom, "\top"] & \mathbb{D}_1 \arrow[l, bend left, "\mathrm{cod}"] \end{tikzcd}$$

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A double cat. is right-connected if:

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The unit ρ has components:

$$\begin{tikzcd} A \arrow[d, "f"', "|" marking] \\ B \end{tikzcd} \longmapsto \begin{tikzcd} A \arrow[r, "Uf"] \arrow[d, "f"', "|" marking] \arrow[dr, phantom, "\rho_f"] & B \arrow[d, "\mathrm{id}_B", "|" marking] \\ B \arrow[r, "1_B"'] & B \end{tikzcd}$$

Idea: Uf is the underlying hor. morph. of f.

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<u>Idea</u>: Uf is the underlying hor. morph. of f.

$$\begin{tikzcd}[column sep=huge]
\mathrm{R_cDBL} \arrow[r, "\mathrm{obj}" description] \arrow[r, phantom, shift left=3, "\top"] \arrow[r, phantom, shift right=3, "\top"] & \mathrm{CAT} \arrow[l, hook', shift right=6, "\mathbb{S}\mathrm{q}"'] \arrow[l, hook, shift left=6, "\mathbb{H}"]
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$\top$ $\top$

The unit has components $U : \mathbb{D} \longrightarrow \mathbb{S}\mathrm{q}(\mathbb{D}_0)$:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] & & A \arrow[r, "h"] \arrow[d, "Uf"'] & C \arrow[d, "Ug"] \\ B \arrow[r, "k"'] & D & \longmapsto & B \arrow[r, "k"'] & D \end{tikzcd}$$

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The unit ρ has components:

$$\begin{tikzcd} A \arrow[d, "f"', "\shortmid" marking] & & A \arrow[r, "Uf"] \arrow[d, "f"', "\shortmid" marking] \arrow[dr, phantom, "\rho_f"] & B \arrow[d, "\mathrm{id}_B", "\shortmid" marking] \\ B & \longmapsto & B \arrow[r, "1_B"'] & B \end{tikzcd}$$

Idea: Uf is the underlying hor. morph. of f.

$$\begin{tikzcd} \mathrm{RcDBL} \arrow[r, "\mathrm{obj}"] \arrow[r, phantom, shift left=2, "\top"] \arrow[r, phantom, shift right=2, "\top"] & \mathrm{CAT} \arrow[l, hook, bend right, "\mathbb{S}q"'] \arrow[l, hook, bend left, "\mathbb{IH}"] \end{tikzcd}$$

The unit has components $U : \mathbb{D} \longrightarrow \mathbb{S}q(D_0)$:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"', "\shortmid" marking] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g", "\shortmid" marking] & & A \arrow[r, "h"] \arrow[d, "Uf"'] & C \arrow[d, "Ug"] \\ B \arrow[r, "k"'] & D & \longmapsto & B \arrow[r, "k"'] & D \end{tikzcd}$$

<u>Thm</u>: $U_1 : D_1 \longrightarrow \mathrm{Sq}(D_0)$ is monadic

$\Longleftrightarrow$ $\mathbb{D} \cong R\text{-}\mathbb{A}\mathrm{lg}$ for an AWFS (L,R) on D_0.

RIGHT-CONNECTED COMPLETION

$$\begin{tikzcd} \mathrm{RcDBL} \arrow[r, hook, shift right=2] \arrow[r, phantom, "\top"] & \mathrm{DBL} \arrow[l, shift right=2, "\Gamma"'] \end{tikzcd}$$

The right-connected completion $\Gamma(\mathbb{D})$ has:

RIGHT-CONNECTED COMPLETION

$$\begin{tikzcd} \mathrm{RcDBL} \arrow[r, hook, bend right] \arrow[r, phantom, "\top"] & \mathrm{DBL} \arrow[l, bend right, "\mathbb{\Gamma}"'] \end{tikzcd}$$

The right-connected completion $\mathbb{\Gamma}(\mathbb{D})$ has:

- same objects & horizontal morphisms as $\mathbb{D}$;

RIGHT-CONNECTED COMPLETION

$$\begin{tikzcd} \mathrm{RcDBL} \arrow[r, hook, bend right] \arrow[r, phantom, "\top"] & \mathrm{DBL} \arrow[l, bend right, "\mathbb{\Gamma}"'] \end{tikzcd}$$

The right-connected completion $\mathbb{\Gamma}(\mathbb{D})$ has:

- same objects & horizontal morphisms as $\mathbb{D}$;
- vertical morphisms are cells in $\mathbb{D}$ of shape:

$$\begin{tikzcd} A \arrow[d, "{(f,f',\alpha)}"', "|" marking] \\ B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "f"', "|" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "\mathrm{id}_B", "|" marking] \\ B \arrow[r, "1_B"'] & B \end{tikzcd}$$

RIGHT-CONNECTED COMPLETION

$$\begin{tikzcd} \mathrm{RcDBL} \arrow[rr, hook, bend right] \arrow[rr, phantom, "\top"] & & \mathrm{DBL} \arrow[ll, bend right, "\mathbb{I}\Gamma"'] \end{tikzcd}$$

The right-connected completion $\mathbb{I}\Gamma(\mathbb{D})$ has:

- same objects & horizontal morphisms as $\mathbb{D}$;
- vertical morphisms are cells in $\mathbb{D}$ of shape:

$$\begin{tikzcd} A \arrow[d, "{(f,f',\alpha)}"', "\shortmid"{marking}] \\ B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "f"', "\shortmid"{marking}] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "\mathrm{id}_B", "\shortmid"{marking}] \\ B \arrow[r, "1_B"'] & B \end{tikzcd}$$

- cells $(f,f',\alpha) \longrightarrow (g,g',\beta)$ are cells θ in $\mathbb{D}$:

$$\begin{tikzcd} \bullet \arrow[r, "h"] \arrow[d, "f"', "\shortmid"{marking}] \arrow[dr, phantom, "\theta"] & \bullet \arrow[r, "g'"] \arrow[d, "g", "\shortmid"{marking}] \arrow[dr, phantom, "\beta"] & \bullet \arrow[d, "\mathrm{id}", "\shortmid"{marking}] \\ \bullet \arrow[r, "k"'] & \bullet \arrow[r, "1"'] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, "f'"] \arrow[d, "f"', "\shortmid"{marking}] \arrow[dr, phantom, "\alpha"] & \bullet \arrow[r, "k"] \arrow[d, "\shortmid"{marking}] \arrow[dr, phantom, "\mathrm{id}_k"] & \bullet \arrow[d, "\mathrm{id}", "\shortmid"{marking}] \\ \bullet \arrow[r, "1"'] & \bullet \arrow[r, "k"'] & \bullet \end{tikzcd}$$

RIGHT-CONNECTED COMPLETION

$$\begin{tikzcd}
\mathrm{RcDBL} \arrow[r, hook, shift right=1.5ex] \arrow[r, phantom, "\top"] & \mathrm{DBL} \arrow[l, shift right=1.5ex, "\mathbb{\Gamma}"']
\end{tikzcd}$$

The **right-connected completion** $\mathbb{\Gamma}(\mathbb{D})$ has:

- same objects & horizontal morphisms as $\mathbb{D}$;
- vertical morphisms are cells in $\mathbb{D}$ of shape:

$$\begin{tikzcd}
A \arrow[d, "(f,f',\alpha)"', "\shortmid" marking] \\
B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f'"] \arrow[d, "f"', "\shortmid" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "\mathrm{id}_B", "\shortmid" marking] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}$$

- cells $(f,f',\alpha) \longrightarrow (g,g',\beta)$ are cells θ in $\mathbb{D}$:

$$\begin{tikzcd}
\bullet \arrow[r, "h"] \arrow[d, "f"', "\shortmid" marking] \arrow[dr, phantom, "\theta"] & \bullet \arrow[r, "g'"] \arrow[d, "g", "\shortmid" marking] \arrow[dr, phantom, "\beta"] & \bullet \arrow[d, "\mathrm{id}", "\shortmid" marking] \\
\bullet \arrow[r, "k"'] & \bullet \arrow[r, "1"'] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "f'"] \arrow[d, "f"', "\shortmid" marking] \arrow[dr, phantom, "\alpha"] & \bullet \arrow[r, "k"] \arrow[d, "\shortmid" marking] \arrow[dr, phantom, "\mathrm{id}_k"] & \bullet \arrow[d, "\mathrm{id}", "\shortmid" marking] \\
\bullet \arrow[r, "1"'] & \bullet \arrow[r, "k"'] & \bullet
\end{tikzcd}$$

Composition of vertical morphisms is:

$$\begin{tikzcd}
\bullet \arrow[d, "(f,f',\alpha)"', "\shortmid" marking] \\
\bullet \arrow[d, "(g,g',\beta)"', "\shortmid" marking] \\
\bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "f'"] \arrow[d, "f"', "\shortmid" marking] \arrow[dr, phantom, "\alpha"] & \bullet \arrow[r, "g'"] \arrow[d, "\mathrm{id}", "\shortmid" marking] & \bullet \arrow[d, "\mathrm{id}", "\shortmid" marking] \\
\bullet \arrow[r, "1"'] \arrow[d, "g"', "\shortmid" marking] & \bullet \arrow[r, "g'"] \arrow[d, "g"', "\shortmid" marking] \arrow[dr, phantom, "\beta"] & \bullet \arrow[d, "\mathrm{id}", "\shortmid" marking] \\
\bullet \arrow[r, "1"'] & \bullet \arrow[r, "1"'] & \bullet
\end{tikzcd}$$

RIGHT-CONNECTED COMPLETION

$$\begin{tikzcd}
\mathrm{RcDBL} \arrow[r, hook, bend right] \arrow[r, phantom, "\top"] & \mathrm{DBL} \arrow[l, bend right, "\mathbb{\Gamma}"']
\end{tikzcd}$$

The right-connected completion $\mathbb{\Gamma}(\mathbb{D})$ has:

- same objects & horizontal morphisms as $\mathbb{D}$;
- vertical morphisms are cells in $\mathbb{D}$ of shape:

$$\begin{tikzcd}
A \arrow[d, "{(f,f',\alpha)}"'] \\
B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "f'"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "\mathrm{id}_B"] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}$$

- cells $(f,f',\alpha)\longrightarrow(g,g',\beta)$ are cells Θ in $\mathbb{D}$:

$$\begin{tikzcd}
\bullet \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\Theta"] & \bullet \arrow[r, "g'"] \arrow[d, "g"] \arrow[dr, phantom, "\beta"] & \bullet \arrow[d, "\mathrm{id}"] \\
\bullet \arrow[r, "k"'] & \bullet \arrow[r, "1"'] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "f'"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & \bullet \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "\mathrm{id}_k"] & \bullet \arrow[d, "\mathrm{id}"] \\
\bullet \arrow[r, "1"'] & \bullet \arrow[r, "k"'] & \bullet
\end{tikzcd}$$

Composition of vertical morphisms is:

$$\begin{tikzcd}
\bullet \arrow[d, "{(f,f',\alpha)}"'] \\
\bullet \arrow[d, "{(g,g',\beta)}"'] \\
\bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[r, "f'"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & \bullet \arrow[r, "g'"] \arrow[d, "\mathrm{id}"] & \bullet \arrow[d, "\mathrm{id}"] \\
\bullet \arrow[r, "1"'] \arrow[d, "g"'] & \bullet \arrow[r, "g'"] \arrow[d, "g"'] \arrow[dr, phantom, "\beta"] & \bullet \arrow[d, "\mathrm{id}"] \\
\bullet \arrow[r, "1"'] & \bullet \arrow[r, "1"'] & \bullet
\end{tikzcd}$$

The counit has components $V:\mathbb{\Gamma}(\mathbb{D})\longrightarrow\mathbb{D}$

with assignment $(f,f',\alpha)\mapsto f$.

EXAMPLES OF THE RIGHT-CONNECTED COMPLETION

$$\mathbb{\Gamma}(\mathbb{K}\ell(\mathcal{C},Q)^{\vee}) \cong \mathbb{S}\mathrm{pEpi}(\mathcal{C},Q)$$

$$\begin{tikzcd}
A \arrow[r, "r"] & B \\
QB \arrow[u, "s"] \arrow[r, equal] & QB \arrow[u, "\varepsilon_B"']
\end{tikzcd}$$

vertical
morphism
is a
Q-split epi

EXAMPLES OF THE RIGHT-CONNECTED COMPLETION

$$\mathbb{\Gamma}(\mathbb{K}\ell(\mathcal{C},Q)^{\vee}) \cong \mathbb{S}\mathrm{pEpi}(\mathcal{C},Q)$$

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vertical morphism is a Q-split epi

Idea: Vertical morphisms in $\mathbb{\Gamma}(\mathbb{D}^{\vee})$ are generalised split epimorphisms.

EXAMPLES OF THE RIGHT-CONNECTED COMPLETION

$$\mathbb{\Gamma}(\mathbb{K}\ell(\mathbb{C},Q)^{\vee}) \cong \mathbb{S}\mathrm{pEpi}(\mathbb{C},Q)$$

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A \arrow[r, "r"] & B \\
QB \arrow[u, "s"] \arrow[r, equal] & QB \arrow[u, "\varepsilon_B"']
\end{tikzcd}$$

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Idea: Vertical morphisms in $\mathbb{\Gamma}(\mathbb{D}^{\vee})$ are generalised split epimorphisms.

$$\mathbb{\Gamma}(\mathbb{R}\mathrm{et}) \cong \mathbb{L}\mathrm{ens}$$

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[d, equal] \\
X \arrow[u, "\varphi"] \arrow[r, "f\varphi"] \arrow[d, "f\varphi"'] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

bijective-on-objects

discrete opfibration

vertical
morphism
is a
delta lens

⇝

generalised
split opfibration

EXAMPLES OF THE RIGHT-CONNECTED COMPLETION

$$\Gamma(\mathbb{K}\ell(\mathcal{C},Q)^{\vee}) \cong \mathbb{S}\mathrm{pEpi}(\mathcal{C},Q)$$

$$\begin{tikzcd}
A \arrow[r, "r"] & B \\
QB \arrow[u, "s"] \arrow[r, equal] & QB \arrow[u, "\varepsilon_B"']
\end{tikzcd}$$

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__Idea__: Vertical morphisms in $\Gamma(\mathbb{D}^{\vee})$ are generalised split epimorphisms.

$$\Gamma(\mathbb{R}\mathrm{et}) \cong \mathbb{L}\mathrm{ens}$$

bijective-on-objects

discrete opfibration

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[d, equal] \\
X \arrow[u, "\varphi"] \arrow[r, "f\varphi"] \arrow[d, "f\varphi"'] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

vertical
morphism
is a
delta lens

$\rightsquigarrow$

generalised
split opfibration

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, shift right=2, "f"'] \arrow[d, phantom, "\dashv"] & \bullet \arrow[d, equal] \\
\bullet \arrow[u, shift right=2, "u"'] \arrow[r, equal] & \bullet
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma(\mathbb{A}\mathrm{dj}(\mathcal{K})) \arrow[d, squiggly] \\
\text{lali in } \mathcal{K}
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[r, "g"] \arrow[d, "f"'] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, equal] \arrow[ur, Rightarrow, shorten=4mm, "\alpha"] & \bullet
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma(\mathbb{Q}(\mathcal{K})) \arrow[d, squiggly] \\
\text{2-cell in } \mathcal{K}
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, "f"'] \arrow[d, equal] \\
\bullet \arrow[u, "f"] \arrow[r, equal] & \bullet \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma(\mathbb{P}\mathrm{bSq}(\mathcal{C})^{vh}) \arrow[d, squiggly] \\
\text{mono in } \mathcal{C}
\end{tikzcd}$$

APPROACH USING THE NERVE

$$\begin{tikzcd}
 & R_c\mathrm{DBL} \arrow[dr, hook] & \\
\mathrm{CAT}_{\ell d} \arrow[rr, "\mathbb{V}"'] & & \mathrm{DBL}
\end{tikzcd}$$

APPROACH USING THE NERVE

$$\begin{tikzcd}
 & \mathrm{R_cDBL} \arrow[dr, hook] & \\
\mathrm{CAT}_{ld} \arrow[ur, "\mathbb{R}c"] \arrow[rr, "\mathbb{V}"'] \arrow[rr, phantom, "\Uparrow j", shift left=3ex] & & \mathrm{DBL}
\end{tikzcd}$$

The right-connected double category $\mathbb{R}c(\mathbb{C})$ is the restriction of $\mathbb{S}q(\mathbb{C})$ to cells of shape:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

The unit has components $j_{\mathbb{C}}: \mathbb{V}(\mathbb{C}) \longrightarrow \mathbb{R}c(\mathbb{C})$.

APPROACH USING THE NERVE

$$\begin{tikzcd}
 & \mathrm{RcDBL} \arrow[dr, hook] \arrow[d, phantom, "\Uparrow j"] & \\
\mathrm{CAT}_{\ell d} \arrow[ur, "\mathbb{R}c"] \arrow[rr, "\mathbb{V}"'] & & \mathrm{DBL}
\end{tikzcd}$$

The right-connected double category $\mathbb{R}c(\mathbb{C})$ is the restriction of $\mathbb{S}q(\mathbb{C})$ to cells of shape:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

The unit has components $j_{\mathbb{C}} : \mathbb{V}(\mathbb{C}) \longrightarrow \mathbb{R}c(\mathbb{C})$.

We have a relative 2-adjunction:

$$\mathrm{RcDBL}(\mathbb{R}c(\mathbb{C}), \mathbb{D}) \cong \mathrm{DBL}(\mathbb{V}(\mathbb{C}), \mathbb{D})$$

APPROACH USING THE NERVE

$$\begin{tikzcd} & \mathrm{RcDBL} \arrow[dr, hook] & \\ \mathrm{CAT}_{\mathrm{ld}} \arrow[ur, "\mathbb{R}\mathrm{c}"] \arrow[rr, "\mathbb{V}"'] \arrow[rr, phantom, shift left=5ex, "\Uparrow j"] & & \mathrm{DBL} \end{tikzcd}$$

The right-connected double category $\mathbb{R}\mathrm{c}(\mathbb{C})$ is the restriction of $\mathbb{S}\mathrm{q}(\mathbb{C})$ to cells of shape:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, equal] \\ B \arrow[r, equal] & B \end{tikzcd}$$

The <u>unit</u> has components $j_{\mathbb{C}} : \mathbb{V}(\mathbb{C}) \longrightarrow \mathbb{R}\mathrm{c}(\mathbb{C})$.

We have a relative 2-adjunction:

$$\mathrm{RcDBL}(\mathbb{R}\mathrm{c}(\mathbb{C}), \mathbb{D}) \cong \mathrm{DBL}(\mathbb{V}(\mathbb{C}), \mathbb{D})$$

The nerve of a r.c. double category

$$\mathrm{RcDBL} \xhookrightarrow{N} [\Delta^{op}, \mathrm{CAT}]$$

is 2-functor $N_{\mathbb{D}} \cong \mathrm{RcDBL}(\mathbb{R}\mathrm{c}(-), \mathbb{D})$.

APPROACH USING THE NERVE

$$\begin{tikzcd}
& \mathrm{RcDBL} \arrow[dr, hook] & \\
\mathrm{CAT}_{\ell d} \arrow[ur, "\mathbb{R}\mathrm{c}"] \arrow[rr, "\mathbb{V}"'] & \Uparrow j & \mathrm{DBL}
\end{tikzcd}$$

The right-connected double category $\mathbb{R}\mathrm{c}(\mathbb{C})$ is the restriction of $\mathbb{S}\mathrm{q}(\mathbb{C})$ to cells of shape:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

The unit has components $j_{\mathbb{C}} : \mathbb{V}(\mathbb{C}) \longrightarrow \mathbb{R}\mathrm{c}(\mathbb{C})$.

We have a relative 2-adjunction:

$$\mathrm{RcDBL}(\mathbb{R}\mathrm{c}(\mathbb{C}), \mathbb{D}) \cong \mathrm{DBL}(\mathbb{V}(\mathbb{C}), \mathbb{D})$$

The nerve of a r.c. double category

$$\mathrm{RcDBL} \xhookrightarrow{N} [\Delta^{op}, \mathrm{CAT}]$$

is 2-functor $N_{\mathbb{D}} \cong \mathrm{RcDBL}(\mathbb{R}\mathrm{c}(-), \mathbb{D})$.

The right-connected completion $\Gamma(\mathbb{D})$ is determined by its nerve:

$$N_{\Gamma(\mathbb{D})} \cong \mathrm{DBL}(\mathbb{R}\mathrm{c}(-), \mathbb{D}) : \Delta^{op} \longrightarrow \mathrm{CAT}$$

APPROACH USING COMMA OBJECTS

$$\begin{tikzcd}
\mathrm{RcDBL} \arrow[r, hook] \arrow[d, swap, "\mathrm{cod} \dashv \mathrm{id}"] & \mathrm{DBL} \arrow[d, "\mathrm{cod}, \mathrm{id}"] \\
\mathrm{LALI} \arrow[r, hook] & \mathrm{SpEPI}
\end{tikzcd}$$

APPROACH USING COMMA OBJECTS

$$\begin{tikzcd}
\mathrm{RcDBL} \arrow[r, hook, shift right] \arrow[r, phantom, "\top"] \arrow[d, "\mathrm{cod} \dashv \mathrm{id}"'] & \mathrm{DBL} \arrow[l, dashed, shift right, "\mathbb{\Gamma}"'] \arrow[d, "\mathrm{cod},\, \mathrm{id}"] \\
\mathrm{LALI} \arrow[r, hook, shift right] \arrow[r, phantom, "\top"] & \mathrm{SpEPI} \arrow[l, shift right]
\end{tikzcd}$$

The right-connected completion $\mathbb{\Gamma}(\mathbb{D})$ arises by constructing the cofree lali on the section-retraction pair $(\mathrm{id}, \mathrm{cod})$ using comma objects in $\mathrm{CAT}/\mathcal{D}_0$.

APPROACH USING COMMA OBJECTS

$$\begin{tikzcd}
\mathrm{RcDBL} \arrow[r, hook, shift right] \arrow[d, "\mathrm{cod} \dashv \mathrm{id}"'] \arrow[r, phantom, "\top"] & \mathrm{DBL} \arrow[l, dashed, shift right, bend right, "\mathbb{\Gamma}"'] \arrow[d, "\mathrm{cod}, \mathrm{id}"] \\
\mathrm{LALI} \arrow[r, hook, shift right] \arrow[r, phantom, "\top"] & \mathrm{SpEPI} \arrow[l, shift right]
\end{tikzcd}$$

The right-connected completion $\mathbb{\Gamma}(\mathbb{D})$ arises by constructing the cofree lali on the section-retraction pair (id, cod) using comma objects in CAT/D_0.

$$\begin{tikzcd}
D_0 \arrow[rr, "\mathrm{id}"] \arrow[dr, "1_{D_0}"'] & & D_1 \arrow[dl, "\mathrm{cod}"] \\
& D_0 &
\end{tikzcd}$$

APPROACH USING COMMA OBJECTS

$$\begin{tikzcd}
\mathrm{RcDBL} \arrow[r, hook] \arrow[r, phantom, shift left=2, "\top"] \arrow[d, "\mathrm{cod} \dashv \mathrm{id}"'] & \mathrm{DBL} \arrow[l, dashed, bend right, "\mathbb{\Gamma}"'] \arrow[d, "\mathrm{cod},\, \mathrm{id}"] \\
\mathrm{LALI} \arrow[r, hook] \arrow[r, phantom, shift left=2, "\top"] & \mathrm{SpEPI} \arrow[l, bend right]
\end{tikzcd}$$

The right-connected completion $\mathbb{\Gamma}(\mathbb{D})$ arises by constructing the cofree lali on the section-retraction pair (id, cod) using comma objects in CAT/D_0.

$$\begin{tikzcd}
& \Gamma_1(\mathbb{D}) \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "V_1"] & \\
D_0 \arrow[rr, "\mathrm{id}"'] \arrow[dr, "1_{D_0}"'] & & D_1 \arrow[dl, "\mathrm{cod}"] \\
& D_0 &
\arrow[from=1-2, to=2-2, phantom, "\Leftarrow"]
\end{tikzcd}$$

APPROACH USING COMMA OBJECTS

$$\begin{tikzcd}
\mathrm{RcDBL} \arrow[r, hook, shift right] \arrow[r, phantom, "\top"] \arrow[d, "\mathrm{cod} \dashv \mathrm{id}"'] & \mathrm{DBL} \arrow[l, dashed, shift right, "\mathbb{\Gamma}"'] \arrow[d, "\mathrm{cod}, \mathrm{id}"] \\
\mathrm{LALI} \arrow[r, hook, shift right] \arrow[r, phantom, "\top"] & \mathrm{SpEPI} \arrow[l, shift right]
\end{tikzcd}$$

The right-connected completion $\mathbb{\Gamma}(\mathbb{D})$ arises by constructing the cofree lali on the section-retraction pair (id, cod) using comma objects in CAT/D_0.

$$\begin{tikzcd}
 & D_0 \arrow[d, dashed, "\mathrm{id}"] \arrow[ddl, bend right, "1_{D_0}"'] \arrow[ddr, bend left, "\mathrm{id}"] & \\
 & \Gamma_1(\mathbb{D}) \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "V_1"] \arrow[d, phantom, "\Leftarrow"] & \\
D_0 \arrow[rr, "\mathrm{id}"'] \arrow[dr, "1_{D_0}"'] & {} & D_1 \arrow[dl, "\mathrm{cod}"] \\
 & D_0 &
\end{tikzcd}$$

APPROACH USING COMMA OBJECTS

$$\begin{tikzcd}
\mathrm{RcDBL} \arrow[r, hook, shift right=1.5] \arrow[r, phantom, "\top"] \arrow[d, "\mathrm{cod} \dashv \mathrm{id}"'] & \mathrm{DBL} \arrow[l, dashed, shift right=1.5, "\mathbb{\Gamma}"'] \arrow[d, "\mathrm{cod},\, \mathrm{id}"] \\
\mathrm{LALI} \arrow[r, hook, shift right=1.5] \arrow[r, phantom, "\top"] & \mathrm{SpEPI} \arrow[l, shift right=1.5]
\end{tikzcd}$$

The right-connected completion $\mathbb{\Gamma}(\mathbb{D})$ arises by constructing the cofree lali on the section-retraction pair (id, cod) using comma objects in CAT/D_0.

$$\begin{tikzcd}
& D_0 \arrow[d, dashed, "\mathrm{id}"] \arrow[ddl, bend right, "1_{D_0}"'] \arrow[ddr, bend left, "\mathrm{id}"] & \\
& \Gamma_1(\mathbb{D}) \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "V_1"] & \\
D_0 \arrow[rr, "\mathrm{id}"'] \arrow[dr, "1_{D_0}"'] & \Leftarrow & D_1 \arrow[dl, "\mathrm{cod}"] \\
& D_0 &
\end{tikzcd}$$

This approach generalises to internal categories $\mathbb{D}$ in any suitable 2-category.

(CO)MONADICITY CONDITION

$$\mathbb{\Gamma}(\mathbb{D})$$

$$\begin{tikzcd}
\Gamma_1(\mathbb{D}) \arrow[d, shift right=8, "\mathrm{dom}"'] \arrow[d, shift left=8, "\mathrm{cod}"] \\
\mathbb{D}_0 \arrow[u, "\mathrm{id}", "\vdash"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "f'"] \arrow[d, "f"', "|" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "\mathrm{id}_B"] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}$$

(CO)MONADICITY CONDITION

$$\begin{tikzcd}
\mathbb{D} & \mathbb{\Gamma}(\mathbb{D}) \arrow[l, "V"'] \\
\mathcal{D}_1 \arrow[d, shift right=3] \arrow[d, shift left=3] & \Gamma_1(\mathbb{D}) \arrow[l, "V_1"'] \arrow[d, shift right=3, "\mathrm{dom}"'] \arrow[d, shift left=3, "\mathrm{cod}"] \\
\mathcal{D}_0 \arrow[u] \arrow[r, equal] & \mathcal{D}_0 \arrow[u, "\mathrm{id}\ \vdash" description]
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[dd, "f"', "|" marking] & & A \arrow[r, "f'"] \arrow[dd, "f"', "|" marking] \arrow[ddr, phantom, "\alpha"] & B \arrow[dd, "\mathrm{id}_B"] \\
{} & {} \arrow[l, mapsto] & & \\
B & & B \arrow[r, "1_B"'] & B
\end{tikzcd}$$

(CO)MONADICITY CONDITION

$$\begin{tikzcd}
\mathbb{D} & & \mathbb{\Gamma}(\mathbb{D}) \arrow[ll, "V"'] \\
D_1 \arrow[d, shift right=3] \arrow[d, shift left=3] & & \Gamma_1(\mathbb{D}) \arrow[ll, "V_1"'] \arrow[d, shift right=4, "\mathrm{dom}"'] \arrow[d, shift left=4, "\mathrm{cod}"] \arrow[d, phantom, shift left=2, "\vdash"] \\
D_0 \arrow[u] \arrow[rr, equal] & & D_0 \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[d, "f"', "\shortmid" marking] & & A \arrow[r, "f'"] \arrow[d, "f"', "\shortmid" marking] \arrow[rd, phantom, "\alpha"] & B \arrow[d, "\mathrm{id}_B"] \\
B & & B \arrow[r, "1_B"'] & B
\arrow[from=1-3, to=1-1, phantom, ""]
\arrow[from=2-3, to=1-1, phantom, ""]
\end{tikzcd} \quad \longleftarrow\!\shortmid$$

<ins>Thm</ins>: $V_1 : \Gamma_1(\mathbb{D}) \longrightarrow D_1$ is comonadic

$\iff$ each fibre $\mathrm{cod}^{-1}\{B\}$ of the functor $\mathrm{cod} : D_1 \longrightarrow D_0$ admits products with the vertical identity morphism $\mathrm{id}_B : B \xrightarrow{\shortmid} B$.

(CO)MONADICITY CONDITION

$$\begin{tikzcd}
\mathbb{D} & \mathbb{\Gamma}(\mathbb{D}) \arrow[l, "V"'] \arrow[r, "U"] & \mathbb{S}q(\mathbb{D}_0) \\
\mathcal{D}_1 \arrow[d, shift right=8] \arrow[d, shift left=8] & \Gamma_1(\mathbb{D}) \arrow[l, "V_1"'] \arrow[r, "U_1"] \arrow[d, shift right=8, "\mathrm{dom}"'] \arrow[d, shift left=8, "\mathrm{cod}"] & Sq(\mathcal{D}_0) \arrow[d, shift right=8] \arrow[d, shift left=8] \\
\mathcal{D}_0 \arrow[u] \arrow[r, equal] & \mathcal{D}_0 \arrow[u, "\mathrm{id}" description] \arrow[r, equal] \arrow[u, phantom, shift left=4, "\vdash"] & \mathcal{D}_0 \arrow[u] \arrow[u, phantom, shift right=4, "\vdash"] \arrow[u, phantom, shift left=4, "\vdash"]
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[dd, "f"', "\shortmid" marking] & & A \arrow[r, "f'"] \arrow[dd, "f"', "\shortmid" marking] & B \arrow[dd, "\mathrm{id}_B"] & & A \arrow[dd, "f'"] \\
& \longmapsfrom & & & \longmapsto & \\
B & & B \arrow[r, "1_B"'] & B & & B
\arrow[from=1-3, to=3-4, phantom, "\alpha"]
\end{tikzcd}$$

<u>Thm</u>: $V_1 : \Gamma_1(\mathbb{D}) \longrightarrow \mathcal{D}_1$ is comonadic

$\Longleftrightarrow$ each fibre $\mathrm{cod}^{-1}\{B\}$ of the functor $\mathrm{cod} : \mathcal{D}_1 \longrightarrow \mathcal{D}_0$ admits products with the vertical identity morphism $\mathrm{id}_B : B \mathrel{\mkern6mu\shortmid\mkern-14mu\longrightarrow} B$.

(CO)MONADICITY CONDITION

$$\begin{tikzcd}
\mathbb{D} & \mathbb{\Gamma}(\mathbb{D}) \arrow[l, "V"'] \arrow[r, "U"] & \mathbb{S}\mathrm{q}(D_0) \\
D_1 \arrow[d, shift right=4] \arrow[d, shift left=4] & \Gamma_1(\mathbb{D}) \arrow[l, "V_1"'] \arrow[r, "U_1"] \arrow[d, shift right=4, "\mathrm{dom}"'] \arrow[d, shift left=4, "\mathrm{cod}"] \arrow[d, phantom, shift left=2, "\vdash"] & \mathrm{Sq}(D_0) \arrow[d, shift right=4] \arrow[d, shift left=4] \arrow[d, phantom, shift right=2, "\vdash"] \arrow[d, phantom, shift left=2, "\vdash"] \\
D_0 \arrow[u] \arrow[r, equal] & D_0 \arrow[u, "\mathrm{id}" description] \arrow[r, equal] & D_0 \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[d, "f"'] & {} & A \arrow[r, "f'"] \arrow[d, "f"'] \arrow[rd, phantom, "\alpha"] & B \arrow[d, "\mathrm{id}_B"] & {} & A \arrow[d, "f'"] \\
B & {} & B \arrow[r, "1_B"'] & B & {} & B
\arrow[from=1-3, to=1-1, mapsto, yshift=-2em, shorten=1.5em]
\arrow[from=1-4, to=1-6, mapsto, yshift=-2em, shorten=1.5em]
\end{tikzcd}$$

Thm: $V_1 : \Gamma_1(\mathbb{D}) \longrightarrow D_1$ is comonadic

$\iff$ each fibre $\mathrm{cod}^{-1}\{B\}$ of the functor $\mathrm{cod} : D_1 \longrightarrow D_0$ admits products with the vertical identity morphism $\mathrm{id}_B : B \mathrel{\rlap{\hskip .5em $\vert$}\longrightarrow} B$.

Suppose that:

- $\mathrm{dom} : D_1 \longrightarrow D_0$ has a LARI,
- $\mathrm{cod} : D_1 \longrightarrow D_0$ is an opfibration

Then $U_1 : \Gamma_1(\mathbb{D}) \longrightarrow \mathrm{Sq}(D_0)$ has left adjoint.

Open question: when is U_1 monadic?

EXAMPLES WHERE (CO)MONADICITY HOLDS

$$\mathbb{K}\ell(\mathcal{C},Q)^{\vee} \xleftarrow{V} \mathbb{S}_p\mathrm{Epi}(\mathcal{C},Q) \xrightarrow{U} \mathbb{S}_q(\mathcal{C})$$

$$\begin{tikzcd}
A & & A & & & A \arrow[dd] \\
& \longmapsfrom & & QB \arrow[ul] \arrow[dl, "\varepsilon_B"] & \longmapsto & \\
QB \arrow[uu] & & B \arrow[from=uu] & & & B
\end{tikzcd}$$

EXAMPLES WHERE (CO)MONADICITY HOLDS

$$\mathbb{K}\ell(\mathcal{C},Q)^{\vee} \xleftarrow{V} \mathbb{S}_p\mathrm{Epi}(\mathcal{C},Q) \xrightarrow{U} \mathbb{S}_q(\mathcal{C})$$

$$\begin{tikzcd}
A & & & A & & & & A \arrow[dd] \\
{} & {} & {} & & QB \arrow[ul] \arrow[dl, "\varepsilon_B"] & {} & {} & \\
QB \arrow[uu] & & & B & & & & B
\arrow[from=2-3, to=2-2, mapsto]
\arrow[from=2-6, to=2-7, mapsto]
\arrow[from=1-4, to=3-4]
\end{tikzcd}$$

- $\mathcal{C}$ has products $\Rightarrow$ V_1 is comonadic
- $\mathcal{C}$ has coproducts $\Rightarrow$ U_1 is monadic

EXAMPLES WHERE (CO)MONADICITY HOLDS

$$\mathbb{K}\ell(\mathcal{C},Q)^{\vee} \xleftarrow{V} \mathbb{S}p\mathbb{E}pi(\mathcal{C},Q) \xrightarrow{U} \mathbb{S}_q(\mathcal{C})$$

$$\begin{tikzcd}
A & & A & & & A \arrow[dd] \\
& \longmapsfrom & & QB \arrow[ul] \arrow[dl, "\varepsilon_B"] & \longmapsto & \\
QB \arrow[uu] & & B \arrow[from=uu, crossing over] & & & B
\end{tikzcd}$$

- $\mathcal{C}$ has products $\Rightarrow$ V_1 is comonadic
- $\mathcal{C}$ has coproducts $\Rightarrow$ U_1 is monadic

$$\mathbb{R}et \xleftarrow{V} \mathbb{L}ens \xrightarrow{U} \mathbb{S}_q(\mathcal{C}at)$$

$$\begin{tikzcd}
A & & & A \arrow[dd] & & & A \arrow[dd] \\
& X \arrow[ul, "\text{b o.o.}"'] \arrow[dl, "\text{d. opf.}"] & \longmapsfrom & & X \arrow[ul, "\text{b o.o.}"'] \arrow[dl, "\text{d. opf.}"] & \longmapsto & \\
B & & & B & & & B
\end{tikzcd}$$

EXAMPLES WHERE (CO)MONADICITY HOLDS

$$\mathbb{K}\ell(\mathcal{C},Q)^\vee \xleftarrow{V} \mathbb{S}\mathrm{pEpi}(\mathcal{C},Q) \xrightarrow{U} \mathbb{S}\mathrm{q}(\mathcal{C})$$

$$\begin{tikzcd}
A & & A \arrow[dd] & & & A \arrow[dd] \\
 & \longmapsfrom & & QB \arrow[ul] \arrow[dl, "\varepsilon_B"] & \longmapsto & \\
QB \arrow[uu] & & B & & & B
\end{tikzcd}$$

- $\mathcal{C}$ has products $\Rightarrow$ V_1 is comonadic
- $\mathcal{C}$ has coproducts $\Rightarrow$ U_1 is monadic

$$\mathbb{R}\mathrm{et} \xleftarrow{V} \mathbb{L}\mathrm{ens} \xrightarrow{U} \mathbb{S}\mathrm{q}(\mathcal{C}at)$$

$$\begin{tikzcd}
A & & & A \arrow[dd] & & & A \arrow[dd] \\
 & X \arrow[ul, "\text{b o.o.}"'] \arrow[dl, "\text{d. opf.}"] & \longmapsfrom & & X \arrow[ul, "\text{b o.o.}"'] \arrow[dl, "\text{d. opf.}"] & \longmapsto & \\
B & & & B & & & B
\end{tikzcd}$$

- V_1 is comonadic ⇝ lenses are retrofunctors with coalgebraic structure.
- U_1 is monadic ⇝ lenses are the R-algebras for an AWFS on $\mathcal{C}at$.

EXAMPLES WHERE (CO)MONADICITY HOLDS

$$\mathbb{K}\ell(\mathcal{C},Q)^{\vee} \xleftarrow{V} \mathbb{S}\mathrm{pEpi}(\mathcal{C},Q) \xrightarrow{U} \mathbb{S}\mathrm{q}(\mathcal{C})$$

$$\begin{tikzcd}
A & & A \arrow[dd] & & & A \arrow[dd] \\
& \longmapsfrom & & QB \arrow[ul] \arrow[dl, "\varepsilon_B"] & \longmapsto & \\
QB \arrow[uu] & & B & & & B
\end{tikzcd}$$

- $\mathcal{C}$ has products $\Rightarrow$ V_1 is comonadic
- $\mathcal{C}$ has coproducts $\Rightarrow$ U_1 is monadic

Other examples giving AWFS include:

$\Gamma(\mathbb{A}\mathrm{dj}(\mathbb{K}))$ $\quad$ $\Gamma(\mathbb{P}\mathrm{b}\mathbb{S}\mathrm{q}(\mathcal{C})^{vh})^{vh}$

$$\mathbb{R}\mathrm{et} \xleftarrow{V} \mathbb{L}\mathrm{ens} \xrightarrow{U} \mathbb{S}\mathrm{q}(\mathcal{C}at)$$

$$\begin{tikzcd}
A & & & A \arrow[dd] & & & A \arrow[dd] \\
& X \arrow[ul, "\text{b o.o.}"'] \arrow[dl, "\text{d. opf.}"] & \longmapsfrom & & X \arrow[ul, "\text{b o.o.}"'] \arrow[dl, "\text{d. opf.}"] & \longmapsto & \\
B & & & B & & & B
\end{tikzcd}$$

- V_1 is comonadic ⇝ lenses are retrofunctors with coalgebraic structure.
- U_1 is monadic ⇝ lenses are the R-algebras for an AWFS on $\mathcal{C}at$.

SUMMARY & FUTURE WORK

$$\begin{tikzcd}
\mathsf{AWFS}_{lax} \arrow[d, hook, "\mathbb{A}\mathrm{lg}"] & & \mathsf{CAT}_{ld} \arrow[dll, "\mathbb{R}\mathrm{c}"'] \arrow[d, "\mathbb{V}"] \arrow[dl, phantom, "\Leftarrow"] \\
\mathsf{R_cDBL} \arrow[rr, hook, shift left=2] \arrow[rr, phantom, "\top"] \arrow[d, "\mathrm{obj}"] \arrow[drr, hook, "\mathbb{D} \mapsto \mathsf{R_cDBL}(\mathbb{R}\mathrm{c}(-), \mathbb{D})"] & & \mathsf{DBL} \arrow[ll, shift left=2, "\mathbb{\Gamma}"] \\
\mathsf{CAT} \arrow[u, hook, bend left=60, "\mathbb{H}"] \arrow[u, hook, bend right=60, "\$_q"'] \arrow[u, phantom, shift right=4, "\dashv"] \arrow[u, phantom, shift left=4, "\dashv"] & & {[\Delta^{op}, \mathsf{CAT}]}
\end{tikzcd}$$

- Constructed the right-connected completion $\mathbb{\Gamma}(\mathbb{D})$ of a double cat $\mathbb{D}$.
- In several examples, this gives an AWFS:

$$\mathbb{\Gamma}(\mathbb{R}\mathrm{et}) \cong \mathbb{L}\mathrm{ens}$$

SUMMARY & FUTURE WORK

$$\begin{tikzcd}
\mathrm{AWFS}_{\mathrm{lax}} \arrow[d, hook, "\mathbb{A}\mathrm{lg}"] & & \mathrm{CAT}_{\mathrm{ld}} \arrow[lld, "\mathbb{R}\mathrm{c}"'] \arrow[d, "\mathbb{V}"] \arrow[lld, phantom, shift right=6ex, "\Leftarrow"] \\
\mathrm{RcDBL} \arrow[rr, hook, shift left] \arrow[rr, phantom, "\top"] \arrow[d, "\mathrm{obj}" description] \arrow[rrd, "\mathbb{D} \mapsto \mathrm{RcDBL}(\mathbb{R}\mathrm{c}(-), \mathbb{D})"] & & \mathrm{DBL} \arrow[ll, shift left, "\mathbb{\Gamma}"] \\
\mathrm{CAT} \arrow[u, hook, shift left=3ex, "\mathbb{H}"] \arrow[u, hook, shift right=3ex, "\mathbb{S}\mathrm{q}"'] \arrow[u, phantom, shift left=1.5ex, "\dashv"] \arrow[u, phantom, shift right=1.5ex, "\dashv"] & & {[\Delta^{\mathrm{op}}, \mathrm{CAT}]}
\end{tikzcd}$$

- Constructed the right-connected completion $\mathbb{\Gamma}(\mathbb{D})$ of a double cat $\mathbb{D}$.
- In several examples, this gives an AWFS:

$$\mathbb{\Gamma}(\mathbb{R}\mathrm{et}) \cong \mathbb{L}\mathrm{ens}$$

- Can we extend $\mathbb{R}\mathrm{c}$ to a left 2-adjoint?
- When is $U_1 : \Gamma_1(\mathbb{D}) \longrightarrow \mathrm{Sq}(\mathcal{D}_0)$ monadic?
- Is there a right 2-adjoint of $\mathrm{AWFS}_{\mathrm{lax}} \hookrightarrow \mathrm{DBL}$?

BONUS: WHAT ABOUT COMPANIONS?

A double category has companions if for each horizontal morphism $f : A \to B$ there is a vertical morphism $f_* : A \nrightarrow B$ and cells

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "f_*"', "|" marking] \arrow[dr, phantom, "\heartsuit"] & \bullet \arrow[d, "\mathrm{id}", "|" marking] \\ \bullet \arrow[r, "1"'] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "1"] \arrow[d, "\mathrm{id}"', "|" marking] \arrow[dr, phantom, "\diamondsuit"] & \bullet \arrow[d, "f_*", "|" marking] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd} \qquad + \text{ axioms}$$

If $\mathbb{D}$ is right-connected, then:

$$\begin{tikzcd} \mathbb{S}\mathrm{q}(\mathcal{D}_0) \arrow[r, hook, "(-)_*"', bend right] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, "U"', bend right] \end{tikzcd}$$

For $\mathbb{D}$ with companions we have

$$\begin{tikzcd} & \mathbb{I}\Gamma(\mathbb{D}) \arrow[dl, "U"'] \arrow[dr, "V"] \arrow[d, phantom, "\Leftarrow"] & \\ \mathbb{S}\mathrm{q}(\mathcal{D}_0) \arrow[rr, hook, "(-)_*"'] & {} & \mathbb{D} \end{tikzcd}$$

the universal colax globular cone over $(-)_*$.

$$\mathbb{C} \longrightarrow \mathbb{I}\Gamma(\mathbb{D})$$

lax double functor

$$\begin{tikzcd} & \mathbb{C} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Leftarrow"] & \\ \mathbb{S}\mathrm{q}(\mathcal{D}_0) \arrow[rr, hook] & {} & \mathbb{D} \end{tikzcd}$$

globular transformation