Spectra of Modelled Spaces à la Coste, Revisited

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Abstract

We provide an alternative construction of **spectra of models of a cartesian theory** in the sense of Coste [3] and prove that it is equivalent to Coste's original construction. This construction can be extended to **spectra of modelled spaces** and used to show that the involved categories of modelled spaces are complete and cocomplete. This work has recently been published as [1].

Introduction

It is well-known that Zariski spectra of commutative rings give rise to a dual adjunction between the categories of commutative rings and of locally ringed spaces:

$$\mathbf{LRS} \underset{\mathrm{Spec}}{\overset{\Gamma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Ring}^{\mathrm{op}}$$

There are several frameworks for synthesizing dual adjunctions between categories of algebras and of "locally algebra-ed" spaces (or toposes). Coste's theory of spectra [3] is such a framework. It was greatly inspired by Cole's theory of topos-theoretic spectra [2], but the conditions involved in the former are much more easily verified. However, most of his proofs remain unpublished, and Osmond [7] has recently filled the details in the case of general (not necessarily spatial) Coste contexts (see the definition below). These previous works heavily rely on topos theory, and it is difficult to follow the arguments and obtain a concrete description of spectra.

Our Aim

Give a simpler description of Coste spectra in the *spatial* case.

The settings

Definition. A **Coste context** is a triple (T_0, T, Λ) such that

- T_0 is a (for simplicity, single-sorted) cartesian $\mathcal{L}$-theory,
- Λ is a set of pairs $(\varphi(\boldsymbol{u}), \psi(\boldsymbol{u}, \boldsymbol{v}))$ of conjunctions of atomic $\mathcal{L}$-formulas with
$$T_0 \models \psi(\boldsymbol{u}, \boldsymbol{v}) \vdash \varphi(\boldsymbol{u}),$$
- T is a coherent $\mathcal{L}$-theory obtained by adding to T_0 some axioms of the form
$$\varphi(\boldsymbol{u}) \vdash \bigvee_i \exists \boldsymbol{v}_i \psi_i(\boldsymbol{u}, \boldsymbol{v}_i)$$
for finitely many pairs $(\varphi(\boldsymbol{u}), \psi_i(\boldsymbol{u}, \boldsymbol{v}_i)) \in \Lambda$.

Moreover, we say a Coste context is **spatial** when each $(\varphi, \psi) \in \Lambda$ satisfies
$$T_0 \models \psi(\boldsymbol{u}, \boldsymbol{v}) \wedge \psi(\boldsymbol{u}, \boldsymbol{v'}) \vdash \boldsymbol{v} = \boldsymbol{v'}.$$

Definition. A homomorphism $\alpha\colon A \to B$ of T_0-models is **admissible** if
$$A \models \varphi(\boldsymbol{a}) \text{ and } B \models \psi(\alpha(\boldsymbol{a}), \boldsymbol{b}) \implies \exists! \boldsymbol{a'} \in A,\ A \models \psi(\boldsymbol{a}, \boldsymbol{a'}) \text{ and } \alpha(\boldsymbol{a'}) = \boldsymbol{b}$$
for each $(\varphi, \psi) \in \Lambda$.

T_0-$\mathbf{ModSp}$ denotes the category of T_0-**modelled spaces**. An object is a pair (X, P) of a topological space X and $P \in T_0\text{-}\mathbf{Mod}(\mathbf{Sh}(X))$ (sheaf of T_0-models). A morphism $f\colon (X, P) \to (Y, Q)$ consists of a continuous map $f\colon X \to Y$ and a morphism $f^\flat\colon f^*Q \to P$ in T_0-$\mathbf{Mod}(\mathbf{Sh}(X))$.

$\mathbb{A}$-$\mathbf{ModSp}$ denotes the subcategory of T_0-$\mathbf{ModSp}$ consisting of T**-modelled spaces and admissible morphisms**. An object is a T_0-modelled space (X, P) for which each stalk P_x is a T-model. A morphism $f\colon (X, P) \to (Y, Q)$ is admissible when each $f^\flat_x\colon Q_{fx} \to P_x$ is admissible.

Examples

The following are Coste contexts for Zariski spectra and prime filter spectra.

- T_0 = the theory of commutative rings.
$\Lambda = \{(u = u, uv = 1),\ (u = u, (1-u)v = 1)\}$.
T = the theory of local rings containing the additional axioms
$$0 = 1 \vdash \bot, \qquad u = u \vdash \exists v(uv = 1) \vee \exists v((1-u)v = 1).$$
Admissible homomorphisms are homomorphisms reflecting units.
- T_0 = the theory of distributive lattices in the language $\{0, 1, \sup, \inf\}$.
$\Lambda = \{(\sup(u, u') = 1, u = 1),\ (\sup(u, u') = 1, u' = 1)\}$.
T = the theory of local lattices containing the additional axioms
$$0 = 1 \vdash \bot, \qquad \sup(u, u') = 1 \vdash u = 1 \vee u' = 1.$$
Admissible homomorphisms are homomorphisms reflecting 1.

Other examples include contexts for field spectra, domain spectra, Pierce spectra, and real spectra of commutative rings (cf. [6, Chap. V]).

Main construction

For a spatial Coste context (T_0, T, Λ) and a T_0-model A, Coste defined a join-semilattice $\mathcal{V}_A$ which is (categorically equivalent to) a subcategory of the coslice category $A \backslash T_0$-$\mathbf{Mod}$. Let A_λ denote the codomain of $\lambda \in \mathcal{V}_A$. Every $\lambda\colon A \to A_\lambda$ is an epimorphism in T_0-$\mathbf{Mod}$.

The underlying space. (essentially identical to Coste's original construction)
$$\mathrm{Spec}(A) := \left\{ I \subseteq \mathcal{V}_A \;\middle|\; I \text{ is an ideal, and } A_I := \varinjlim_{\lambda \in I} A_\lambda \text{ is a } T\text{-model.} \right\}$$
The topology is generated by the basic open sets $D_\lambda := \{ I \in \mathrm{Spec}(A) \mid \lambda \in I \}$ for $\lambda \in \mathcal{V}_A$. Then $\mathrm{Spec}(A)$ is a spectral space.

The structure sheaf. For any open $U \subseteq \mathrm{Spec}(A)$,
$$SU := \left\{ s \in \prod_{I \in U} A_I \;\middle|\; \begin{array}{l} \forall J \in U,\ \exists \lambda \in J \text{ (i.e. } J \in D_\lambda),\ \exists a \in A_\lambda, \\ D_\lambda \subseteq U, \text{ and } \forall I \in D_\lambda,\ s_I = a_I \end{array} \right\},$$
where a_I is the image of a under $A_\lambda \to A_I$. We obtain the structure sheaf S.

Notice that the canonical morphism $A_\lambda \to S(D_\lambda)$ need not be an isomorphism.

Key observation: For any $I \in \mathrm{Spec}(A)$, the stalk S_I is isomorphic to A_I.

Theorem. Spec is a right adjoint to the global section functor Γ.

$$\mathbb{A}\text{-}\mathbf{ModSp} \underset{\mathrm{Spec}}{\overset{\Gamma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} T_0\text{-}\mathbf{Mod}^{\mathrm{op}}$$

These can be proved by diagram chasing in T_0-$\mathbf{Mod}$, and we believe it is easier to follow the proof than Coste's original one. The following theorem is more involved.

Theorem. The above construction yields a T-modelled topos $(\mathbf{Sh}(\mathrm{Spec}(A)), S)$ which is equivalent to Coste's original spectrum $(\mathbf{Sh}(\mathcal{V}_A^{\mathrm{op}}, C), \tilde{P})$, where $\tilde{P}$ is the associated sheaf of $P\colon \mathcal{V}_A \to T_0\text{-}\mathbf{Mod}$ $(\lambda \mapsto A_\lambda)$.

Limits, colimits, and spectra of modelled spaces

The above construction is extended to relative spectra of modelled spaces.

Theorem. For any T-modelled space (X, P), there exists an adjunction:

$$\mathbb{A}\text{-}\mathbf{ModSp}/(X, P) \underset{\mathrm{Spec}}{\overset{\text{forgetful}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} T_0\text{-}\mathbf{ModSp}/(X, P)$$

Relativization of spectra is crucial to yield the limit part of the following theorem:

Theorem. T_0-$\mathbf{ModSp}$ and $\mathbb{A}$-$\mathbf{ModSp}$ have small limits and colimits.

We would like to emphasize that a spatial Coste context is an appropriate setting for treating limits and colimits of modelled spaces. Completeness and cocompleteness of $\mathbf{LRS}$ were proved in [5] and [4], respectively. As an interesting corollary of the above theorem, we can see that the category of ringed spaces whose stalks are fields is complete and cocomplete.

Future directions

We are interested in models represented by T-modelled spaces.

- When is the Coste adjunction idempotent?
- If the adjunction is idempotent, then how well does the subcategory $\mathrm{Im}(\Gamma) = \mathrm{Fix}(\eta) \subseteq T_0$-$\mathbf{Mod}$ behave? For example, when is it a (pseudo-)elementary class?

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