

Sweedler theory for double categories

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5 July 2023

Outline

1. Motivation and background
2. Sweedler theory for monoidal double categories
3. Oplax monoidal double structure
4. Further directions

Fibered versus enriched categories

- A \mathcal{V} -enriched category has hom-objects that belong to \mathcal{V} , and composition rule is a morphism in \mathcal{V} , for a monoidal category \mathcal{V}
- A category fibered over a category \mathcal{B} is an ordinary category, whose objects and morphisms lie over specified objects and morphisms in \mathcal{B} , with certain (cartesian) liftings

... a common pattern in what follows is

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{\text{enriched}} & \mathcal{D} \\
 \text{fibered} \downarrow & & \downarrow \text{opfibered} \\
 \mathcal{A} & \xrightarrow{\text{enriched}} & \mathcal{B}
 \end{array}$$

★ Bunge / Shulman: *enriched indexed categories*, over a fixed base. Above picture? Theory of *enriched fibrations*, over different bases.

Monoids and comonoids

Suppose $(\mathcal{V}, \otimes, I)$ is monoidal category.

► A *monoid* is an object A together with maps $\mu: A \otimes A \rightarrow A$ and $\eta: I \rightarrow A$ which are associative and unital:

$$\begin{array}{ccc}
 A \otimes A \otimes A & \xrightarrow{\mu \otimes 1} & A \otimes A \\
 1 \otimes \mu \downarrow & & \downarrow \mu \\
 A \otimes A & \xrightarrow{\mu} & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 I \otimes A & \xrightarrow{\eta \otimes 1} & A \otimes A & \xleftarrow{1 \otimes \eta} & A \otimes I \\
 & \searrow & \downarrow \mu & & \swarrow \\
 & & A & &
 \end{array}$$

► Dually, a *comonoid* is an object C together with maps $\delta: C \rightarrow C \otimes C$ and $\epsilon: C \rightarrow I$ which are coassociative and counital.

★ These form categories Mon and Comon, with maps preserving structure.

When $(\mathcal{V}, \otimes, I, \sigma)$ is braided, Mon and Comon are monoidal, with I and

$$A \otimes B \otimes A \otimes B \xrightarrow{1 \otimes \sigma \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{\mu \otimes \mu} A \otimes B$$

■ If \mathcal{V} is also closed, induced $[-, -]: \text{Comon}(\mathcal{V})^{\text{op}} \times \text{Mon}(\mathcal{V}) \rightarrow \text{Mon}(\mathcal{V})$ makes $[C, A]$ into a monoid via *convolution*

$$\begin{array}{ccc}
 [C, A] \otimes [C, A] \otimes C & \xrightarrow{1 \otimes \delta} & [C, A] \otimes [C, A] \otimes C \otimes C \xrightarrow{1 \otimes \sigma \otimes 1} [C, A] \otimes C \otimes [C, A] \otimes C \\
 & & \downarrow \text{ev} \otimes \text{ev} \\
 & & A \otimes A \\
 & & \downarrow \mu \\
 & & A
 \end{array}$$

★ In Vect_k , linear dual $C^* = \text{Hom}_k(C, k)$ for a k -coalgebra is a k -algebra – while A^* for a k -algebra is a k -coalgebra only if it is finite dimensional.

Sweedler dual ‘fixes’ that: $A^\circ = \{f \in A^* \mid \ker f \text{ contains cofinite ideal}\}$ is a k -coalgebra, and $\text{Alg}(A, C^*) \cong \text{Coalg}(C, A^\circ)$ by adjunction

$$\begin{array}{ccc}
 & (-)^* & \\
 \text{Coalg}^{\text{op}} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \text{Alg} \\
 & ((-)^{\circ})^{\text{op}} &
 \end{array}$$

'Sweedler theory' for monoidal categories

► *Universal measuring k -coalgebra* $\text{Alg}(A, [C, B]) \cong \text{Coalg}(C, P(A, B))$, algebraically as terminal object in category of k -coalgebras that *measure*.

★ Moving to general context of braided monoidal closed categories, local presentability gives passage from Vect_k to $(d)\text{gVect}_k$, Mod_R & many more.

Suppose \mathcal{V} is a braided monoidal closed and locally presentable category. There is a parameterized adjunction between

$$\begin{aligned} [-, -] : \text{Comon}^{\text{op}} \times \text{Mon} &\rightarrow \text{Mon} && \text{convolution} \\ P(-, -) : \text{Mon}^{\text{op}} \times \text{Mon} &\rightarrow \text{Comon} && \text{universal measuring} \end{aligned}$$

$P(A, B) = \int^C \text{Mon}(A, [C, B]) \cdot C$ in Set is $\text{Mon}(A, B)$; in Vect_k contains k -algebra maps as grouplike; in dgVect_k , relates to bar-cobar adjunction.

► Convolution $[-, -]$ is an *action* of the monoidal $\text{Comon}^{(\text{op})}$ on Mon .

Any parameterized adjoint of an action $\bullet: \mathcal{V} \times \mathcal{C} \rightarrow \mathcal{C}$ gives rise to a \mathcal{V} -enriched structure on \mathcal{C} , and all tensored \mathcal{V} -categories arise this way.

If \mathcal{V} further symmetric, Mon is enriched in symmetric monoidal Comon .

Digression: *semi-Hopf* \mathcal{V} -categories generalize bimonoids

- $H(x, y) \otimes H(y, z) \rightarrow H(z, x)$, $I \rightarrow H(x, x)$ ‘global’ multipl
- $H(a, b) \rightarrow H(a, b) \otimes H(a, b)$, $H(a, b) \rightarrow I$ ‘local’ comultipl

$$\begin{array}{ccc}
 H_{x,y} \otimes H_{y,z} & \xrightarrow{\delta_{xy} \otimes \delta_{yz}} & H_{x,y} \otimes H_{x,y} \otimes H_{y,z} \otimes H_{y,z} \\
 \mu_{xyz} \downarrow & & \downarrow (\mu_{xyz} \otimes \mu_{xyz}) \circ (1 \otimes \sigma \otimes 1) \\
 H_{x,z} & \xrightarrow{\delta_{xz}} & H_{x,z} \otimes H_{x,z}
 \end{array}$$

The category of monoids in \mathcal{V} is a semi-Hopf \mathcal{V} -category.

(Co)modules enter the picture

► For monoid A , an A -module M comes with associative and unital $\mu: A \otimes M \rightarrow M$, and dually a C -comodule X comes with $\chi: X \rightarrow C \otimes X$. With maps preserving (co)actions, categories ${}_A\text{Mod}$ and ${}_C\text{Comod}$.

■ 'Global' categories Mod , Comod of (co)modules for any (co)monoid, maps for Mod are $g: {}_A M \rightarrow {}_B N$ in \mathcal{V} with $f: A \rightarrow B$ in Mon that

$$\begin{array}{ccc} A \otimes M & \xrightarrow{\mu} & M \\ 1 \otimes g \downarrow & & \downarrow g \\ A \otimes N & \xrightarrow{f \otimes 1} B \otimes N \xrightarrow{\mu} & N \end{array}$$

★ Naturally form fibration $\text{Mod} \rightarrow \text{Mon}$ and opfibration $\text{Comod} \rightarrow \text{Comon}$.

If \mathcal{V} symmetric monoidal closed & locally presentable, adjunction between

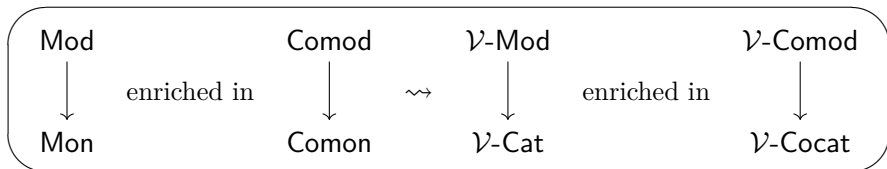
$$[-, -]: \text{Comod}^{\text{op}} \times \text{Mod} \rightarrow \text{Mod} \quad \text{convolution}$$

$$Q(-, -): \text{Mod}^{\text{op}} \times \text{Mod} \rightarrow \text{Comod} \quad \text{universal measuring}$$

and Mod is enriched in the symmetric monoidal Comod .

From one to many objects

Goal: generalize from monoids in \mathcal{V} , to \mathcal{V} -categories!



Method: work ‘bottom-up’ (technical but direct), or identify general framework and work ‘top-down’

- ▶ A \mathcal{V} -module M consists of $\{M(x)\}_{x \in X}$ in \mathcal{V} with an action $(\sum_y)A(x, y) \otimes M(y) \rightarrow M(x)$ for A some \mathcal{V} -category ($A \rightsquigarrow \mathcal{I}$).
- ▶ For \mathcal{V} with coproducts preserved by \otimes , a \mathcal{V} -cocategory C consists of $\{C(x, z)\}_X$ in \mathcal{V} with $C(x, z) \rightarrow \sum_y C(x, y) \otimes C(y, z)$, $C(x, x) \rightarrow I$.
- ▶ For \mathcal{V} with coproducts preserved by \otimes , a \mathcal{V} -comodule K consists of $\{K(x)\}_{x \in X}$ in \mathcal{V} with a coaction $K(x) \rightarrow \sum_y C(x, y) \otimes K(y)$.

From monoidal to double categories

★ Opcategories = \mathcal{V}^{op} -categories? Not as convenient, formally!

▶ A double category \mathbb{D} has \mathbb{D}_0 (0-cells & vertical 1-cells), \mathbb{D}_1 (horizontal 1-cells & 2-maps) and $\mathbb{D}_0 \xrightarrow{1} \mathbb{D}_1$, $\mathbb{D}_1 \xrightarrow[t]{s} \mathbb{D}_0$, $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\circ} \mathbb{D}_1$ + coherent isos.

▶ A monad in a double category \mathbb{D} is $A: X \rightarrow X$ with associative, unital

$$\begin{array}{ccccc}
 X & \xrightarrow{A} & X & \xrightarrow{A} & X & & X & \xrightarrow{1_X} & X \\
 \parallel & & \downarrow \mu & & \parallel & & \parallel & \downarrow \eta & \parallel \\
 X & \xrightarrow{\quad} & X & \xrightarrow{\quad} & X & & X & \xrightarrow{A} & X
 \end{array}$$

Dually, comonad $C: X \rightarrow X$. These form categories $\text{Mnd}(\mathbb{D})$ & $\text{Cmd}(\mathbb{D})$.

★ Morphisms are different than those for (co)monads in bicategories.

For $\mathbb{D} = \mathcal{V}\text{-Mat}$ of sets, functions and \mathcal{V} -matrices $S: X \rightarrow Y$ i.e.

$$\{S(x, y)\} \in \mathcal{V} \text{ with } (S \circ T)(x, z) = \sum_y T(x, y) \otimes S(y, z),$$

$$\text{Mnd}(\mathcal{V}\text{-Mat}) = \mathcal{V}\text{-Cat} \text{ and } \text{Cmd}(\mathcal{V}\text{-Mat}) = \mathcal{V}\text{-Cocat}.$$

► A module in \mathbb{D} is $M: Z \rightarrow X$ with an action of a monad $A: X \rightarrow X$

$$\begin{array}{ccccc} Z & \xrightarrow{M} & X & \xrightarrow{A} & X \\ \parallel & & \downarrow & & \parallel \\ Z & \xrightarrow{\quad M \quad} & X & & X \end{array}$$

Dually, comodule $Q: W \rightarrow X$ with globular $Q \Rightarrow C \circ Q$ for comonad $C: X \rightarrow X$. These form categories $\text{Mod}(\mathbb{D})$ & $\text{Comod}(\mathbb{D})$, with maps

$$\begin{array}{ccc} \begin{array}{ccc} Z & \xrightarrow{M_A} & X \\ g \downarrow & \downarrow \beta & \downarrow f \\ T & \xrightarrow{N_B} & Y \end{array} & \text{s.t.} & \begin{array}{ccc} Z & \xrightarrow{M} & X & \xrightarrow{A} & X \\ \parallel & & \downarrow & & \parallel \\ Z & \xrightarrow{\quad M \quad} & X & & X \\ g \downarrow & & \downarrow \beta & & \downarrow f \\ T & \xrightarrow{\quad N \quad} & Y & & Y \end{array} = \begin{array}{ccc} Z & \xrightarrow{M} & X & \xrightarrow{A} & X \\ g \downarrow & \downarrow \beta & \downarrow f & \downarrow \alpha & \downarrow f \\ T & \xrightarrow{N} & Y & \xrightarrow{B} & Y \\ \parallel & & \downarrow & & \parallel \\ T & \xrightarrow{\quad N \quad} & Y & & Y \end{array} \end{array}$$

★ Subcategories of interest: fixed-monad ${}_A\text{Mod}(\mathbb{D})$, fixed-dom ${}^Z\text{Mod}(\mathbb{D})$, bicat modules ${}^Z_A\text{Mod}(\mathbb{D}) = \text{Mod}(\mathcal{H}(\mathbb{D}))$...several monadicity results.

$\{*\}\text{Mod}(\mathcal{V}\text{-Mat}) = \mathcal{V}\text{-Mod}$ and $\{*\}\text{Comod}(\mathcal{V}\text{-Mat}) = \mathcal{V}\text{-Comod}$.
 $\text{Mod}(\mathcal{V}\text{-Mat})$ has $\{M(x, z)\}_{X \times Z}$ with $A(x, x') \otimes M(x', z) \rightarrow M(x, z)$.

For \mathbb{D} a double category, $\text{Mnd}(\mathbb{D})$ is enriched in $\text{Cmd}(\mathbb{D})$ and $\text{Mod}(\mathbb{D})$ is enriched in $\text{Comod}(\mathbb{D})$, *under certain conditions*.

■ Fibrant: vertical 1-cells f turn to horizontal, companion \hat{f} & conjoint \check{f} .

In $\mathcal{V}\text{-Mat}$, $f: X \rightarrow Y$ gives matrices $\hat{f}(x, y) = \check{f}(y, x) = \begin{cases} 1 & \text{if } fx = y \\ 0 & \text{if } fx \neq y \end{cases}$

$\text{Mod}(\mathbb{D}) \rightarrow \text{Mnd}(\mathbb{D})$ is a fibration, with reindexing $\check{f} \circ -: {}_B\text{Mod} \rightarrow {}_A\text{Mod}$.
 $\text{Comod}(\mathbb{D}) \rightarrow \text{Cmd}(\mathbb{D})$ is an opfibration, with $\hat{f} \circ -: {}_C\text{Comod} \rightarrow {}_D\text{Comod}$.

■ Monoidal: \mathbb{D}_0 & \mathbb{D}_1 monoidal, $(M \otimes N) \circ (M' \otimes N') \cong (M \circ M') \otimes (N \circ N')$.

In $\mathcal{V}\text{-Mat}$, $(X \otimes Y) = X \times Y$ & $(S \otimes T)((x, y), (z, w)) = S(x, z) \otimes T(y, w)$.

$\text{Cmd}(\mathbb{D})$ and $\text{Comod}(\mathbb{D})$ are monoidal, with $Q_C \otimes P_D = (Q \otimes P)_{C \otimes D}$ via $Q \otimes P \rightarrow (C \circ Q) \otimes (D \circ P) \cong (C \otimes D) \circ (Q \otimes P)$. Symmetry is inherited.

■ Locally closed monoidal: lax double functor $H: \mathbb{D}^{\text{op}} \times \mathbb{D} \rightarrow \mathbb{D}$ such that \mathbb{D}_0 & \mathbb{D}_1 monoidal closed, $\mathfrak{s}, \mathfrak{t}$ maps of adjunctions.

In $\mathcal{V}\text{-Mat}$, $[X, Y] = Y^X$ and $H(S, T)(f, g) = \prod_{x,y} [S(x, y), T(fx, gy)]$.

Lax double H induce functors $\text{Cmd}(\mathbb{D})^{\text{op}} \times \text{Mnd}(\mathbb{D}) \rightarrow \text{Mnd}(\mathbb{D})$ and $\text{Comod}(\mathbb{D})^{\text{op}} \times \text{Mod}(\mathbb{D}) \rightarrow \text{Mod}(\mathbb{D})$ which are actions. *convolution*

■ Locally presentable: \mathbb{D}_0 & \mathbb{D}_1 locally presentable, $\mathfrak{s}, \mathfrak{t}$ accessible right adjoints, $\mathbf{1}$ accessible, $- \circ -$ accessible in each variable.

In $\mathcal{V}\text{-Mat}$, Set is locally presentable & $\mathcal{V}\text{-Mat}_1$ is too (Limit theorem...)

Induced functors H have adjoints $\text{Mnd}(\mathbb{D})^{\text{op}} \times \text{Mnd}(\mathbb{D}) \rightarrow \text{Cmd}(\mathbb{D})$ and $\text{Mod}(\mathbb{D})^{\text{op}} \times \text{Mod}(\mathbb{D}) \rightarrow \text{Comod}(\mathbb{D})$. *universal measuring*

◇ Obtain enrichment of $\text{Mnd}(\mathbb{D})$ in $\text{Cmd}(\mathbb{D})$ & of $\text{Mod}(\mathbb{D})$ in $\text{Comod}(\mathbb{D})$.

Moving to other contexts

★ For \mathcal{V} symmetric monoidal closed & locally presentable, $\mathbb{D} = \mathcal{V}\text{-Mat}$ is a fibrant, locally closed monoidal & locally presentable double category.

- Enrichment of \mathcal{V} -categories in \mathcal{V} -cocategories
- Enrichment of \mathcal{V} -modules in \mathcal{V} -comodules

Goal: employ/*extend* theory to apply to $\mathbb{D} = \mathcal{V}\text{-Sym}$



operads

- objects are sets X, Y, \dots
- vertical 1-cells are functions f, g, \dots
- horizontal 1-cells are coloured symmetric sequences $M: SX \times Y \rightarrow \mathcal{V}$
- 2-maps are $M(x_1, \dots, x_n; y) \rightarrow N(fx_1, \dots, fx_n; gy)$

Horizontal composition is generalization of substitution for species...

■ $\mathcal{V}\text{-Sym}$ is fibrant; but not monoidal double anymore!

Oplax monoidal double structure

- A double category is *oplax* monoidal with comparison maps

$$(N \circ M) \otimes (N' \circ M') \rightarrow (M \otimes M') \circ (N \otimes N'), \quad 1_X \otimes 1_{X'} \rightarrow 1_{X \otimes X'}$$

$$l_1 \rightarrow l_1 \circ l_1, \quad l_1 \rightarrow 1_l$$

making $\otimes: \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D}$, $l: \mathbf{1} \rightarrow \mathbb{D}$ into oplax double functors.

An oplax monoidal double category with a single object and vertical arrow is precisely a duoidal category.

- *Normality* condition (l pseudo, \otimes pseudo in each variable) reduces to normal duoidal structure & gives passage to ‘oplax monoidal bicategories’.

Idea: show that $\mathcal{V}\text{-Sym}$ (in fact $\mathcal{V}\text{-CatSym}$...) is normal oplax monoidal, by expressing it as a **Kleisli double category** of monoidal double $\mathcal{V}\text{-Prof}$.

Monoidal double monads

- A *vertical double monad* is a double functor $T: \mathbb{D} \rightarrow \mathbb{D}$ with vertical transformations $m: TT \Rightarrow T$, $e: 1 \Rightarrow T$ with

$$\begin{array}{ccc}
 TTX & \xrightarrow{TTM} & TTY \\
 m_X \downarrow & \Downarrow m_M & \downarrow m_Y \\
 TX & \xrightarrow{TM} & TY
 \end{array}
 \qquad
 \begin{array}{ccc}
 X & \xrightarrow{M} & Y \\
 e_X \downarrow & \Downarrow e_M & \downarrow e_Y \\
 TX & \xrightarrow{TM} & TY
 \end{array}$$

- It is *pseudomonoidal* when T lax monoidal and m, e pseudomonoidal

$$\begin{array}{ccc}
 TTX_1 \otimes TTX_2 & \xrightarrow{1} & TTX_1 \otimes TTX_2 \\
 \downarrow & & \downarrow m \otimes m \\
 \text{e.g. } TT(X_1 \otimes X_2) & \Downarrow & TX_1 \otimes TX_2 \\
 m \downarrow & & \downarrow \\
 T(X_1 \otimes X_2) & \xrightarrow{1} & T(X_1 \otimes X_2)
 \end{array}
 \qquad
 \begin{array}{l}
 \text{is invertible in} \\
 \text{vertical 2-category}
 \end{array}$$

- A vertical double monad is *special* when m_X, e_X have companions in \mathbb{D} , and each 2-cell transpose of m_M, e_M is invertible, e.g.

$$\begin{array}{ccccc}
 X & \xrightarrow{M} & Y & \xrightarrow{\hat{e}_Y} & TY \\
 \parallel & & \Downarrow \hat{e}_M & & \parallel \\
 X & \xrightarrow{\hat{e}_X} & TX & \xrightarrow{TM} & TY
 \end{array}
 =
 \begin{array}{ccccccc}
 X & \xrightarrow{1} & X & \xrightarrow{M} & Y & \xrightarrow{\hat{e}_Y} & TY \\
 \parallel & & \Downarrow e_X & & \Downarrow e_M & & \parallel \\
 X & \xrightarrow{\hat{e}_X} & TX & \xrightarrow{TM} & TY & \xrightarrow{1} & TY
 \end{array}$$

- ★ 'Free symmetric strict monoidal 2-category monad' $S: \mathcal{V}\text{-Cat} \rightarrow \mathcal{V}\text{-Cat}$

$$\text{with } S_n(C)((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sum_{\sigma} \prod_{1 \leq i \leq n} C(x_{\sigma(i)}, y_i)$$

extends to vertical double monad on $\mathcal{V}\text{-Prof}$, special and pseudomonoidal:

$$((x_1, \dots, x_m), (y_1, \dots, y_n)) \mapsto ((x_1, y_1), (x_1, y_2), \dots, (x_m, y_n)) := \vec{x} \boxtimes \vec{y}$$

Note: \mathcal{V} needs to be cartesian monoidal.

Kleisli double category

■ Each special vertical double monad $T: \mathbb{D} \rightarrow \mathbb{D}$ gives double $\mathbb{Kl}(T)$

· $\mathbb{Kl}(T)_0$ is \mathbb{D}_0

· $M: X \rightsquigarrow Y$ are horizontal $M: X \rightarrow TY$ in \mathbb{D}

· 2-morphisms $\begin{array}{ccc} X & \overset{M}{\rightsquigarrow} & Y \\ f \downarrow & \Downarrow & \downarrow g \\ Z & \overset{N}{\rightsquigarrow} & W \end{array}$ are $\begin{array}{ccc} X & \xrightarrow{M} & TY \\ f \downarrow & \Downarrow & \downarrow Tg \\ Z & \xrightarrow{N} & TW \end{array}$ in \mathbb{D}

· horizontal composition is $X \xrightarrow{M} TY \xrightarrow{TN} TTZ \xrightarrow{\hat{m}_Z} TZ$

For \mathbb{D} monoidal double category and T pseudomonoidal special vertical double monad, if structure maps $I \rightarrow TI$ and $TX \otimes TY \xrightarrow{\tau} T(X \otimes Y)$ have companions, then $\mathbb{Kl}(T)$ is an oplax monoidal double category.

★ Induced tensor is $M \boxtimes N = X \otimes Z \xrightarrow{M \otimes N} TY \otimes TW \xrightarrow{\hat{\tau}} T(Y \otimes W)$.

Arithmetic product of coloured symmetric sequences

★ Acquired $S: \mathcal{V}\text{-Prof} \rightarrow \mathcal{V}\text{-Prof}$ satisfies above conditions!

Kleisli double category is $\mathcal{V}\text{-CatSym}$ of *categorical symmetric sequences* $M: SY^{\text{op}} \times X \rightarrow \mathcal{V}$, with ‘discrete’ case $\mathcal{V}\text{-Sym}$.

▶ Horizontal composition is many-object generalisation of substitution

$$(N \circ M)(\vec{z}, x) = \int^{SZ, SY} SZ[\vec{z}, \bigotimes_i \vec{w}^i] \times \prod N(\vec{w}^i, y_i) \times M(\vec{y}, x)$$

▶ Oplax monoidal structure is many-object generalisation of ‘arithmetic product’ of species

$$(M \boxtimes N)(\vec{a}, (x, z)) = \int^{\vec{y}, \vec{w}} S(Y \times W)(\vec{a}, \vec{y} \boxtimes \vec{w}) \times M(\vec{y}, x) \times N(\vec{w}, z)$$

Some future directions

- ▶ Extend ‘Sweedler theory’ from monoidal double to oplax monoidal double categories: do we still obtain an enrichment of monads in comonads, and of modules in comodules?
- ▶ Further explore structure of $\mathcal{V}\text{-Sym}$: is it locally monoidal closed and locally presentable as a double category, for introduced definitions?

[One-object case] If \mathcal{V} is symmetric monoidal closed and loc presentable,

- positive operads are enriched in positive cooperads, if \mathcal{V} has biproducts;
 - symmetric operads are enriched in symmetric cooperads, if \mathcal{V} is cartesian.
- ▶ Boardman-Vogt tensor of bimodules of symmetric coloured operads and bimodules: abstract double categorical framework behind that?

Thank you for your attention!



- *Aravantinos-Sotiropoulos, Vasilakopoulou*, “Enriched duality in double categories II: \mathcal{V} -modules and \mathcal{V} -comodules”, in preparation
- *Gambino, Garner, Vasilakopoulou*, “Monoidal Kleisli bicategories and the arithmetic product of symmetric sequences”, arXiv:2206.06858