When are there enough model isomorphisms?
Representing groupoids for classifying toposes

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On topological groupoids that represent theories, arXiv:2306.16331

Main result and overview

A theory can be *represented* by certain groupoids of its models.

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Overview

I. Recall the definition of the topos of sheaves on a groupoid and the classifying topos of a theory.

II. Review an example representing groupoid.

III. Define elimination of parameters.

IV. Technically restate the main theorem, and give some applications.

Topological groupoids

Definition

A (small) groupoid $\mathbb{X} = (X_1 \rightrightarrows X_0)$ consists of a diagram of sets

$$X_1 \times_{X_0} X_1 \xrightarrow{m} X_1 \underset{s}{\overset{t}{\underset{\xleftarrow{e}}{\rightrightarrows}}} X_0, \qquad i : X_1 \to X_1$$

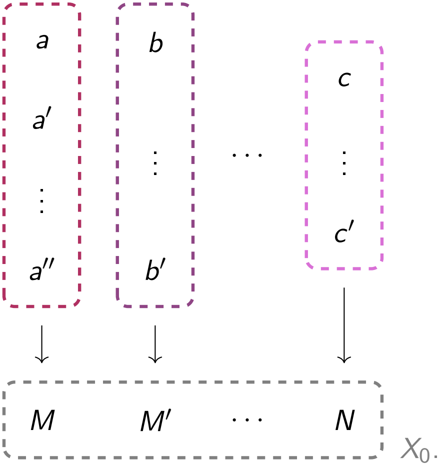
satisfying the 'obvious' equations.

A *topological groupoid* consists of topologies on X_0 and X_1 such that the above maps are continuous.

We say that $\mathbb{X}$ is *open* if s (equivalently, t) is open.

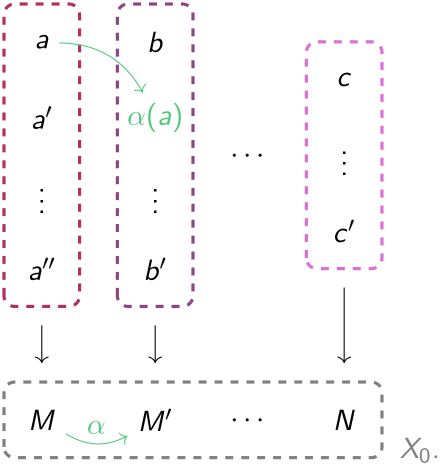
Equivariant sheaves on a groupoid

Given a groupoid $\mathbb{X}$, a discrete *bundle* on $\mathbb{X}$ consists of a map $q\colon Y \to X_0$,



Equivariant sheaves on a groupoid

Given a groupoid $\mathbb{X}$, a discrete *bundle* on $\mathbb{X}$ consists of a map $q\colon Y \to X_0$, equipped with an X_1-action $\beta\colon Y \times_{X_0} X_1 \to Y$,



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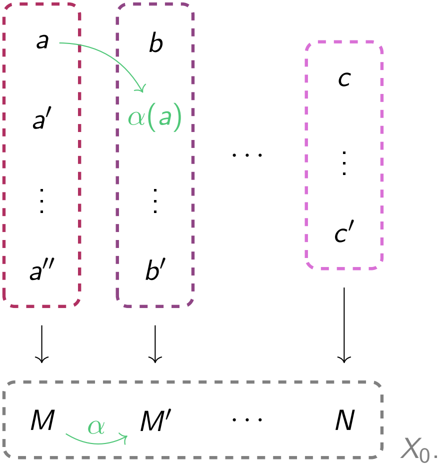

If $\mathbb{X}$ is endowed with topologies, we say that a bundle is a *sheaf* if

(i) $q\colon Y \to X_0$ is a local homeomorphism,

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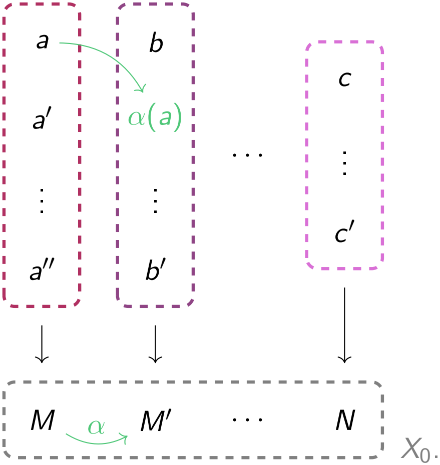

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A *morphism* of sheaves is a continuous map $f\colon Y \to Y'$ such that the following commute:

$$\begin{array}{ccc} Y \times_{X_0} X_1 & \xrightarrow{f \times_{X_0} \mathrm{id}_{X_1}} & Y' \times_{X_0} X_1 \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta'} \\ Y & \xrightarrow{f} & Y', \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle q'} & \\ & X_0. & \end{array}$$

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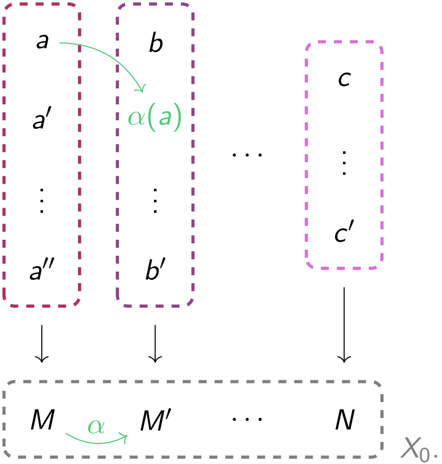

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Definition – topos of equivariant sheaves

The category of sheaves and their morphisms define a topos $\mathbf{Sh}(X)$.

Classifying topos

If a topos is ‘like’ a generalised space, then a *classifying topos* is ‘like’ a space whose points are models.

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Definition

If $\mathbb{X}$ is a (open) topological groupoid for which

$$\mathbf{Sh}(\mathbb{X}) \simeq \mathcal{E}_{\mathbb{T}},$$

we say that $\mathbb{X}$ *represents* $\mathbb{T}$.

Indexed structures

Let $\mathbb{T}$ be a theory over a signature Σ whose set-based models are conservative.

We would expect the groupoid of *all* models to represent $\mathbb{T}$.

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Definition

Let M be a structure over a signature Σ.

Given a set $\mathfrak{K}$ of *parameters*, a $\mathfrak{K}$-*indexing* of M consists of:

(i) a subset $\mathfrak{K}' \subseteq \mathfrak{K}$,

(ii) and an *expansion* of M to the signature $\Sigma \cup \{ c_m \mid m \in \mathfrak{K}' \}$ such that M satisfies

$$\top \vdash_x \bigvee_{m \in \mathfrak{K}} x = c_m,$$

i.e. every $n \in M$ is the interpretation of some parameter $m \in \mathfrak{K}$.

Equivalently, this is a choice of partial surjection $\mathfrak{K} \rightharpoondown M$.

The groupoid of all indexed models

Definition

Let $\mathbf{Ind}(\mathfrak{K})$ denote the groupoid:

(i) whose objects are $\mathfrak{K}$-indexed models of $\mathbb{T}$,

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Theorem (Awodey–Forssell [1], [4])

Let $\mathfrak{K}$ be infinite. For suitable topologies on $\mathbf{Ind}(\mathfrak{K})$, there is an equivalence

$$\mathbf{Sh}(\mathbf{Ind}(\mathfrak{K})) \simeq \mathcal{E}_{\mathbb{T}}$$

if and only if the $\mathfrak{K}$-indexed models are *conservative* –

– if two formulae are interpreted identically, then they are $\mathbb{T}$-provably equivalent.

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This is just one example among many representing groupoids for $\mathbb{T}$.

In the following sections, we develop our characterisation.

Definable subsets of a single model

Let M be a model of $\mathbb{T}$ with an indexing $\mathfrak{K} \twoheadrightarrow M$.

(i) A *definable subset* is a subset of the form

$$[\![\vec{x} : \varphi]\!]_M = \{ \vec{n} \subseteq M \mid M \vDash \varphi(\vec{n}) \} \subseteq M^n$$

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(ii) A *definable subset with parameters* is a subset of the form

$$[\![\vec{x}, \vec{m} : \psi]\!]_M = \{ \vec{n} \subseteq M \mid M \vDash \psi(\vec{n}, \vec{m}) \} \subseteq M^n$$

for some formula $\{ \vec{x}, \vec{y} : \psi \}$ and a tuple of parameters $\vec{m} \subseteq \mathfrak{K}$.

Definables for a groupoid of models

For a groupoid $\mathbb{X}$ of $\mathbb{T}$-models, a $\mathfrak{K}$-indexing of $\mathbb{X}$ is a choice of $\mathfrak{K}$-indexing $\mathfrak{K} \rightharpoonup M$ for each $M \in \mathbb{X}$.

(i) A *definable* or *definable without parameters* is a subset of the form

$$[\![\, \vec{x} : \varphi \,]\!]_{\mathbb{X}} = \{\, \langle \vec{n}, M \rangle \mid \vec{n} \subseteq M \in X_0,\ M \vDash \varphi(\vec{n}) \,\} \subseteq \textstyle\coprod_{M \in X_0} M^n$$

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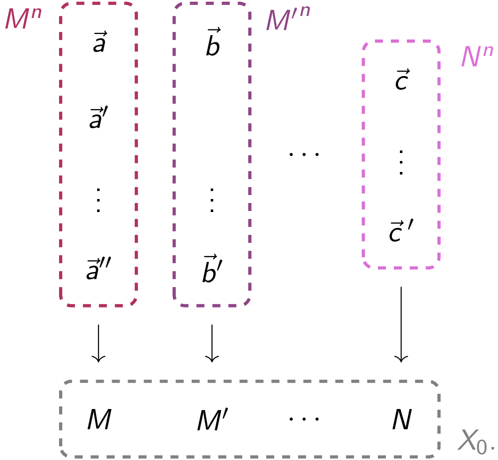
Interpreting definables and elimination of parameters

For each n, there is a bundle



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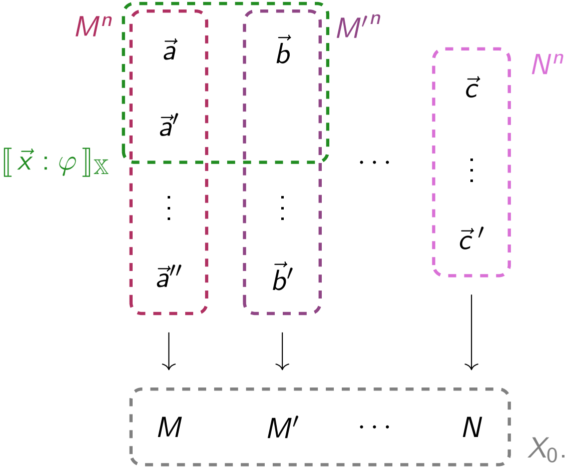
Each definable defines a subset



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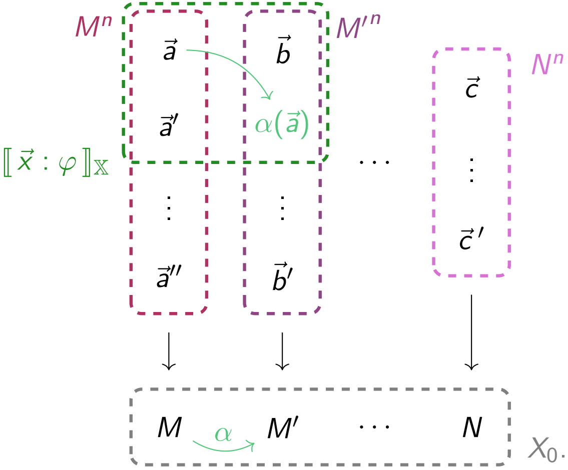

Note that $[\![\vec{x} : \varphi]\!]_{\mathbb{X}}$ is *stable* under the X_1-action.

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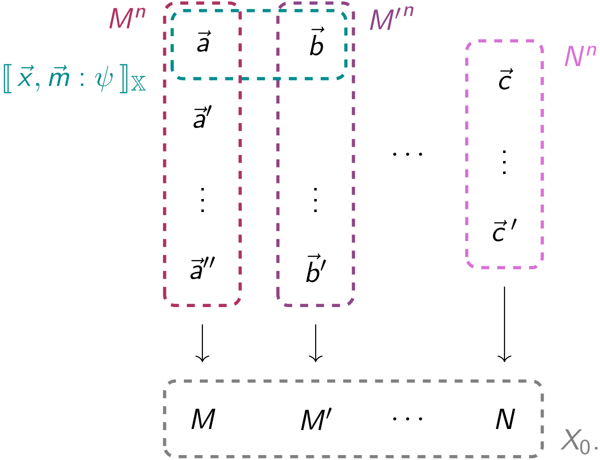
Each definable with parameters also defines a subset



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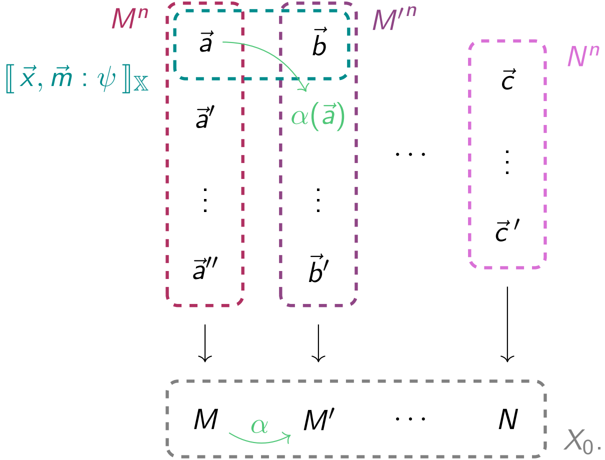
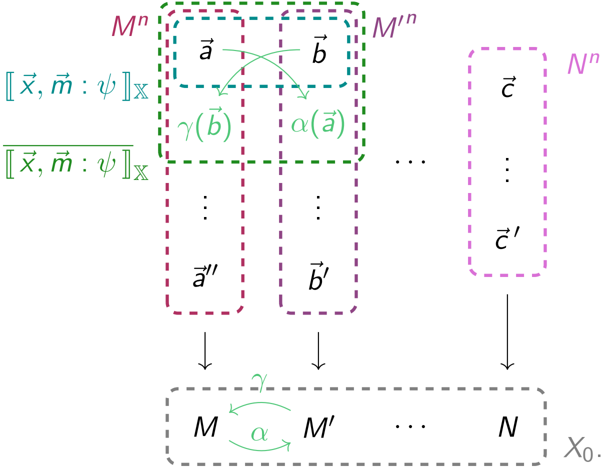


However, $[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}$ is not stable under the X_1-action.

Interpreting definables and elimination of parameters

We can consider the closure of $[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}$ under the X_1-action



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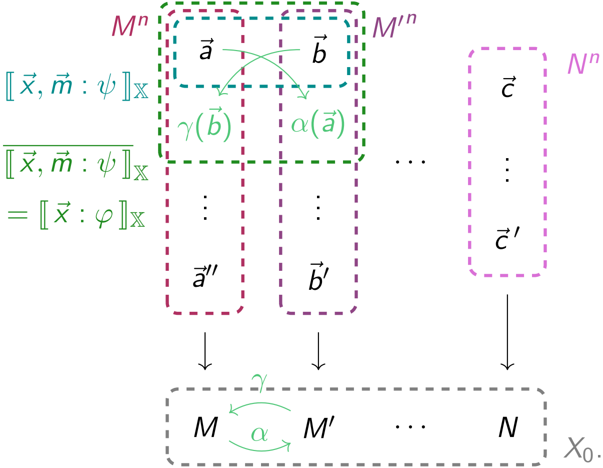

In this case, $\overline{[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}} = [\![\vec{x} : \varphi]\!]_{\mathbb{X}}$.

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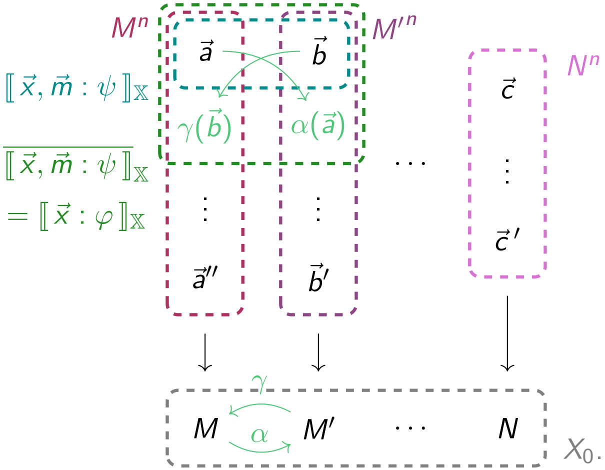

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Main Definition

Given a groupoid $\mathbb{X}$ of $\mathbb{T}$-models and an indexing $\mathfrak{K} \twoheadrightarrow \mathbb{X}$,

$\mathbb{X}$ *eliminates parameters* if, for every ψ and $\vec{m}$, there exists some *geometric* formula φ such that

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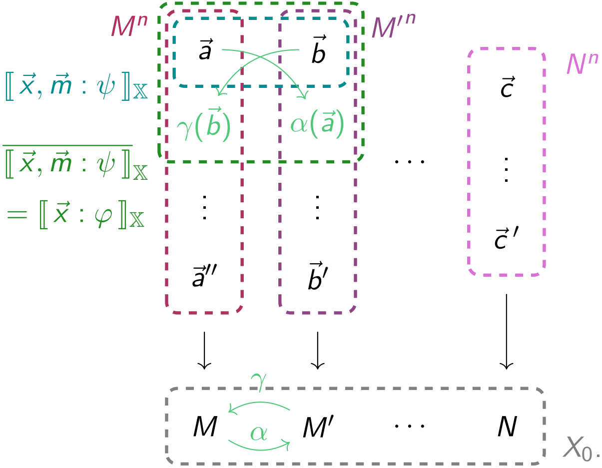

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Example

The automorphism group $\mathrm{Aut}\left(\overline{\mathbb{Q}}\right)$ eliminates parameters.

For each $a \in \overline{\mathbb{Q}}$,

$$\overline{[\![x = a]\!]_{\mathrm{Aut}(\overline{\mathbb{Q}})}} = [\![x : p_a(x) = 0]\!]_{\mathrm{Aut}(\overline{\mathbb{Q}})},$$

where p_a is the minimal polynomial for a.

Classification result

Main Theorem (J.W.)

Let $\mathbb{T}$ be a geometric theory and let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a small groupoid of $\mathbb{T}$-models.

We can endow $\mathbb{X}$ with the structure of an **open** topological groupoid for which

$$\mathbf{Sh}(\mathbb{X}) \simeq \mathcal{E}_{\mathbb{T}}$$

if and only if

(i) X_0 is a conservative set –

$$[\![\vec{x} : \varphi]\!]_{\mathbb{X}} = [\![\vec{x} : \chi]\!]_{\mathbb{X}} \implies \varphi \equiv^{\mathbb{T}}_{\vec{x}} \chi,$$

(ii) there is an indexing of $\mathbb{X}$ by parameters $\mathfrak{K}$ for which $\mathbb{X}$ eliminates parameters –

$$\overline{[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}} = [\![\vec{x} : \varphi]\!]_{\mathbb{X}}.$$

Applications

Proposition (cf. Awodey–Forssell [1],[4], Butz–Moerdijk [2], Caramello [3])

(i) The groupoid of all $\mathfrak{K}$-indexed models eliminates parameters.

(ii) The groupoid of all $\mathfrak{K}$-*enumerated* –

every element is indexed by infinitely many parameters

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(iii) If $\mathbb{T}$ is an atomic theory, then

$$\mathrm{Aut}(M) \text{ eliminates parameters} \iff M \text{ is } \textit{ultrahomogeneous},$$

i.e. every finite partial isomorphism of M extends to a total isomorphism,

$$\begin{array}{ccc} \vec{n} & \xrightarrow{\sim} & \vec{n}' \\ \downarrow & & \downarrow \\ M & \dashrightarrow^{\sim} & M. \end{array}$$

Étale complete groupoids

Definition (Moerdijk [5])

Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a groupoid of $\mathbb{T}$-models.

We say that $\mathbb{X}$ is *étale complete* if for each pair $M, N \in X_0$, every isomorphism $M \xrightarrow{\alpha} N$ is in X_1.

The *étale completion* $\hat{\mathbb{X}}$ of $\mathbb{X}$ is the étale complete groupoid with objects X_0.

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Corollary (cf. Joyal–Tierney [6])

If $\mathbb{X}$ eliminates parameters, then adding isomorphisms to $\mathbb{X}$ preserves elimination of parameters.

Thus, for every open topological groupoid, there is an equivalence

$$\mathbf{Sh}(\mathbb{X}) \simeq \mathbf{Sh}(\hat{\mathbb{X}}).$$

Thank you for listening

The preprint:

On topological groupoids that represent theories, arXiv:2306.16331

Other references:

[1] Awodey, S.; Forssell, H. First-order logical duality. *Ann. Pure Appl. Logic* 164 (2013), no. 3, 319–348.

[2] Butz, C.; Moerdijk, I. Representing topoi by topological groupoids. *J. Pure Appl. Algebra* 130 (1998), no. 3, 223–235.

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[4] Forssell, H. Topological representation of geometric theories. *Math. Log. Q.* 58 (2012), no. 6, 380–393.

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[6] Joyal, A.; Tierney, M. An extension of the Galois theory of Grothendieck. *Mem. Amer. Math. Soc.* 51 (1984), no. 309, v–71.