Sheafification as a geometric tripos-to-topos adjunction

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Tripos-to-topos

$\mathsf{A}^{(-)} \colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{Hey}$ is a **localic tripos**

$P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Hey}$ is a **tripos**

$$\mathsf{A}^{(-)} \xmapsto{\text{T-t-T}} \mathsf{Sh}(A)$$

$$P \xmapsto{\text{T-t-T}} \mathsf{T}_P$$

J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts (1980), *Tripos theory*, Math. Proc. Camb. Phil. Soc.
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Tripos-to-topos

$\mathsf{A}^{(-)} \colon \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ is a **localic tripos**

$P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ is a **tripos**

$$\mathsf{A}^{(-)} \xmapsto{\text{T-t-T}} \mathrm{Sh}(A)$$

$$\mathrm{Sh}(A) \underset{\longleftarrow}{\overset{\hookrightarrow}{\bot}} \mathrm{PSh}(A)$$

$$P \xmapsto{\text{T-t-T}} \mathsf{T}_P$$

Tripos-to-topos

$A^{(-)} : \mathsf{Set}^{op} \longrightarrow \mathsf{Hey}$ is a **localic tripos**

$P : \mathcal{C}^{op} \longrightarrow \mathsf{Hey}$ is a **tripos**

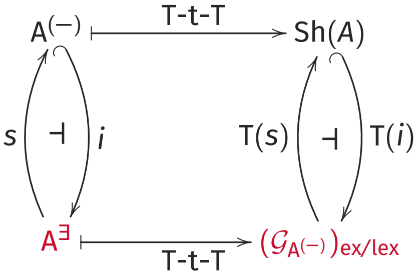
$$\begin{array}{ccc} A^{(-)} & \xmapsto{\text{T-t-T}} & \mathrm{Sh}(A) \\ & & \uparrow \dashv \downarrow \\ & & \mathrm{PSh}(A) \equiv (\mathcal{G}_{A^{(-)}})_{ex/lex} \end{array}$$

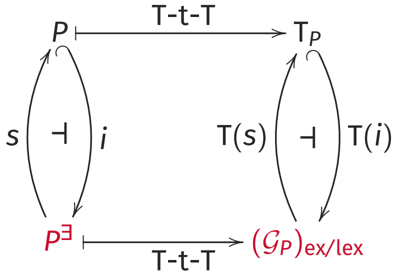
$$P \xmapsto{\text{T-t-T}} \mathrm{T}_P$$

Tripos-to-topos

$\mathsf{A}^{(-)} \colon \mathsf{Set}^{op} \longrightarrow \mathsf{Hey}$ is a **localic tripos**

$P \colon \mathcal{C}^{op} \longrightarrow \mathsf{Hey}$ is a **∃-sheaf tripos**





Main references

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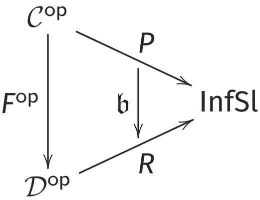
Full primary doctrines

Definition (full primary doctrine)

A **full primary doctrine** is a functor $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSl}$ from the opposite of a category $\mathcal{C}$ with finite limits to the category of inf-semilattices.

A **full primary morphism** of doctrines is given by a pair $(F, \mathfrak{b})$



where $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a finite limits preserving functor and $\mathfrak{b}\colon P \longrightarrow R \circ F$ is a natural transformation.

Full existential doctrines

Definition (full existential doctrine)

A full primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSl}$ is called a **full existential doctrine** if for every arrow $f\colon A \longrightarrow B$ of $\mathcal{C}$ the functor P_f has a left adjoint $\exists_f$ and these satisfy Beck-Chevalley condition and Frobenius reciprocity.

Definition (full existential morphism of doctrines)

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSl}$ and $R\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathrm{InfSl}$ be two full existential doctrines. A full primary morphism of doctrines $(F, \mathfrak{b})$ is said **full existential** if for every arrow $f\colon A \longrightarrow B$ of $\mathcal{C}$ we have that

$$\exists_{Ff}\mathfrak{b}_A(\alpha) = \mathfrak{b}_B(\exists_f(\alpha))$$

for every element α of $P(A)$.

Full hyperdoctrines and full triposes

Definition (full hyperdoctrine)

A full existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSl}$ is said **full hyperdoctrine** if

- for every object A of $\mathcal{C}$, the poset $P(A)$ is a Heyting algebra and for every arrow $f\colon A \longrightarrow B$, $P_f\colon P(B) \longrightarrow P(A)$ is a morphism of Heyting algebras;
- for every arrow $f\colon A \longrightarrow B$, the functor P_f has a right adjoint $\forall_f$ and these satisfy Beck-Chevalley condition.

Definition (full tripos)

A full hyperdoctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSl}$ is said **full tripos** if for every object X of $\mathcal{C}$ there exists an object $\mathrm{P}X$ and an element $\in_X$ of $P(X \times \mathrm{P}X)$ such that for every α of $P(X \times Y)$ there exists an arrow $\{\alpha\}_X\colon Y \longrightarrow \mathrm{P}X$ such that $\alpha = P_{\mathrm{id}_X \times \{\alpha\}_X}(\in_X)$.

Examples

Example

Let A be a locale. The representable functor $\mathsf{A}^{(-)} : \mathsf{Set}^{op} \longrightarrow \mathsf{InfSl}$ assigning to a set X the poset A^X of functions from X to A with the pointwise order is a full tripos.

Example

Given a pca $\mathbb{A}$, we can consider the realizability tripos $\mathcal{P}_{\mathbb{A}} : \mathsf{Set}^{op} \longrightarrow \mathsf{InfSl}$ over Set. For each set X, the partial ordered set $(\mathcal{P}_{\mathbb{A}}(X), \leq)$ is defined as the set of functions $P(\mathbb{A})^X$ from X to the powerset $P(\mathbb{A})$ of $\mathbb{A}$. Given two elements α and β of $\mathcal{P}_{\mathbb{A}}(X)$, we say that $\alpha \leq \beta$ if there exists an element $\overline{a} \in \mathbb{A}$ such that for all $x \in X$ and all $a \in \alpha(x)$, $\overline{a} \cdot a$ is defined and it is an element of $\beta(x)$.

Examples

Example

Let $\mathcal{C}$ be a category with finite limits. The full existential doctrine of **weak subobjects (or variations)** is given by the functor

$$\Psi_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$$

where $\Psi_{\mathcal{C}}(A)$ is the poset reflection of the slice category $\mathcal{C}/A$. For an arrow $B \xrightarrow{f} A$, the homomorphism $\Psi_{\mathcal{D}}([f]) \colon \Psi_{\mathcal{D}}(A) \longrightarrow \Psi_{\mathcal{D}}(B)$ is given by the equivalence class of a pullback of an arrow $X \xrightarrow{g} A$ with f. This doctrine is a full tripos if and only if $\mathcal{C}$ has weak dependent products and a generic proof.

Full triposes and Presheaves

Definition

Let $P\colon \mathcal{C}^{op} \longrightarrow \mathrm{InfSl}$ be a full tripos. The **Grothendieck category** $\mathcal{G}_P$ of P is given by the following objects and arrows:

- objects are pairs (A, α), where A is an object of $\mathcal{C}$ and $\alpha \in P(A)$;
- a morphism $f\colon (A, \alpha) \longrightarrow (B, \beta)$ is an arrow $f\colon A \longrightarrow B$ of $\mathcal{C}$ such that $\alpha \leq P_f(\beta)$.

Definition

Let $P\colon \mathcal{C}^{op} \longrightarrow \mathrm{InfSl}$ be a full tripos. We define the category of P-**presheaves** as the category $\mathrm{PSh}(P) := (\mathcal{G}_P)_{\mathrm{ex/lex}}$.

Examples

Example

Let A be a locale and the localic tripos $A^{(-)} \colon \mathrm{Set}^{op} \longrightarrow \mathrm{InfSl}$. We have the equivalence $\mathrm{PSh}(A) \equiv (A_+)_{ex/lex} \equiv (\mathcal{G}_{A^{(-)}})_{ex/lex}$.

Example

Let $\mathbb{A}$ be a pca, and let us consider the realizability tripos $\mathcal{P} \colon \mathrm{Set}^{op} \longrightarrow \mathrm{InfSl}$. The category $\mathcal{G}_{\mathcal{P}}$ can be described as follows: they are pairs (X, α), where X is a set and $\alpha \subseteq X \times \mathbb{A}$ is a relation. A morphism $f \colon (X, \alpha) \longrightarrow (B, \beta)$ is given by a function $f \colon X \longrightarrow Y$ such that there exists an element $a \in \mathbb{A}$ that tracks f.

$$\mathrm{RT}(\mathbb{A}) \hookrightarrow (\mathcal{G}_{\mathcal{P}})_{ex/lex} \equiv \mathrm{PSh}(\mathcal{P}).$$

Tripos-to-topos

Tripos-to-topos. Given a full tripos $P\colon \mathcal{C}^{op} \longrightarrow \mathrm{InfSl}$, the topos T_P consists of:

- **objects:** are pairs (A, ρ) where A is an object of $\mathcal{C}$ and ρ is an element of $P(A \times A)$ satisfying:
  1. *symmetry:* $a_1, a_2 : A \mid \rho(a_1, a_2) \vdash \rho(a_2, a_1)$;
  2. *transitivity:* $a_1, a_2, a_3 : A \mid \rho(a_1, a_2) \wedge \rho(a_2, a_3) \vdash \rho(a_1, a_3)$;
- **arrows:** $\phi\colon (A, \rho) \longrightarrow (B, \sigma)$ are objects ϕ of $P(A \times B)$ such that:
  1. $a : A, b : B \mid \phi(a, b) \vdash \rho(a, a) \wedge \sigma(b, b)$;
  2. $a_1, a_2 : A, b : B \mid \rho(a_1, a_2) \wedge \phi(a_1, b) \vdash \phi(a_2, b)$;
  3. $a : A, b_1, b_2 : B \mid \sigma(b_1, b_2) \wedge \phi(a, b_1) \vdash \phi(a, b_2)$;
  4. $a : A, b_1, b_2 : B \mid \phi(a, b_1) \wedge \phi(a, b_2) \vdash \sigma(b_1, b_2)$;
  5. $a : A \mid \rho(a, a) \vdash \exists b.\phi(a, b)$.

J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts (1980), *Tripos theory*, Math. Proc. Camb. Phil. Soc.

A.M. Pitts (2002), *Tripos theory in retrospect*, Math. Struct. in Comp. Science

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Full existential completion

Full existential completion. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSl}$ be a full primary doctrine. For every object A of $\mathcal{C}$ consider the preorder $P^{\exists}(A)$ defined by:

- **objects:** pairs $(B \xrightarrow{f} A, \alpha)$, where $B \xrightarrow{f} A$ is an arrow of $\mathcal{C}$ and $\alpha \in P(B)$;
- **order:** $(B \xrightarrow{f} A, \alpha) \leq (C \xrightarrow{g} A, \beta)$ if there exists an arrow $h\colon B \longrightarrow C$ of $\mathcal{C}$ such that the diagram

  $$\begin{array}{ccc} & & B \\ & \swarrow^{h} & \downarrow^{f} \\ C & \xrightarrow[g]{} & A \end{array}$$

  commutes and $\alpha \leq P_h(\beta)$.

The doctrine $P^{\exists}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSl}$ is called the **full existential completion** of P.

D. Trotta (2020), *The existential completion*, Theory and Applications of Categories

Examples of full existential completion

The following doctrines are instances of the full existential completion:

1. the **realizability doctrine** $\mathcal{P}_{\mathbb{A}} \colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ (for a given a pca $\mathbb{A}$);
2. the **localic doctrine** $\mathrm{A}^{(-)} \colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ when A is a supercoherent locale;
3. the **weak subobjects doctrine** $\Psi_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ for a lex category $\mathcal{C}$.

M.E. Maietti and D. Trotta (2023), , *A characterization of generalized existential completions*, Ann. Pure Appl. Log.

A characterization of the tripos-to-topos construction

Theorem

Let $P\colon \mathcal{C}^{\text{op}} \longrightarrow \mathsf{InfSl}$ *be a full tripos. If P is the full existential completion of a full primary doctrine* $P'\colon \mathcal{C}^{\text{op}} \longrightarrow \mathsf{InfSl}$ *then:*

$$\mathsf{T}_P \cong (\mathcal{G}_{P'})_{\text{ex/lex}}.$$

Categories obtained as **full existential completion + tripos-to-topos**:

Example

- realizability toposes $\mathrm{RT}(\mathbb{A})$ for a given pca $\mathbb{A}$;
- toposes of presheaves $\mathrm{PreSh}(\mathrm{A})$ for a given locale A;
- toposes of sheaves $\mathrm{Sh}(\mathrm{A})$ for a given supercoherent locale A;
- the exact completion $(\mathcal{C})_{\text{ex/lex}}$ of a lex category $\mathcal{C}$.

Sheaf triposes

Definition ($\exists$-sheaf tripos)

A full tripos $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSl}$ is said a $\exists$-**sheaf tripos** if the Grothendieck category $\mathcal{G}_P$ has weak dependent products and a generic proof.

Theorem

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSl}$ *be a full tripos. Then the following are equivalent:*

1. $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSl}$ *is a* $\exists$*-sheaf tripos;*
2. $\Psi_{\mathcal{G}_P}\colon \mathcal{G}_P^{\mathrm{op}} \longrightarrow \mathrm{InfSl}$ *is a full tripos;*
3. $P^{\exists}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSl}$ *is a full tripos;*
4. $\mathrm{PSh}(P)$ *is a topos.*

Theorem

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSl}$ *be a* $\exists$*-sheaf tripos. Then there is an adjunction of triposes*

$$P^{\exists} \underset{i}{\overset{s}{\rightleftarrows}} P \qquad (s \dashv i)$$

such that $si \cong \mathrm{id}_P$*, and* s *is a full existential morphism. Moreover, this induces an adjunction of toposes*

$$\mathsf{T}_{P^{\exists}} \underset{\mathsf{T}(i)}{\overset{\mathsf{T}(s)}{\rightleftarrows}} \mathsf{T}_P \qquad (\mathsf{T}(s) \dashv \mathsf{T}(i))$$

such that $\mathsf{T}(s)\mathsf{T}(i) \cong \mathrm{id}_{\mathsf{T}_P}$.

M.E. Maietti and D. Trotta (2023), , *A characterization of generalized existential completions*, Ann. Pure Appl. Log.

J. Frey (2015), *Triposes, q-toposes and toposes*, Ann. of Pure and Appl. Logic

Tripos-to-topos of ∃-sheaf triposes

Corollary

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSl}$ *be a* ∃*-sheaf tripos. Then there exists a Lawvere-Tierney topology* $j^{\exists}$ *on* $\mathsf{T}_{P^{\exists}}$ *such that* $\mathsf{T}_P \equiv \mathrm{Sh}_{j^{\exists}}(\mathsf{T}_{P^{\exists}})$.

Sufficient conditions for ∃-sheaf triposes

Theorem

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ *be a full tripos such that*

- $\mathcal{C}$ *has weak dependent products;*
- *the weak predicate classifier* Ω *has a power object in* $\mathcal{C}$*;*
- $\mathcal{C}$ *admits a proper factorization system* $\langle \mathcal{E}, \mathcal{M} \rangle$ *and every epi of* $\mathcal{E}$ *splits.*

Then $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ *is a* ∃*-sheaf tripos.*

Corollary

Every full tripos whose base category is Set *(with the axiom of choice) is a* ∃*-sheaf tripos.*

Example

The localic tripos $\mathsf{A}^{(-)} : \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ is a $\exists$-sheaf tripos. The adjunction

$$\mathsf{PSh}(A) \underset{\hookleftarrow}{\overset{\to}{\bot}} \mathsf{Sh}(A)$$

is exactly the so-called sheafification.

Example

The realizability tripos $\mathcal{P}_{\mathbb{A}} : \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ is a $\exists$-sheaf tripos. Then, we have

$$\mathsf{PSh}(\mathcal{P}_{\mathbb{A}}) \underset{\hookleftarrow}{\overset{\to}{\bot}} \mathsf{RT}(\mathbb{A})$$

and hence that $\mathsf{RT}(\mathbb{A}) \equiv \mathsf{Sh}_{j^{\exists}}(\mathsf{T}_{\mathcal{P}_{\mathbb{A}}{}^{\exists}})$.