D-K theorem for model categories

joint work with David White

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Introduction: Classical result

Theorem 1 (Dwyer-Kan, [1]). *Let $f\colon (A,U)\to(B,V)$ be a map of relative categories, then the adjunction $\mathrm{Lan}_f\colon \mathcal{S}^{A,U}\to\mathcal{S}^{B,V} : f^*$ between the categories of homotopy functors is a Quillen equivalence iff the induced map of simplicial localizations $Lf\colon L(A,U)\to L(B,V)$ is an* r-equivalence *of simplicial categories.*

Definition 2. A map $f\colon A\to B$ of simplicial categories is an *r-equivalence* if

- $\forall a_1,a_2\in A$, the induced map $\hom_A(a_1,a_2)\to\hom_B(fa_1,fa_2)$ is a weak equivalence, and
- $\forall b\in\pi_0 B \quad \exists a\in\pi_0 A$, s.t. b is a retract of fa.

Preliminaries: Induced Quillen maps

Let $L\colon\mathcal{A}\to\mathcal{B}$ be a simplicial *accessible* functor between simplicial *combinatorial* model categories, then there is an adjunction

$$\mathrm{Lan}_L\colon \mathcal{S}^{\mathcal{A}}\leftrightarrows\mathcal{S}^{\mathcal{B}}:L^*$$

Moreover, the adjunction $\mathrm{Lan}_L\dashv L^*$ is a Quillen map if the categories of *small functors* equipped with

- the projective model structure;
- the fibrant-projective model structure iff L preserves fibrant objects;
- the bifibrant-projective model structure if L preserves both fibrant and cofibrant objects.

Generalizing Homotopy functors

Classical case: All objects are cofibrant

Homotopy functors are functors preserving weak equivalences.

Let $\mathcal{A}$ be a model category, if we localize the projective model structure on $\mathcal{S}^{\mathcal{A}}$ with respect to

$$\mathcal{F}_{\mathcal{A}}=\{\hom(a_1,-)\to\hom(a_2,-)\,|\,a_1\tilde{\to}a_2\},$$

then the local objects are precisely the fibrant homotopy functors.

Theorem 3. *Let $\mathcal{A}$ be a simplicial combinatorial model category with* all objects cofibrant*, then there exists the left Bousfield localization $\mathcal{S}^{\mathcal{A}}[\mathcal{F}_{\mathcal{A}}^{-1}]$, s.t. {fibrant objects}={proj. fibrant homotopy functors}.*

General case: Not all objects are cofibrant

More generally, $\forall F\in S^{\mathcal{A}}$ there is an $\mathcal{F}_{\mathcal{A}}$-equivalence $F\to QF$ with QF a homotopy functor.

Conjecture 4. For every simplicial combinatorial model category $\mathcal{A}$ there exists the homotopy model structure on $\mathcal{S}^{\mathcal{A}}$.

Assuming the above conjecture,

Theorem 5. *Let $\mathcal{A}$ be a simplicial combinatorial model, then there is a Quillen equivalence*

$$\mathrm{Id}\colon \mathcal{S}^{\mathcal{A}}_{\textit{bifib-proj}}\leftrightarrows\mathcal{S}^{\mathcal{A}}_{\textit{ho}}:\mathrm{Id}$$

Main result: Dwyer-Kan theorem for model categories

Definition 6. A simplicial functor $f\colon\mathcal{A}\to\mathcal{B}$ between simplicial model categories is an *r-equivalence* if

- $\forall a_1,a_2\in\mathcal{A}$ bifibrant, $\hom(a_1,a_2)\tilde{\to}\hom(f(a_1),f(a_2))$;
- $\forall b\in\mathcal{B}$ bifibrant $\exists a\in\mathcal{A}$, s.t. b is a homotopy retract of $f(a)$.

Example 7. Let $L\colon\mathcal{A}\leftrightarrows\mathcal{B}:R$ be a Quillen equivalence of simplicial combinatorial model categories. Then $f=\widehat{L}$ is an r-equivalence.

Theorem 8 ([2]). *Let $f\colon\mathcal{A}\to\mathcal{B}$ be an accessible functor between simplicial combinatorial model categories* preserving fibrant and cofibrant objects. *Then the Quillen pair*

$$\mathrm{Lan}_f\colon\mathcal{S}^{\mathcal{A}}_{\textit{bifib-proj}}\leftrightarrows\mathcal{S}^{\mathcal{B}}_{\textit{bifib-proj}}:f^*$$

is a Quillen equivalence iff f is an r-equivalence.

Application: Invariance of Goodwillie calculus under Quillen equivalence

Theorem 9. *Let $L\colon\mathcal{A}\leftrightarrows\mathcal{B}:R$ be a Quillen equivalence between simplicial combinatorial model categories. Put $f=\widehat{L}$. Then*

$$\mathrm{Lan}_f\colon\mathcal{S}^{\mathcal{A}}\leftrightarrows\mathcal{S}^{\mathcal{B}}:f^*$$

is a Quillen equivalence of bifibrant-projective model structures. Moreover, if we localize both sides so that the local objects are the n-excisive functors and the adjunction $\mathrm{Lan}_f\dashv f^*$ is a Quillen equivalence between the n-polynomial functors on $\mathcal{A}$ and $\mathcal{B}$ taking values in simplicial sets.

References

[1] W. G. Dwyer and D. M. Kan. Equivalences between homotopy theories of diagrams. In *Algebraic topology and algebraic K-theory (Princeton, N.J., 1983)*, volume 113 of *Ann. of Math. Stud.*, pages 180–205. Princeton Univ. Press, Princeton, NJ, 1987.

[2] B. Chorny and D. White. A variant of Dwyer-Kan Theorem for model categories. *Algebraic and Geometric Topology*, to appear.