# The dependent Gödel fibration

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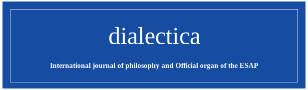


ÜBER EINE BISHER NOCH NICHT BENÜTZTE ERWEITERUNG DES FINITEN STANDPUNKTES

von Kurt GÖDEL, Princeton

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To define the Dialectica translation for formulas, let $\varphi^D = \exists x \forall y \, \varphi_D$ and $\psi^D = \exists u \forall v \, \psi_D$, and define inductively:

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(\exists z \, \varphi(z)) &\mapsto (\exists z \, \varphi(z))^D = \exists z, x \, \forall y \, \varphi_D(x, y, z) \\
(\varphi \to \psi) &\mapsto \boldsymbol{(\varphi \to \psi)^D = \exists U, Y \, \forall x, v \, (\varphi_D(x, Y(x, v)) \to \psi_D(U(x), v))}
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**Intuition:** The new function $U$ assigns to any *witness* $x$ of $\varphi_D$ a witness $U(x)$ of $\psi_D$. The new function $Y$ assigns to any *counterexample* $v$ to $\psi_D$ a counterexample $Y(x, v)$ to $\varphi_D$.

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- *Hence:* Characterization theorem shows that a certain **external** description (Dialectica fibration as a bicompletion) coincides with an **internal** one (Gödel fibration via qf objects)

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- Generalizes non-dependent, cartesian closed structure $\times$ and $\times$ by dependent, locally cartesian closed structure $\Sigma$ and $\Pi$, and eventually tricartesian fibration structure $\coprod$ and $\prod$
- **Intuition:**

$$
\begin{array}{ccccccc}
 & \coprod & \dashv & (-)^* & \dashv & \prod & \\
 & \uparrow & & & & \uparrow & \\
 & \Sigma & & & & \Pi & \\
\nearrow & \uparrow & & & & \uparrow & \nwarrow \\
\exists & \times & & & & \to & \forall
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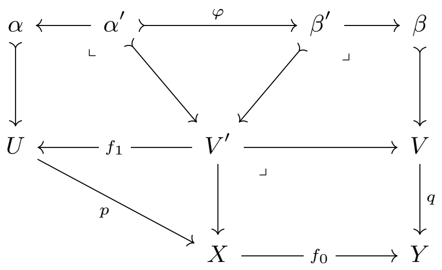
- morphisms:

$$\varphi : \prod_{x:X} \prod_{v:V(f_0(x))} \alpha(x, f_1(v)) \implies \beta(f_0(x), v)$$

$$f_1 : \prod_{x:X} V(f_0(x)) \to U(x)$$

$$f_0 : X \to Y$$



# Display map categories

**Definition (Display map categories, after Taylor '99)**

Let B be a category. A class $\mathcal{F} \subseteq \mathrm{Mor}(\mathsf{B})$ is said to be a class of **display maps**, if $\mathcal{F}$ contains all isomorphisms in B and is closed under pullbacks along arbitrary maps in B, namely pullbacks along arrows of $\mathcal{F}$ exist and belong to $\mathcal{F}$. If an arrow $u : K \to I$ is in $\mathcal{F}$ we write it as $u : K \twoheadrightarrow I$. A **display map category (dmc)** is a pair $\langle \mathsf{B}, \mathcal{F} \rangle$ where B is a category and $\mathcal{F}$ a class of display maps.

Given a display map category $\langle \mathsf{B}, \mathcal{F} \rangle$, we will denote by $\mathcal{F}^{\to}$ the subcategory of the arrow category $\mathsf{B}^{\to}$ whose objects are arrows of $\mathcal{F}$.

**Definition**

Let $\langle \mathsf{B}, \mathcal{F} \rangle$ be a display map category. We denote by $\mathsf{cod} \colon \mathcal{F}^{\to} \longrightarrow \mathsf{B}$ the (full) subfibration of the codomain fibration $\mathsf{cod} \colon \mathsf{B}^{\to} \longrightarrow \mathsf{B}$.

# Dialectica fibrations

- Let $\mathrm{p}\colon \mathrm{E} \longrightarrow \mathrm{B}$ be a Grothendieck fibration over a display map category $\langle \mathrm{B}, \mathcal{F} \rangle$. The category $\mathfrak{Dial}_{\mathcal{F}}(\mathrm{E})$ has

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$$\left(\alpha \overset{\alpha \in \mathsf{E}_U}{=\!=\!=\!=\!=} U \xrightarrow{p \in \mathcal{F}} X \xrightarrow{r \in \mathcal{F}} I\right) \qquad \hat{=} \qquad \exists i, x\, \forall u\, \alpha(i, x, u)$$

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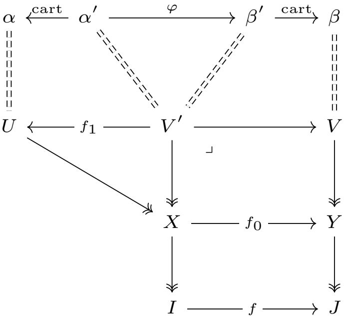
$$f : I \to J$$



# Dialectica fibrations (cont'd)

- The (dependent) *Dialectica fibration* $\mathfrak{Dial}_{\mathcal{F}}(\mathsf{p})\colon \mathfrak{Dial}_{\mathcal{F}}(\mathsf{E}) \longrightarrow \mathsf{B}$ is defined as the first projection
$$\mathfrak{Dial}_{\mathcal{F}}(\mathsf{E})(I, U, X, \alpha) := I, \quad \mathfrak{Dial}_{\mathcal{F}}(\mathsf{p})(f, f_0, f_1, \varphi) := f.$$

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- We obtain the classical dependent Dialectica category $\mathrm{Dial}(\mathsf{C})$ as the fiber over $1$ of the Dialectica fibration of the subobject (or mono) fibration of $\mathsf{C}$ (with $\mathcal{F} = \mathsf{Mor}(\mathsf{C})$).

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- In fact, it arises by iterating free completions w.r.t. those, cf. Hofstra, von Glehn, Moss

# Dialectica fibrations (cont'd)

- The (dependent) *Dialectica fibration* $\mathfrak{Dial}_{\mathcal{F}}(\mathsf{p})\colon \mathfrak{Dial}_{\mathcal{F}}(\mathsf{E}) \longrightarrow \mathsf{B}$ is defined as the first projection
$$\mathfrak{Dial}_{\mathcal{F}}(\mathsf{E})(I, U, X, \alpha) := I, \quad \mathfrak{Dial}_{\mathcal{F}}(\mathsf{p})(f, f_0, f_1, \varphi) := f.$$
- We obtain the classical dependent Dialectica category $\mathrm{Dial}(\mathsf{C})$ as the fiber over $1$ of the Dialectica fibration of the subobject (or mono) fibration of $\mathsf{C}$ (with $\mathcal{F} = \mathsf{Mor}(\mathsf{C})$).
- The Dialectica fibration is *bicomplete*, *i.e.*, it possesses fibered coproducts and products along maps in $\mathcal{F}$.
- In fact, it arises by iterating free completions w.r.t. those, cf. Hofstra, von Glehn, Moss (though it is **not** the free bicompletion).

# Fibered coproducts in a fibration

**Definition (Fibered $\mathcal{F}$-coproducts in a fibration)**

Consider a display map category $\langle B, \mathcal{F} \rangle$ and a Gothendieck $\mathsf{p} \colon \mathsf{E} \longrightarrow \mathsf{B}$ be a Grothendieck fibration. Then $\mathsf{p}$ is said to **have (fibered or internal) $\mathcal{F}$-coproducts** if:

1. $\mathsf{p}$ is also an *opfibration*, so any $f \colon J \to I$ gives rise to an adjunction $\coprod_f \dashv f^*$,
2. satisfying the **Beck–Chevalley conditions (BCC)**: for each pullback in $\mathsf{B}$ of the form

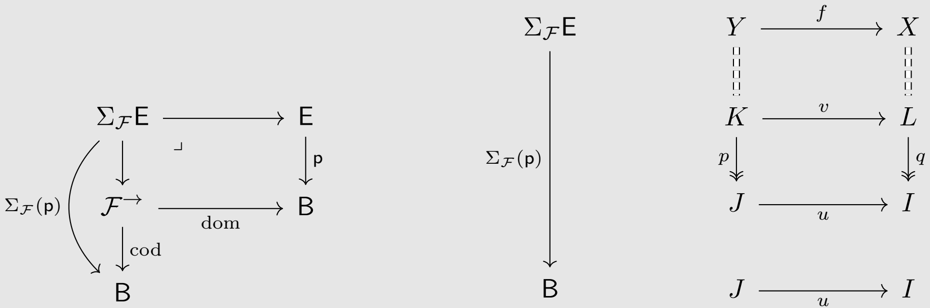
$$\begin{array}{ccc} K & \xrightarrow{\;u\;} & I \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ L & \xrightarrow[\;v\;]{} & J \end{array}$$

   where $u, v \in \mathcal{F}$ the canonical natural transformation $\coprod_u g^* \Longrightarrow f^* \coprod_v$ is an isomorphism.

# Family fibration

**Definition (Family fibration)**

Let $\langle \mathsf{B}, \mathcal{F} \rangle$ be a display map category and $\mathsf{p} \colon \mathsf{E} \longrightarrow \mathsf{B}$ a fibration. The **family fibration of** $\mathsf{p}$ **relative to** $\mathcal{F}$ is the functor $\Sigma_{\mathcal{F}}(\mathsf{p})$ constructed as follows:



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- B has $\mathcal{F}$-dependent coproducts if and only if the codomain fibration $\mathsf{cod}\colon \mathcal{F}^{\to} \longrightarrow \mathsf{B}$ has $\mathcal{F}$-coproducts.

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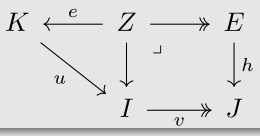
- A category C has small $\mathbf{Set}$-indexed coproducts if and only if the fibration $\mathrm{Fam}(\mathsf{C}) \to \mathbf{Set}$ has fibered coproducts.

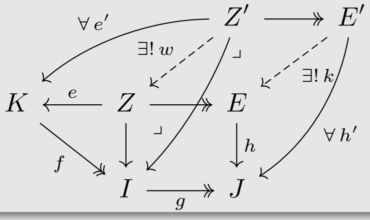
# Dependent products in a dmc

**Definition ($\mathcal{F}$-dependent products in a dmc)**

Let B be a category and let $\mathcal{F}$ be a class of display maps. An $\mathcal{F}$-**dependent product** of an arrow $f : K \to I$ of $\mathcal{F}$ along an arrow $g : I \twoheadrightarrow J$ of $\mathcal{F}$ consists of a commutative diagram as below left, such that:




$\rightsquigarrow$



# Fibered products in a fibration

**Definition (Fibered $\mathcal{F}$-products in a fibration)**

A fibration $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ over a display map category $\langle \mathsf{B}, \mathcal{F} \rangle$ has **fibered $\mathcal{F}$-products** if and only if the *dual fibration* $\mathsf{p}^{(\mathrm{op})}\colon \mathsf{E}^{(\mathrm{op})} \longrightarrow \mathsf{B}$ has fibered $\mathcal{F}$-coproducts.

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- $\mathsf{B}$ has dependent $\mathcal{F}$-products if and only if the codomain fibration $\mathsf{cod}\colon \mathcal{F}^{\rightarrow} \longrightarrow \mathsf{B}$ has fibered $\mathcal{F}$-products.

# Dialectica fibrations as completions

**Theorem (after e.g. Hofstra '11)**

*Let* $\langle \mathsf{B}, \mathcal{F} \rangle$ *be a display map category and* $\mathsf{p} \colon \mathsf{E} \longrightarrow \mathsf{B}$ *be a Grothendieck fibration. Then there is a fibered isomorphism* $\mathfrak{Dial}_{\mathcal{F}}(\mathsf{p}) \cong \Sigma_{\mathcal{F}}\big(\Pi_{\mathcal{F}}(\mathsf{p})\big)$.

# Quantifier-freeness: Intuitition

- An element $\alpha$ in a fiber of p is *quantifier-free* if: Given a proof $\pi$ of a statement $\exists i\, \beta(i)$, then there exists a witness $t$, which depends on the proof $\pi$, together with a proof of $\beta(t)$.

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- This has been categorified, in the non-dependent case, by Spadetto–Trotta–de Paiva.
- We're presenting a dependent generalization, subsuming the notion of *decomposable object* in a cocomplete category (cf. e.g. Carboni–Vitale '98).

# Quantifier splittings

**Definition** ($(\mathcal{F}, \coprod)$-quantifier splitting objects)

Let $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ be a fibration with all $\mathcal{F}$-coproducts. For $A \in \mathsf{B}$, an object $\alpha \in \mathsf{E}_A$ in the fiber is called **(generalised)** $(\mathcal{F}, \coprod)$**-quantifier splitting** he following holds:

- Given $\beta \in \mathsf{E}_B$ for some $B \in \mathsf{B}$,
- together with a vertical map $h \in \mathsf{E}_A(\alpha, \coprod_u \beta)$, where $u : B \twoheadrightarrow A$ is an arrow of $\mathcal{F}$

there uniquely exist the following:

1. a section $g\colon A \to B$ of $u\colon B \twoheadrightarrow A$
2. and a vertical arrow $\overline{h} : \alpha \to g^*\beta$ in $\mathsf{E}_A$ such that the vertical arrow $h$ decomposes as

$$\begin{array}{ccc} f^*\alpha & \xrightarrow{\quad h \quad} & g^*(u^* \coprod_u \beta) \cong \coprod_u \beta \\ & \overset{\overline{h}}{\searrow} & \uparrow g^*\eta_\beta \\ & & g^*\beta \end{array}$$

# Quantifier-free elements

**Definition (($\mathcal{F}, \coprod$)-quantifier free objects)**

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**Definition (Enough $(\mathcal{F}, \coprod)$-quantifier free objects)**

Let $\langle \mathsf{B}, \mathcal{F} \rangle$ be a display map category. A fibration $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ is said to have **enough** $(\mathcal{F}, \coprod)$**-quantifier free objects** if it has all $\mathcal{F}$-coproducts and the following property holds: for all $I \in \mathsf{B}$ and $\alpha \in \mathsf{E}_I$ there exists some object $A \in \mathsf{B}$, an arrow $f : A \twoheadrightarrow I$ in $\mathcal{F}$, and $\beta \in \mathsf{E}_A$ such that $\alpha \cong \coprod_f(\beta)$.

# Skolem fibrations

We want to recover a categorified version of the *Skolem principle*

$$\forall x \exists y \varphi(x, y) \leftrightarrow \exists f \forall x \varphi(x, fx),$$

**Definition (Skolem fibration)**

Let $\langle \mathsf{B}, \mathcal{F} \rangle$ be a display map category. A fibration $\mathsf{p} \colon \mathsf{E} \longrightarrow \mathsf{B}$ is called a *(dependent) Skolem fibration* if

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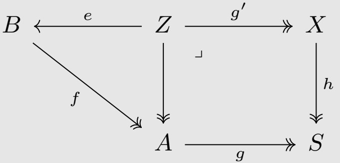
- its base category B has dependent products along $\mathcal{F}$;
- the fibration p has fibred products and coproducts along $\mathcal{F}$;
- the fibration p has enough $(\mathcal{F}, \coprod)$-quantifier-free objects;
- $(\mathcal{F}, \coprod)$-quantifier free objects are stable under $\mathcal{F}$-products, *i.e.*, if for any $(\mathcal{F}, \coprod)$-quantifier free object $\alpha \in \mathsf{E}_I$, $I \in \mathsf{B}$, the object $\prod_f (\alpha) \in \mathsf{E}_J$ is $(\mathcal{F}, \coprod)$-quantifier free, too, for any map $f : I \twoheadrightarrow J$ in $\mathcal{F}$.

# Skolemization

**Theorem (Skolemization, Trotta–W–de Paiva)**

*Let $\langle \mathsf{B}, \mathcal{F}\rangle$ be a display map category and $\mathsf{p}\colon \mathsf{E} \longrightarrow \mathsf{B}$ a Skolem fibration over it. Let $g : A \to S$ and $f : B \to A$ be maps in $\mathcal{F}$. Consider the $\mathcal{F}$-dependent product of $f$ along $g$, as given by the diagram:*




*Then there is a natural isomorphism*

$$\prod_g \coprod_f \cong \coprod_h \prod_{g'} e^*$$

*between functors from $\mathsf{E}_B$ to $\mathsf{E}_S$.*

# Gödel fibrations

Following the non-dependent setting of Spadetto–Trotta–de Paiva, we introduce, a *dependent* notion of Gödel fibration. The idea is to consider Skolem fibrations with the additional property that every formula $\alpha(i)$ can be brought to prenex normal form, *i.e.*, there exists a quantifier-free formula $\beta(x, y, i)$ s.t. $\alpha(i) \cong \beta(x, y, i)$.

**Definition (Dependent Gödel fibration)**

Let $\langle \mathsf{B}, \mathcal{F} \rangle$ be a dmc and $\mathsf{p} \colon \mathsf{E} \longrightarrow \mathsf{B}$ be a Skolem fibration over it. Then $\mathsf{p}$ is called a **(dependent) Gödel fibration** if its subfibration of $(\mathcal{F}, \coprod)$-quantifier free elements has enough $(\mathcal{F}, \prod)$-quantifier free objects.

# Main result: Dependent Gödel = dependent Dialectica

With Davide Trotta and Valeria de Paiva we are currently working towards the following results:

**Theorem (Internal characterization of $\Sigma_{\mathcal{F}}$)**

*A fibration with $\mathcal{F}$-coproducts is an instance of an $\mathcal{F}$-coproduct completion if and only if it has enough $(\mathcal{F}, \coprod)$-quantifier free objects.*

**Theorem (Gödel = Dialectica, dependently)**

*Let $\langle \mathsf{B}, \mathcal{F} \rangle$ be a dmc such that $\mathsf{B}$ has dependent $\mathcal{F}$-products. Consider a fibration $\mathsf{p} \colon \mathsf{E} \longrightarrow \mathsf{B}$ a fibration with $\mathcal{F}$-products and $\mathcal{F}$-coproducts.*
*Then the following are equivalent:*

1. *There exists a fibration $\mathsf{p}'$ over $\langle \mathsf{B}, \mathcal{F} \rangle$ such that $\mathsf{p} \cong \mathfrak{Dial}_{\mathcal{F}}(\mathsf{p}')$.*
2. *$\mathsf{p}$ is a Gödel fibration.*