Lax epimorphisms in CAT, $\mathcal{V}$-Cat and everywhere

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Joint work with Fernando Lucatelli Nunes

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Outline

1. Lax epimorphisms in 2-categories
2. in $\mathcal{V}$-Cat
3. in CAT

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A functor $A \xrightarrow{f} B$ is fully faithful iff the hom-functor

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Equivalently, for every 2-cell $\alpha : gf \to hf$,

$$\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ A \arrow[ur, "f"] \arrow[dr, "f"'] & & X \\ & B \arrow[ur, "h"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\overline{\alpha}"] & X \end{tikzcd}$$

for a unique $\overline{\alpha} : g \to h$, $\alpha = \overline{\alpha} * f$

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6. In Grp, with 2-cells given by $\alpha : f \Rightarrow g : A \to B$, with $\alpha \in B$ and $f(x) \circ \alpha = \alpha \circ f(x)$ for all $x \in A$,
   Lax epimorphisms $=$ Regular epimorphisms

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- are preserved by left 2-adjoints.

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For every small 2-category $\mathbb{S}$, every weight $W : \mathbb{S}^{\mathsf{op}} \to \mathsf{Cat}$, and every 2-natural transformation $\lambda : D \to D' : \mathbb{S} \to \mathbb{A}$, if λ is a pointwise lax epimorphism, then the induced morphism

$$\mathrm{colim}\,(W, \lambda) : \mathrm{colim}(W, D) \to \mathrm{colim}(W, D')$$

is a lax epimorphism in $\mathbb{A}$.

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$\mathcal{E}$, $\mathcal{M} \subseteq \mathrm{Mor}(\mathbb{A})$, closed under composition with isomorphisms from the left and from the right, respectively.

$(\mathcal{E}, \mathcal{M})$ forms an orthogonal factorization system provided that:

(i) Every morphism f has an $(\mathcal{E}, \mathcal{M})$ factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\mathcal{E}\ni e"'] & & B \\ & \bullet \arrow[ur, "m\in\mathcal{M}"'] & \end{tikzcd}$$

(ii) For every $A \xrightarrow{e} B$ in $\mathcal{E}$ and $C \xrightarrow{m} D$ in $\mathcal{M}$,

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, dashed, "t"] \\ C \arrow[r, "m"'] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "e"] \arrow[d, bend right, "f"'] \arrow[d, bend left, "f'"] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & \arrow[d, phantom, "\overset{\theta}{\Rightarrow}"] & B \arrow[dll, dashed, "t"'] \arrow[d, bend right, "g'"'] \arrow[d, bend left, "g"] \arrow[dl, phantom, "\overset{\beta}{\Leftarrow}"] \\ C \arrow[rr, "m"'] & & D \arrow[ll, dashed, "t'"'] \end{tikzcd}$$

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$\longrightarrow$ If e is a lax epi, the second of (ii) comes for free.

A 1-cell $m : C \to D$ is a lax strong monomorphism if it has the diagonal fill-in property w.r.t. lax epimorphisms e:

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Proposition. In a 2-category:

1. Every inserter is a lax strong monomorphism.
2. In the presence of coequifiers, every lax strong monomorphism is faithful, i.e., a morphism m such that $\mathbb{A}(X, m)$ is faithful for all X.

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   Lax strong monomorphisms = Order-embeddings = Inserters
2. In general, Inserters $\neq$ Lax Strong Monos
3. Neither equifiers nor equalizers are, in general, lax strong monomorphisms. (Counter-examples in Cat)
4. In Cat, every coproduct injection $A_j \xrightarrow{\nu_j} \coprod A_i$ is a lax strong mono.

Cat is <u>not</u> cowellpowered w.r.t. lax epimorphisms.

But it is <u>almost cowellpowered</u> w.r.t. lax epimorphisms:

$\mathcal{E} \subseteq \mathrm{Mor}(\mathbb{A})$, $A \xrightarrow{f} B$

$$f \,|\, \mathcal{E}$$

consists of

$\mathcal{E}$-factorizations

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\mathcal{E} \ni e"'] & & B \\ & E \arrow[ur, "g"'] & \end{tikzcd}$$

and morphisms between them

$$\begin{tikzcd} & E \arrow[dr, "g"] \arrow[dd, "d"] & \\ A \arrow[ur, "e"] \arrow[dr, "e'"'] & & B \\ & E' \arrow[ur, "g'"'] & \end{tikzcd}$$

$\mathbb{A}$ is said to be almost cowellpowered w.r.t. $\mathcal{E}$, if $f \,|\, \mathcal{E}$ has a weakly terminal set for every morphism f.

Theorem. Let $\mathbb{A}$ be a 2-category with conical colimits. $\mathbb{A}$ has an orthogonal $(\mathit{LaxEpi}, \mathit{LaxStrongMono})$-factorization system if and only if it is almost cowellpowered w.r.t. lax epimorphisms.

2. In $\mathcal{V}$-Cat

$\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ symmetric monoidal closed (co)complete category

$\mathcal{V}$-Cat $=$ 2-category of (small) $\mathcal{V}$-categories, $\mathcal{V}$-funct., $\mathcal{V}$-nat. transf.

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$\mathcal{V}$-Cat$[\mathcal{A}, \mathcal{B}]$ $=$ $\mathcal{V}$-category of $\mathcal{V}$-functors with hom-objects

$$\mathcal{V}\text{-Cat}[\mathcal{A}, \mathcal{B}](F, G) = \int_{A \in \mathcal{A}} \mathcal{B}(FA, GA)$$

$[\mathcal{A}, \mathcal{B}] := \mathcal{V}\text{-Cat}[\mathcal{A}, \mathcal{B}]$

A $\mathcal{V}$-functor $F : \mathcal{A} \to \mathcal{B}$ is $\mathcal{V}$-fully faithful if the maps

$$F_{AA'} : \mathcal{A}(A, A') \to \mathcal{B}(FA, FA')$$

are all isomorphisms in $\mathcal{V}_0$.

A $\mathcal{V}$-functor $F : \mathcal{A} \to \mathcal{B}$ is:

- a lax epimorphism if the functor

$$\mathcal{V}\text{-Cat}(F, \mathcal{C}) : \mathcal{V}\text{-Cat}(\mathcal{B}, \mathcal{C}) \to \mathcal{V}\text{-Cat}(\mathcal{A}, \mathcal{C})$$

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$$\mathcal{V}\text{-Cat}\,[F, \mathcal{C}] : \mathcal{V}\text{-Cat}\,[\mathcal{B}, \mathcal{C}] \to \mathcal{V}\text{-Cat}\,[\mathcal{A}, \mathcal{C}]$$

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Theorem. For a $\mathcal{V}$-functor $F : \mathcal{A} \to \mathcal{B}$ in $\mathcal{V}$-Cat, the following conditions are equivalent:

(a) F is a $\mathcal{V}$-lax epimorphism.

(b) The functor $\mathcal{V}\text{-Cat}(F, \mathcal{V}) : \mathcal{V}\text{-Cat}(\mathcal{B}, \mathcal{V}) \to \mathcal{V}\text{-Cat}(\mathcal{A}, \mathcal{V})$ is fully faithful.

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[Lucatelli Nunes, Prezado, S., Bull. Belg. Math. Soc., 2023]

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For $\mathcal{V} = \mathrm{Set}$, (b)$\Leftrightarrow$ (c) $\Leftrightarrow$ (e) is in
[Adámek, El Bashir, Sobral, Velebil, TAC, 2001]

3. In CAT

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Orthogonal factorization systems in Cat:

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Next:

- (Lax Epimorphisms, Discrete Splitting Bifibrations)

Functor $F : \mathbb{A} \to \mathbb{B}$, morphism $g : b \to c$ of $\mathbb{B}$

Category $g//F$: has objects $b \xrightarrow{h} Fa \xrightarrow{k} b$ with $kh = g$
and morphisms $f : (h, k) \to (h', k')$ given by

$$\begin{tikzcd}
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For $\begin{tikzcd} b \arrow[r, "r"] \arrow[rr, bend left, "g"] & Fa \arrow[r, "s"] & c \end{tikzcd}$, $[(r, s)]$:= connected component of (r, s) in $g//F$

Let $P : \mathbb{E} \to \mathbb{B}$ be a functor.

A P-split diagram is a rectangle (blue)

$$\begin{tikzcd}
b \arrow[r, "h"] \arrow[d, "g"'] \arrow[rr, bend left, "1_b"] & Pe \arrow[r, "k"] \arrow[d, squiggly, no head] & b \arrow[d, "g"] \\
c \arrow[r, "h'"'] \arrow[rr, bend right, "1_c"'] & Pe' \arrow[r, "k'"'] & c
\end{tikzcd}$$

with $[(1_{Pe}, hk)] = [(hk, 1_{Pe})]$ in $hk//P$

with $[(1_{Pe'}, h'k')] = [(h'k', 1_{Pe'})]$ in $h'k'//P$

and $[(h, gk)] = [(h'g, k')]$ in $g//P$.

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and $[(h, gk)] = [(h'g, k')]$ in $g//P$.

$P : \mathbb{E} \to \mathbb{B}$ is a discrete splitting bifibration if every P-split diagram as above can be uniquely lifted to the category $\mathbb{E}$, that is, there is a unique rectangle

$$\begin{tikzcd}
b_0 \arrow[r, "h_0"] \arrow[d, "g_0"'] & e \arrow[r, "k_0"] & b_0 \arrow[d, "g_0"] \\
c \arrow[r, "h'_0"'] & e' \arrow[r, "k'_0"'] & c
\end{tikzcd}$$

whose image by P is the above blue rectangle.

Proposition. Every discrete splitting bifibration

1. is faithful,
2. is conservative, and
3. reflects identities.

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Description of the (LaxEpi,DSB)-factorization

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "F"] \arrow[dr, "E"'] & & \mathbb{B} \\
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ob($\mathbb{E}$): $\quad (b, B)$ with $b \in \mathbb{B}$ and B a connected component of $1_b // F$ containing some $b \xrightarrow{h} Fa \xrightarrow{k} b$ with $[(1_{Fa}, hk)] = [(hk, 1_{Fa})]$

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Special examples

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Proposition. Every topological functor is a lax epimorphism.

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Proposition. If $F : \mathbb{A} \to \mathbb{B}$ is a full functor, the $(LaxEpi, DSB)$-factorization is given by the Lawvere closure of $\mathbb{A}$ over F (i.e. the full subcategory of $\mathbb{B}$ consisting of all b such that there is $b \xrightarrow{h} Fa \xrightarrow{k} b$ with $kh = 1_b$).

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "F"] \arrow[dr] & & \mathbb{B} \\
& \bar{\mathbb{A}} \arrow[ur] &
\end{tikzcd}$$

In case F is also a right adjoint, it coincides with the Eilenberg-Moore factorization.

A construction of the LaxEpi-factorization in Cat

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$\mathcal{A} \xrightarrow{F} \mathcal{B}$ is a lax epimorphism iff

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "F"'] \arrow[dr, phantom, "id \Nearrow"] & \mathcal{B} \arrow[d, "Y"] \\
\mathcal{B} \arrow[r, "Y"'] & {[\mathcal{B}^{\mathrm{op}}, \mathrm{Set}]}
\end{tikzcd}$$

exhibits Y as a left Kan extension of YF along F.

A construction of the LaxEpi-factorization in Cat

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] \arrow[d, "F"'] & \mathbb{B} \arrow[d, "L"] \arrow[dr, bend left, "Y"] \arrow[d, phantom, shift left=4ex, "\overset{\mu}{\Rightarrow}"] & \\
\mathbb{B} \arrow[r, "Y"'] \arrow[ur, Rightarrow, "\lambda"] & {[\mathbb{B}^{op}, \mathsf{Set}]} &
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$(L, \lambda) = \mathsf{Lan}_F(YF)$

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$$\begin{tikzcd}
& \mathbb{B} \arrow[dr, "L"] & \\
\mathbb{I} \arrow[ur, "I"] \arrow[dr, "I"'] & \arrow[u, phantom, "\Uparrow \alpha"] & {[\mathbb{B}^{op}, \mathsf{Set}]} \\
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$(I, \alpha) =$ inserter of (Y, L)

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& & \mathbb{B} \arrow[dr] \arrow[dr, phantom, shift left=4ex, "\overset{\mu}{\Rightarrow}"] \arrow[dr, phantom, shift right=4ex, "L"] \arrow[dr, bend left, "Y"] & \\
\mathbb{D} \arrow[r, "D"] & \mathbb{I} \arrow[ur, "I"] \arrow[dr, "I"'] & \arrow[u, phantom, "\Uparrow \alpha"] & {[\mathbb{B}^{op}, \mathrm{Set}]} \\
& & \mathbb{B} \arrow[ur, "Y"'] &
\end{tikzcd}$$

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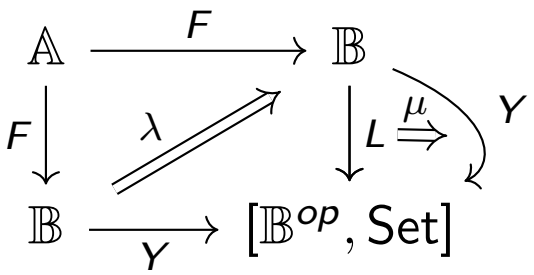

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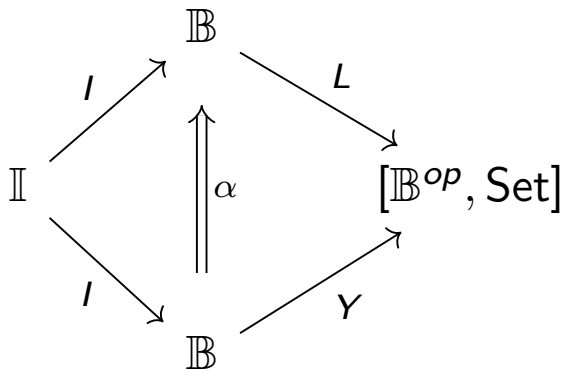

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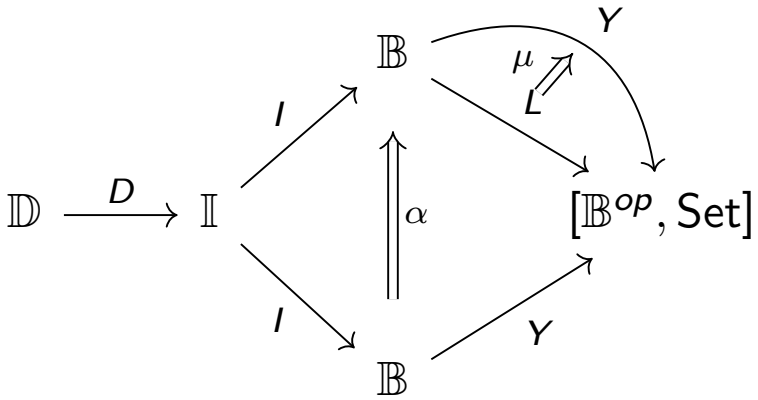

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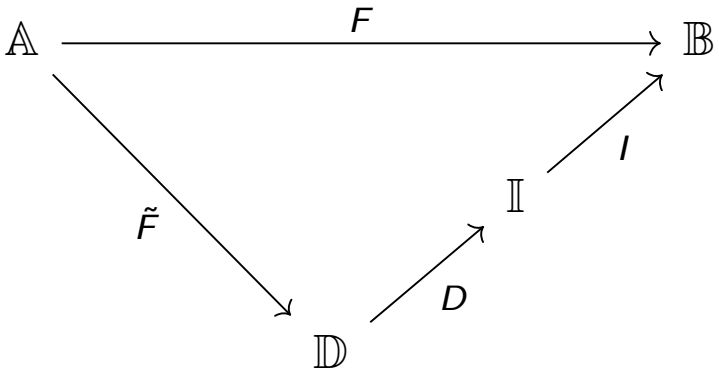

A construction of the LaxEpi-factorization in Cat

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] \arrow[d, "F"'] \arrow[dr, phantom, "\overset{\lambda}{\Nearrow}"] & \mathbb{B} \arrow[d, "L"] \arrow[dr, phantom, "\overset{\mu}{\Rightarrow}"] \arrow[dr, bend left, "Y"] & \\
\mathbb{B} \arrow[r, "Y"'] & {[\mathbb{B}^{op}, \mathrm{Set}]} &
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$(L, \lambda) = \mathrm{Lan}_F(YF)$

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$$\begin{tikzcd}
 & \mathbb{B} \arrow[dr, "L"] & \\
\mathbb{I} \arrow[ur, "I"] \arrow[dr, "I"'] & \arrow[u, phantom, "\overset{\alpha}{\Uparrow}"] & {[\mathbb{B}^{op}, \mathrm{Set}]} \\
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$(I, \alpha) =$ inserter of (Y, L)

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\mathbb{D} \arrow[r, "D"] & \mathbb{I} \arrow[ur, "I"] \arrow[dr, "I"'] & \arrow[u, phantom, "\overset{\alpha}{\Uparrow}"] & {[\mathbb{B}^{op}, \mathrm{Set}]} \\
 & \mathbb{B} \arrow[ur, "Y"'] & &
\end{tikzcd}$$

$D =$ identifier of $(\mu I)\alpha$

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "F"] \arrow[dr, "\tilde{F}"'] & & \mathbb{B} \\
 & \mathbb{D} \arrow[r, "D"] & \mathbb{I} \arrow[u, "I"']
\end{tikzcd}$$

Using the Lawvere closure of $\tilde{F}$,

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "F"] \arrow[dr, "\overline{\tilde{F}}"'] & & \mathbb{B} \\
 & \overline{\mathbb{A}} \arrow[ur] &
\end{tikzcd}$$