The Univalence Principle

Univalent foundations for the formalization of (higher) category theory

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Univalent foundations for the formalization of (higher) category theory

Univalent foundations A particular foundation of mathematics, developed by V. Voevodsky and others

Formalization Formulation of reasoning in a precise (possibly machine-checkable) format ⇝ computer proof assistants

Category theory . . .

Goal of my talk

Show how univalent foundations provide a suitable foundation for (computer) formalization of category theory

Univalent foundations for category theory — in a nutshell

1. **Sameness** is a fundamental concept in mathematics
2. Set-theoretic equality is often **not** the desired notion of sameness
3. Category theory provides a suitable notion of sameness — isomorphism — for mathematical objects
4. Univalent foundations allow one to express reasoning exactly modulo this kind of sameness

In short

In set theory foundational and mathematical sameness **diverge**

In UF foundational and mathematical sameness **coincide**

Good vs Evil

Good category theory

Objects:	up to isomorphism
Sameness:	equivalence of categories

Evil category theory

Objects:	up to equality
Sameness:	isomorphism of categories

Good vs Evil

Good category theory

Objects: up to isomorphism

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Evil category theory

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Sameness: isomorphism of categories

How to know good from evil?

Separating Sense from Nonsense

- Blanc, Freyd: first-order logic which is invariant under equivalence of categories
- Makkai's FOLDS: language on top of set theory for good statements

Makkai, *Towards a Categorical Foundation of Mathematics*

> *The basic character of the Principle of Isomorphism is that of a* ***constraint on the language*** *of Abstract Mathematics; a welcome one, since it provides for the separation of sense from nonsense.*

Univalent foundations and the univalence principle

Vladimir Voevodsky's vision

- Univalent Foundations as an "invariant language"
- Any construction on objects in UF can be transported along equivalences of objects

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- Any construction on objects in UF can be transported along equivalences of objects

UP for groups (Coquand, Danielsson, 2012)

$$(G \cong H) \quad \to \quad \forall P, (P(G) \longleftrightarrow P(H))$$

UP for categories (A., Kapulkin, Shulman, 2013)

$$(A \simeq B) \quad \to \quad \forall P, (P(A) \longleftrightarrow P(B))$$

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In this talk

UP for mathematical structures specified by a theory

Outline

1. A Brief Description of Univalent Foundations
2. Univalence Principle for Categories
3. Univalence Principle for Mathematical Structures

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Overview of univalent foundations

- Primitive objects are ∞-groupoids/types/spaces A, B,...
- Objects/0-cells

$$x : A, y : B, \ldots$$

- Higher cells

$$f, f' : x =_A x'$$
$$\alpha : f =_{(x=x')} f'$$

$x =_A x'$ the "identity type" from x to x'

- Type $A \to B$ of functions/functors
- Type $\mathcal{U}$ ("universe") of types

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Homotopy levels: stratification of types $(A, =_A, =_{=_A}, \ldots)$

- **Propositions** are types with at most one object
- **Sets** are discrete types

Voevodsky's Univalence Axiom

Constructions are invariant under identities

$$\mathsf{transport} : x = y \to \prod_{B:A\to\mathcal{U}} \big(B(x) \simeq B(y)\big)$$

Univalence axiom

Specifies the identity type of a universe:

$$(X =_{\mathcal{U}} Y) \to (X \simeq Y)$$
$$\mathsf{refl}(X) \mapsto \mathrm{I}_X$$

is an equivalence

Constructions are invariant under equivalence of types

$$\mathsf{transport} : (X \simeq Y) \to \prod_{B:\mathcal{U}\to\mathcal{U}} \big(B(X) \simeq B(Y)\big)$$

Outline

Definition of category, set-based

A **strict category** $\mathscr{C}$ is given by

- a **set** $\mathscr{C}_0 : \mathscr{U}$ of objects
- for any $a, b : \mathscr{C}_0$, a **set** $\mathscr{C}(a,b) : \mathscr{U}$ of morphisms
- operations: identity & composition

$$\begin{aligned} 1_a &: \mathscr{C}(a,a) \\ (\circ)_{a,b,c} &: \mathscr{C}(b,c) \to \mathscr{C}(a,b) \to \mathscr{C}(a,c) \end{aligned}$$

- axioms: unitality & associativity

$$1 \circ f = f \qquad f \circ 1 = f \qquad (h \circ g) \circ f = h \circ (g \circ f)$$

Definition of category, space-based

A **univalent category** $\mathscr{C}$ is given by

- a **space** $\mathscr{C}_o : \mathscr{U}$ of objects
- for any $a, b : \mathscr{C}_o$, a **space** $\mathscr{C}(a, b) : \mathscr{U}$ of morphisms
- operations: identity & composition

$$\begin{aligned} \mathrm{I}_a &: \mathscr{C}(a, a) \\ (\circ)_{a,b,c} &: \mathscr{C}(b, c) \to \mathscr{C}(a, b) \to \mathscr{C}(a, c) \end{aligned}$$

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$$\mathrm{I} \circ f = f \qquad f \circ \mathrm{I} = f \qquad (h \circ g) \circ f = h \circ (g \circ f)$$

- $\mathscr{C}(a, b)$ is a discrete space (a set in the sense of UF)
- **completeness:** $a = b \to \mathsf{iso}(a, b)$ is an equivalence

Categories in univalent foundations

In UF, we have two kinds of categories:

Strict: category whose objects and morphisms consist of **discrete** spaces

- Come with equality on objects
- Sameness of categories: isomorphism

Univalent: objects are given by a 1-groupoid, morphisms are discrete spaces

- Do not come with equality on objects
- Sameness of categories: equivalence

Univalence principle for univalent categories

Univalence Principle (A., Kapulkin, Shulman)

$$(A = B) \simeq (A \simeq B)$$

$$(A \simeq B) \to \prod_{T:\mathsf{uCat}\to\mathcal{U}} \big(T(A) \simeq T(B)\big)$$

Univalent category theory is automatically good.

Remarks

- Many categories (sets, groups, . . .) are univalent
- "Free completion" operation to build univalent categories
- Ess. surj. and f. f. functor admits a quasi-inverse without AC

Outline

How to generalize UP from categories to other things

The Univalence Principle (A., North, Shulman, Tsementzis)

- Signature

$\rightsquigarrow$ Structures of a signature

$\rightsquigarrow$ Equivalence ($\simeq$) of structures

$\rightsquigarrow$ Univalence of structures

$+$ Notion of axiom $\rightsquigarrow$ Theory and models

Theorem (Univalence Principle)

Given a theory $\mathscr{T}$ and univalent models M, N of $\mathscr{T}$,

$$(M = N) \simeq (M \simeq N)$$

Example (Theory of categories)

- Equivalences: essentially surjective and fully faithful functors
- Univalent models: univalent categories

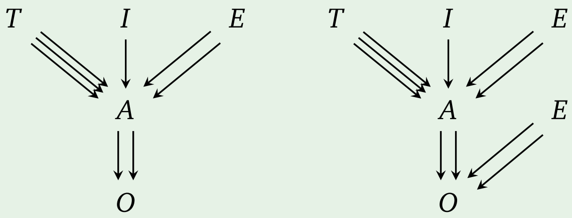
Signatures for mathematical structures

Two notions of signature

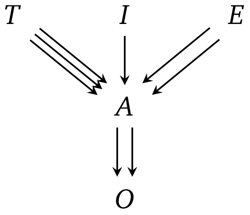
Diagram
- **Sorts** and dependencies between them
- Inspired by Makkai's FOLDS signatures

Functorial
- In terms of functors and natural transformations
- More general than diagram signatures

Example (Signatures for univalent and strict categories)



Structures of the signature of univalent categories



Structure M consists of

$O : \mathcal{U}$

$A : O \times O \to \mathcal{U}$

$I : \prod_{x:O} A(x,x) \to \mathcal{U}$

$T : \prod_{x,y,z:O} A(x,y) \times A(y,z) \times A(x,z) \to \mathcal{U}$

$E : \prod_{x,y,z:O} A(x,y) \times A(x,y) \to \mathcal{U}$

A category is a structure M satisfying the axioms:

1. Existence & uniqueness of composition
2. Existence of identity arrows
3. Categorical axioms
4. E is congruence for I, T
5. E is $=$
6. I, T, E pointwise propositions
7. A is pointwise discrete
8. $(a = b) \simeq (a \cong b)$ for $a, b : O$

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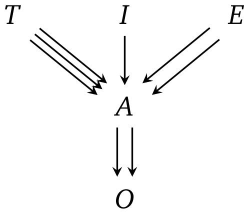

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A category is a **univalent** structure M satisfying the axioms:

1. Existence & uniqueness of composition
2. Existence of identity arrows
3. Categorical axioms
4. E is congruence for I, T

Steps to defining **univalence** for structures

1. Reformulate the type of isomorphisms $a \cong b$ in a category
   - $\rightsquigarrow$ **Indiscernibilities** $a \asymp b$
   - Definition of indiscernibility only depends on the **signature** of categories
2. Definition of indiscernibilities can be applied to each sort in any signature
3. Define univalence for models of any theory by requiring

$$(a = b) \to (a \asymp b)$$

   to be an equivalence

Given $a, b : O$, what is an indiscernibility $i : a \asymp b$?

1. For each $x : O$, an equivalence $\phi_{x\bullet} : A(x,a) \simeq A(x,b)$.
2. For each $z : O$, an equivalence $\phi_{\bullet z} : A(a,z) \simeq A(b,z)$.
3. An isomorphism $\phi_{\bullet\bullet} : A(a,a) \simeq A(b,b)$.

$$T_{x,y,a}(f,g,h) \simeq T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$$
$$T_{x,a,z}(f,g,h) \simeq T_{x,b,z}(\phi_{x\bullet}(f), \phi_{\bullet z}(g), h)$$
$$T_{a,z,w}(f,g,h) \simeq T_{b,z,w}(\phi_{\bullet z}(f), g, \phi_{\bullet w}(h))$$
$$T_{x,a,a}(f,g,h) \simeq T_{x,b,b}(\phi_{x\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{x\bullet}(h))$$
$$T_{a,x,a}(f,g,h) \simeq T_{b,x,b}(\phi_{\bullet x}(f), \phi_{x\bullet}(g), \phi_{\bullet\bullet}(h))$$
$$T_{a,a,x}(f,g,h) \simeq T_{b,b,x}(\phi_{\bullet\bullet}(f), \phi_{\bullet x}(g), \phi_{\bullet x}(h))$$
$$T_{a,a,a}(f,g,h) \simeq T_{b,b,b}(\phi_{\bullet\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{\bullet\bullet}(h))$$
$$I_{a,a}(f) \simeq I_{b,b}(\phi_{\bullet\bullet}(f))$$
$$E_{x,a}(f,g) \simeq E_{x,b}(\phi_{x\bullet}(f), \phi_{x\bullet}(g))$$
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$$
\begin{aligned}
T_{x,y,a}(f,g,h) &\simeq T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h)) \\
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\end{aligned}
$$

Theorem

$$(a \asymp b) \simeq (a \cong b)$$

Indiscernibilities and univalence condition

“Definition”

Given a signature $\mathscr{L}$ with a sort K, and a structure M of $\mathscr{L}$ and $a, b : K$, an **indiscernibility** $i : a \asymp b$ is a family of equivalences of fibers over a and b in M.

Example (Indiscernibilities in a category)

- For $a, b : O$ in a category, $(a \asymp b) \simeq (a \cong b)$
- For $f, g : A(a, b)$, $(f \asymp g) \simeq E(f, g)$
- For $p, q : T(f, g, h)$, $(p \asymp q) \simeq 1$

Definition (Univalence of structures)

Given an $\mathscr{L}$-structure M and sort K of $\mathscr{L}$, M is **univalent at** K if for any $a, b : K$, the function $(a = b) \;\to\; (a \asymp b)$ is an equivalence.

Univalent categories

Example

A structure for categories is univalent if

- For $a, b : O$ in a category, $(a = b) \simeq (a \cong b)$
- For $f, g : A(a, b)$, $(f = g) \simeq E(f, g)$
- For $p, q : T(f, g, h)$, $(p = q) \simeq 1$

Given a model M for the theory of categories,

5. E is $=$
6. I, T, E pointwise propositions
7. A is pointwise discrete
8. $(a = b) \simeq (a \cong b)$ for any $a, b : O$

follow from M being univalent at I, T, E, A, and O.

Theorem

The univalent models of the theory of categories are exactly univalent categories as defined by AKS13.

Univalence principle and examples

Theorem (Univalence Principle)

Given a theory $\mathscr{T}$ and univalent models M, N of $\mathscr{T}$,

$$(M = N) \ \simeq \ (M \simeq N)$$

- First-order logic (with equality)
- Higher-order logic, e.g., topological spaces, suplattices
- Categories
- Dagger categories
- (Ana)functors
- Profunctors
- Displayed categories / Fibrations
- Bicategories
- Double (bi)categories
- ...

Conclusion

Summary

- Univalence Principle: constructions in UF are invariant under equivalence of univalent structures
- Univalence condition on structures encompasses completeness condition $(a = b) \simeq (a \cong b)$ for categories
- Many proofs by induction on height of a signature

Open questions

- Completion operation for structures other than categories?
- What about structures of infinite height?

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- What about structures of infinite height?

Thanks for your attention!

References

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