The representing localic groupoid of a geometric theory

Graham Manuell
graham@manuell.me
University of Coimbra

Joint work with Joshua Wrigley

CT2023

Geometric theories

A geometric theory is defined by basic sorts X, basic relations $R \subseteq X_1 \times \cdots \times X_n$ (including basic propositions for $n = 0$) and axioms.

Geometric theories

A geometric theory is defined by basic sorts X, basic relations $R \subseteq X_1 \times \cdots \times X_n$ (including basic propositions for $n = 0$) and axioms. Formulae in the axioms are built up from basic relations and the equality relation using finite conjunctions, arbitrary disjunctions, and existential quantification over sorts.

Geometric theories

A geometric theory is defined by basic sorts X, basic relations $R \subseteq X_1 \times \cdots \times X_n$ (including basic propositions for $n = 0$) and axioms. Formulae in the axioms are built up from basic relations and the equality relation using finite conjunctions, arbitrary disjunctions, and existential quantification over sorts.

For example, consider the theory of *inhabited total orders*. This has one sort X, one relation $\leq \subseteq X \times X$ and the following axioms.

$$
\begin{aligned}
&\vdash_{x : X} && x \leq x && \text{(reflexivity)} \\
x \leq y \wedge y \leq z &\vdash_{x,y,z : X} && x \leq z && \text{(transitivity)} \\
x \leq y \wedge y \leq x &\vdash_{x,y : X} && x = y && \text{(antisymmetry)} \\
&\vdash_{x,y : X} && x \leq y \vee y \leq x && \text{(totality)} \\
&\vdash_{\emptyset} && \exists x : X.\ \top && \text{(inhabitedness)}
\end{aligned}
$$

Classifying toposes

We can define models of a geometric theory in any (Grothendieck) topos.

Classifying toposes

We can define models of a geometric theory in any (Grothendieck) topos.

In fact, Grothendieck toposes are themselves a presentation-independent way to describe geometric theories.

For each geometric theory $\mathbb{T}$, there is a classifying topos $\mathbf{Set}[\mathbb{T}]$ satisfying $\mathrm{Geom}(\mathcal{E}, \mathbf{Set}[\mathbb{T}]) \simeq \mathrm{Mod}_{\mathbb{T}}(\mathcal{E})$.

Classifying toposes

We can define models of a geometric theory in any (Grothendieck) topos.

In fact, Grothendieck toposes are themselves a presentation-independent way to describe geometric theories.

For each geometric theory $\mathbb{T}$, there is a classifying topos $\textbf{Set}[\mathbb{T}]$ satisfying $\mathrm{Geom}(\mathcal{E}, \textbf{Set}[\mathbb{T}]) \simeq \mathrm{Mod}_{\mathbb{T}}(\mathcal{E})$.

There is a generic model $M^{\mathbb{T}}$ in $\textbf{Set}[\mathbb{T}]$ such that any model $M' \in \mathrm{Mod}_{\mathbb{T}}(\mathcal{E})$ can be obtained from some geometric morphism $F \colon \mathcal{E} \to \textbf{Set}[\mathbb{T}]$ by 'pulling back' $M^{\mathbb{T}}$ along F.

Propositional theories and locales

A propositional geometric theory is one with no sorts.

Propositional theories and locales

A propositional geometric theory is one with no sorts.

For example, for a fixed set X the theory of *partial functions from* $\mathbb{N}$ *to* X has a proposition $[f(n) = x]$ for each $n \in \mathbb{N}$ and $x \in X$ and an axiom

$$[f(n) = x] \wedge [f(n) = y] \vdash x = y$$

for each $n \in \mathbb{N}$ and $x, y \in X$.

Propositional theories and locales

A propositional geometric theory is one with no sorts.

For example, for a fixed set X the theory of *partial functions from* $\mathbb{N}$ *to* X has a proposition $[f(n) = x]$ for each $n \in \mathbb{N}$ and $x \in X$ and an axiom

$$[f(n) = x] \wedge [f(n) = y] \vdash x = y$$

for each $n \in \mathbb{N}$ and $x, y \in X$.

In this case it is natural to equip the set of models with a *topology*. The open sets correspond to definable propositions.

Propositional theories and locales

A propositional geometric theory is one with no sorts.

For example, for a fixed set X the theory of *partial functions from* $\mathbb{N}$ *to* X has a proposition $[f(n) = x]$ for each $n \in \mathbb{N}$ and $x \in X$ and an axiom

$$[f(n) = x] \wedge [f(n) = y] \vdash x = y$$

for each $n \in \mathbb{N}$ and $x, y \in X$.

In this case it is natural to equip the set of models with a *topology*. The open sets correspond to definable propositions.

More precisely, a propositional theory defines a *'point-free'* space called a locale. The category **Loc** of locales is the opposite of the a certain algebraic category of lattices ('of opens') called frames.

Frames

A frame is a lattice with finite meets and arbitrary joins satisfying the distributivity condition $a \wedge \bigvee_\alpha b_\alpha = \bigvee_\alpha a \wedge b_\alpha$.

Frames are the Lindenbaum–Tarski algebras for propositional geometric theories. We can easily turn propositional theories into presentations: basic propositions give generators and axioms give relations.

Frames

A frame is a lattice with finite meets and arbitrary joins satisfying the distributivity condition $a \wedge \bigvee_\alpha b_\alpha = \bigvee_\alpha a \wedge b_\alpha$.

Frames are the Lindenbaum–Tarski algebras for propositional geometric theories. We can easily turn propositional theories into presentations: basic propositions give generators and axioms give relations.

The lattice of opens of a topological space is a frame. However, not every frame comes from a space.

Frames

A frame is a lattice with finite meets and arbitrary joins satisfying the distributivity condition $a \wedge \bigvee_\alpha b_\alpha = \bigvee_\alpha a \wedge b_\alpha$.

Frames are the Lindenbaum–Tarski algebras for propositional geometric theories. We can easily turn propositional theories into presentations: basic propositions give generators and axioms give relations.

The lattice of opens of a topological space is a frame. However, not every frame comes from a space.

Fix a set X and consider the theory of *partial surjections from* $\mathbb{N}$ *to* X. This adds the axiom $\top \vdash \bigvee_{n \in \mathbb{N}} [f(n) = x]$ for each $x \in X$ to the theory of partial functions from $\mathbb{N}$ to X.

Frames

A frame is a lattice with finite meets and arbitrary joins satisfying the distributivity condition $a \wedge \bigvee_\alpha b_\alpha = \bigvee_\alpha a \wedge b_\alpha$.

Frames are the Lindenbaum–Tarski algebras for propositional geometric theories. We can easily turn propositional theories into presentations: basic propositions give generators and axioms give relations.

The lattice of opens of a topological space is a frame. However, not every frame comes from a space.

Fix a set X and consider the theory of *partial surjections from* $\mathbb{N}$ *to* X. This adds the axiom $\top \vdash \bigvee_{n \in \mathbb{N}} [f(n) = x]$ for each $x \in X$ to the theory of partial functions from $\mathbb{N}$ to X.

If X is sufficiently large this has no **Set**-models (i.e. points).

Frames

A frame is a lattice with finite meets and arbitrary joins satisfying the distributivity condition $a \wedge \bigvee_\alpha b_\alpha = \bigvee_\alpha a \wedge b_\alpha$.

Frames are the Lindenbaum–Tarski algebras for propositional geometric theories. We can easily turn propositional theories into presentations: basic propositions give generators and axioms give relations.

The lattice of opens of a topological space is a frame. However, not every frame comes from a space.

Fix a set X and consider the theory of *partial surjections from* $\mathbb{N}$ *to* X. This adds the axiom $\top \vdash \bigvee_{n \in \mathbb{N}} [f(n) = x]$ for each $x \in X$ to the theory of partial functions from $\mathbb{N}$ to X.

If X is sufficiently large this has no **Set**-models (i.e. points). However, the frame is always nontrivial in a strong sense.

Sheaves

If $X_{\mathbb{T}}$ is the locale obtained from a propositional theory $\mathbb{T}$, then $\mathbf{Set}[\mathbb{T}]$ is the topos $\mathrm{Sh}(X_{\mathbb{T}})$ of sheaves on $X_{\mathbb{T}}$.

A sheaf on a locale X can be defined as a *local homeomorphism* into X. A morphism of sheaves is morphism in the slice category $\mathbf{Loc}/X$.

Sheaves

If $X_{\mathbb{T}}$ is the locale obtained from a propositional theory $\mathbb{T}$, then $\mathbf{Set}[\mathbb{T}]$ is the topos $\mathrm{Sh}(X_{\mathbb{T}})$ of sheaves on $X_{\mathbb{T}}$.

A sheaf on a locale X can be defined as a *local homeomorphism* into X. A morphism of sheaves is morphism in the slice category $\mathbf{Loc}/X$.

Sh embeds **Loc** into the 2-category **Topos** of (Grothendieck) toposes.

Sheaves

If $X_{\mathbb{T}}$ is the locale obtained from a propositional theory $\mathbb{T}$, then $\mathbf{Set}[\mathbb{T}]$ is the topos $\mathrm{Sh}(X_{\mathbb{T}})$ of sheaves on $X_{\mathbb{T}}$.

A sheaf on a locale X can be defined as a *local homeomorphism* into X. A morphism of sheaves is morphism in the slice category $\mathbf{Loc}/X$.

Sh embeds **Loc** into the 2-category **Topos** of (Grothendieck) toposes.

Not every topos is localic. This is a pity, since it is easier to work with locales than toposes.

Sheaves

If $X_{\mathbb{T}}$ is the locale obtained from a propositional theory $\mathbb{T}$, then $\mathbf{Set}[\mathbb{T}]$ is the topos $\mathrm{Sh}(X_{\mathbb{T}})$ of sheaves on $X_{\mathbb{T}}$.

A sheaf on a locale X can be defined as a *local homeomorphism* into X. A morphism of sheaves is morphism in the slice category $\mathbf{Loc}/X$.

Sh embeds **Loc** into the 2-category **Topos** of (Grothendieck) toposes.

Not every topos is localic. This is a pity, since it is easier to work with locales than toposes. However, Joyal and Tierney showed that this is almost true — every topos is a topos of sheaves on a localic groupoid.

A locale of models for a general geometric theory

Let $\mathbb{T}$ be a geometric theory. How can we describe a locale of models?

A locale of models for a general geometric theory

Let $\mathbb{T}$ be a geometric theory. How can we describe a locale of models?

The problem is encoding the *sorts* of $\mathbb{T}$. A model of $\mathbb{T}$ has a set M_X for every sort X. So there may be a proper class of models of $\mathbb{T}$.

A locale of models for a general geometric theory

Let $\mathbb{T}$ be a geometric theory. How can we describe a locale of models?

The problem is encoding the *sorts* of $\mathbb{T}$. A model of $\mathbb{T}$ has a set M_X for every sort X. So there may be a proper class of models of $\mathbb{T}$.

But recall that the locale of partial surjections from $\mathbb{N}$ to any set X is nontrivial. So in some sense *"every set is a subquotient of* $\mathbb{N}$".

A locale of models for a general geometric theory

Let $\mathbb{T}$ be a geometric theory. How can we describe a locale of models?

The problem is encoding the *sorts* of $\mathbb{T}$. A model of $\mathbb{T}$ has a set M_X for every sort X. So there may be a proper class of models of $\mathbb{T}$.

But recall that the locale of partial surjections from $\mathbb{N}$ to any set X is nontrivial. So in some sense *"every set is a subquotient of* $\mathbb{N}$".

This suggests representing sorts by partial equivalence relations on $\mathbb{N}$.

A locale of models for a general geometric theory

Let $\mathbb{T}$ be a geometric theory. How can we describe a locale of models?

The problem is encoding the *sorts* of $\mathbb{T}$. A model of $\mathbb{T}$ has a set M_X for every sort X. So there may be a proper class of models of $\mathbb{T}$.

But recall that the locale of partial surjections from $\mathbb{N}$ to any set X is nontrivial. So in some sense *"every set is a subquotient of* $\mathbb{N}$".

This suggests representing sorts by partial equivalence relations on $\mathbb{N}$. These have a propositional geometric theory with basic propositions $[n \sim m]$ for $n, m \in \mathbb{N}$ and the following axioms.

- $[n \sim m] \vdash [m \sim n]$ (symmetry)
- $[n \sim m] \wedge [m \sim \ell] \vdash [n \sim \ell]$ (transitivity)

The locale of models $G_0^{\mathbb{T}}$

For a geometric theory $\mathbb{T}$, we define $G_0^{\mathbb{T}}$ to be the classifying locale of a propositional geometric theory $P[\mathbb{T}]$, defined as follows.

- For each sort X of $\mathbb{T}$, add a copy of the theory of partial equivalence relations on $\mathbb{N}$ (i.e. add basic propositions $[n \sim^X m]$ and axioms for symmetry and transitivity).

The locale of models $G_0^{\mathbb{T}}$

For a geometric theory $\mathbb{T}$, we define $G_0^{\mathbb{T}}$ to be the classifying locale of a propositional geometric theory $P[\mathbb{T}]$, defined as follows.

- For each sort X of $\mathbb{T}$, add a copy of the theory of partial equivalence relations on $\mathbb{N}$ (i.e. add basic propositions $[n \sim^X m]$ and axioms for symmetry and transitivity).
- For each relation symbol $R \subseteq X^1 \times \cdots \times X^k$ of $\mathbb{T}$, and for each $n_1, \ldots, n_k \in \mathbb{N}$ and $m_1, \ldots, m_k \in \mathbb{N}$, add a proposition $[(n_1, \ldots, n_k) \in R]$ and axioms
    - $[(n_1, \ldots, n_k) \in R] \wedge [n_1 \sim^{X^1} m_1] \wedge \cdots \wedge [n_k \sim^{X^k} m_k] \vdash [(m_1, \ldots, m_k) \in R]$,
    - $[(n_1, \ldots, n_k) \in R] \vdash [n_1 \sim^{X^1} n_1] \wedge \cdots \wedge [n_k \sim^{X^k} n_k]$.

The locale of models $G_0^{\mathbb{T}}$

- For each axiom $\varphi \vdash_{x_1,\ldots,x_k} \psi$ of $\mathbb{T}$, we add an axiom

$$\bigwedge_{i=1}^{k} [n_i \sim^{X^i} n_i] \wedge \varphi_{n_1,\ldots,n_k} \vdash \psi_{n_1,\ldots,n_k}$$

  for each $n_1, \ldots, n_k \in \mathbb{N}$, where $\varphi_{n_1,\ldots,n_k}$ and $\psi_{n_1,\ldots,n_k}$ are obtained from φ and ψ by replacing:

  - each free variable x_i by a (fixed) natural number n_i,

The locale of models $G_0^{\mathbb{T}}$

- For each axiom $\varphi \vdash_{x_1,\dots,x_k} \psi$ of $\mathbb{T}$, we add an axiom

$$\bigwedge_{i=1}^{k} [n_i \sim^{X^i} n_i] \wedge \varphi_{n_1,\dots,n_k} \vdash \psi_{n_1,\dots,n_k}$$

for each $n_1, \dots, n_k \in \mathbb{N}$, where $\varphi_{n_1,\dots,n_k}$ and $\psi_{n_1,\dots,n_k}$ are obtained from φ and ψ by replacing:

  - each free variable x_i by a (fixed) natural number n_i,
  - each subformula of the form $x = y$ with $[n_x \sim^X n_y]$,

The locale of models $G_0^{\mathbb{T}}$

- For each axiom $\varphi \vdash_{x_1,\ldots,x_k} \psi$ of $\mathbb{T}$, we add an axiom

$$\bigwedge_{i=1}^{k} [n_i \sim^{X^i} n_i] \wedge \varphi_{n_1,\ldots,n_k} \vdash \psi_{n_1,\ldots,n_k}$$

  for each $n_1, \ldots, n_k \in \mathbb{N}$, where $\varphi_{n_1,\ldots,n_k}$ and $\psi_{n_1,\ldots,n_k}$ are obtained from φ and ψ by replacing:
  - each free variable x_i by a (fixed) natural number n_i,
  - each subformula of the form $x = y$ with $[n_x \sim^X n_y]$,
  - each subformula $(y_1, \ldots, y_\ell) \in R$ with $[(n_{y_1}, \ldots, n_{y_\ell}) \in R]$,

The locale of models $G_0^{\mathbb{T}}$

- For each axiom $\varphi \vdash_{x_1,\ldots,x_k} \psi$ of $\mathbb{T}$, we add an axiom

$$\bigwedge_{i=1}^{k} [n_i \sim^{X^i} n_i] \wedge \varphi_{n_1,\ldots,n_k} \vdash \psi_{n_1,\ldots,n_k}$$

  for each $n_1, \ldots, n_k \in \mathbb{N}$, where $\varphi_{n_1,\ldots,n_k}$ and $\psi_{n_1,\ldots,n_k}$ are obtained from φ and ψ by replacing:
    - each free variable x_i by a (fixed) natural number n_i,
    - each subformula of the form $x = y$ with $[n_x \sim^X n_y]$,
    - each subformula $(y_1, \ldots, y_\ell) \in R$ with $[(n_{y_1}, \ldots, n_{y_\ell}) \in R]$,
    - and each quantifier $\exists x \colon X.\ \chi(x, \ldots)$ by a join $\bigvee_{n_x \in \mathbb{N}} \chi(n_x, \ldots)$.

The locale of models $G_0^{\mathbb{T}}$

- For each axiom $\varphi \vdash_{x_1,\dots,x_k} \psi$ of $\mathbb{T}$, we add an axiom

$$\bigwedge_{i=1}^{k} [n_i \sim^{X^i} n_i] \wedge \varphi_{n_1,\dots,n_k} \vdash \psi_{n_1,\dots,n_k}$$

  for each $n_1, \dots, n_k \in \mathbb{N}$, where $\varphi_{n_1,\dots,n_k}$ and $\psi_{n_1,\dots,n_k}$ are obtained from φ and ψ by replacing:
  - each free variable x_i by a (fixed) natural number n_i,
  - each subformula of the form $x = y$ with $[n_x \sim^X n_y]$,
  - each subformula $(y_1, \dots, y_\ell) \in R$ with $[(n_{y_1}, \dots, n_{y_\ell}) \in R]$,
  - and each quantifier $\exists x \colon X.\ \chi(x, \dots)$ by a join $\bigvee_{n_x \in \mathbb{N}} \chi(n_x, \dots)$.

Different subquotients of $\mathbb{N}$ might correspond to isomorphic models.

The locale of models $G_0^{\mathbb{T}}$

- For each axiom $\varphi \vdash_{x_1,\dots,x_k} \psi$ of $\mathbb{T}$, we add an axiom

$$\bigwedge_{i=1}^{k} [n_i \sim^{X^i} n_i] \wedge \varphi_{n_1,\dots,n_k} \vdash \psi_{n_1,\dots,n_k}$$

  for each $n_1, \dots, n_k \in \mathbb{N}$, where $\varphi_{n_1,\dots,n_k}$ and $\psi_{n_1,\dots,n_k}$ are obtained from φ and ψ by replacing:
  - each free variable x_i by a (fixed) natural number n_i,
  - each subformula of the form $x = y$ with $[n_x \sim^{X} n_y]$,
  - each subformula $(y_1, \dots, y_\ell) \in R$ with $[(n_{y_1}, \dots, n_{y_\ell}) \in R]$,
  - and each quantifier $\exists x \colon X.\ \chi(x, \dots)$ by a join $\bigvee_{n_x \in \mathbb{N}} \chi(n_x, \dots)$.

Different subquotients of $\mathbb{N}$ might correspond to isomorphic models. To deal with this we construct a locale of isomorphisms.

The locale of isomorphisms $G_1^{\mathbb{T}}$

We define a locale $G_1^{\mathbb{T}}$ as the classifying locale of the theory $P[\mathbb{T}_{\cong}]$ where $\mathbb{T}_{\cong}$ is the theory of *isomorphisms* of $\mathbb{T}$-models.

The theory $\mathbb{T}_{\cong}$ is obtained by simply taking two copies of each sort, relations and axioms of $\mathbb{T}$, adding relations corresponding to bijections between corresponding sorts, and adding axioms that force relations on each copy to map into each other via the appropriate bijections.

The locale of isomorphisms $G_1^{\mathbb{T}}$

We define a locale $G_1^{\mathbb{T}}$ as the classifying locale of the theory $P[\mathbb{T}_{\cong}]$ where $\mathbb{T}_{\cong}$ is the theory of *isomorphisms* of $\mathbb{T}$-models.

The theory $\mathbb{T}_{\cong}$ is obtained by simply taking two copies of each sort, relations and axioms of $\mathbb{T}$, adding relations corresponding to bijections between corresponding sorts, and adding axioms that force relations on each copy to map into each other via the appropriate bijections.

We can now equip $G_0^{\mathbb{T}}$ and $G_1^{\mathbb{T}}$ with locale morphisms to give an internal groupoid $G^{\mathbb{T}}$ in **Loc**.

$$G_1^{\mathbb{T}} \times_{G_0^{\mathbb{T}}} G_1^{\mathbb{T}} \xrightarrow{\;m\;} \overset{\overset{i}{\circlearrowright}}{G_1^{\mathbb{T}}} \;\begin{array}{c} \xrightarrow{\;s\;} \\ \xleftarrow{\;e\;} \\ \xrightarrow[\;t\;]{} \end{array}\; G_0^{\mathbb{T}}$$

This is (essentially) the localic groupoid of Joyal and Tierney.

Sheaves on a localic groupoid

We can recover the topos **Set**$[\mathbb{T}]$ by taking sheaves on $G^{\mathbb{T}}$.

Sheaves on a localic groupoid

We can recover the topos $\mathbf{Set}[\mathbb{T}]$ by taking sheaves on $G^{\mathbb{T}}$.

A sheaf on a localic groupoid G is a sheaf on the locale of objects $E \to G_0$ equipped with an action by the locale of morphisms G_1.

Sheaves on a localic groupoid

We can recover the topos $\mathbf{Set}[\mathbb{T}]$ by taking sheaves on $G^{\mathbb{T}}$.

A sheaf on a localic groupoid G is a sheaf on the locale of objects $E \to G_0$ equipped with an action by the locale of morphisms G_1.

This can also be understood as specifying a 'topologically discrete' internal discrete (op)fibration into $G^{\mathbb{T}}$ — intuitively, a 'continuous' functor from $G^{\mathbb{T}}$ to $\mathbf{Set}$.

Sheaves on a localic groupoid

We can recover the topos $\mathbf{Set}[\mathbb{T}]$ by taking sheaves on $G^{\mathbb{T}}$.

A sheaf on a localic groupoid G is a sheaf on the locale of objects $E \to G_0$ equipped with an action by the locale of morphisms G_1.

This can also be understood as specifying a 'topologically discrete' internal discrete (op)fibration into $G^{\mathbb{T}}$ — intuitively, a 'continuous' functor from $G^{\mathbb{T}}$ to **Set**.

The category $\mathrm{Sh}(G)$ of sheaves over G and 'equivariant' maps between them is a topos.

Sheaves on a localic groupoid

We can recover the topos **Set**$[\mathbb{T}]$ by taking sheaves on $G^{\mathbb{T}}$.

A sheaf on a localic groupoid G is a sheaf on the locale of objects $E \to G_0$ equipped with an action by the locale of morphisms G_1.

This can also be understood as specifying a 'topologically discrete' internal discrete (op)fibration into $G^{\mathbb{T}}$ — intuitively, a 'continuous' functor from $G^{\mathbb{T}}$ to **Set**.

The category $\mathrm{Sh}(G)$ of sheaves over G and 'equivariant' maps between them is a topos.

The Joyal–Tierney result shows that $\mathrm{Sh}(G^{\mathbb{T}}) \simeq \mathbf{Set}[\mathbb{T}]$.

The generic sheaves

A $\mathbb{T}$-model in $\mathrm{Sh}(G)$ corresponds to a sheaf on G for each sort of $\mathbb{T}$ together with open sublocales of products of these (over G) for each relation satisfying the axioms.

The generic sheaves

A $\mathbb{T}$-model in $\mathrm{Sh}(G)$ corresponds to a sheaf on G for each sort of $\mathbb{T}$ together with open sublocales of products of these (over G) for each relation satisfying the axioms.

In particular, we can express the generic model of $\mathbf{Set}[\mathbb{T}]$ in terms of structures on $G^{\mathbb{T}}$.

For a sort X the corresponding étale space is given by the locale classifying pairs $(x, [n])$ with $x \in G_0^{\mathbb{T}}$ and $[n] \in \mathbb{N}/E_X(x)$ where E_X is the value of the partial equivalence relation $\sim^X$ at the point x.

The generic sheaves

A $\mathbb{T}$-model in $\mathrm{Sh}(G)$ corresponds to a sheaf on G for each sort of $\mathbb{T}$ together with open sublocales of products of these (over G) for each relation satisfying the axioms.

In particular, we can express the generic model of $\mathbf{Set}[\mathbb{T}]$ in terms of structures on $G^{\mathbb{T}}$.

For a sort X the corresponding étale space is given by the locale classifying pairs $(x, [n])$ with $x \in G_0^{\mathbb{T}}$ and $[n] \in \mathbb{N}/E_X(x)$ where E_X is the value of the partial equivalence relation $\sim^X$ at the point x.

The generic $\mathbb{T}$-model over $G^{\mathbb{T}}$ then satisfies a universal property, not only in the 2-category of toposes, but in a bicategory of all localic groupoids (and anafunctors between them).

Analogous constructions

Our construction can be understood as giving universal étale bundles (with certain extra properties and structure).

This suggests other similar constructions. For instance, we might replace local homeomorphisms with proper separated locale maps.

Analogous constructions

Our construction can be understood as giving universal étale bundles (with certain extra properties and structure).

This suggests other similar constructions. For instance, we might replace local homeomorphisms with proper separated locale maps.

- Then instead of using (open) partial equivalence relations on $\mathbb{N}$, we use *closed* partial equivalence relations on $2^{\mathbb{N}}$.

Analogous constructions

Our construction can be understood as giving universal étale bundles (with certain extra properties and structure).

This suggests other similar constructions. For instance, we might replace local homeomorphisms with proper separated locale maps.

- Then instead of using (open) partial equivalence relations on $\mathbb{N}$, we use *closed* partial equivalence relations on $2^{\mathbb{N}}$.
- Instead of the locale of partial surjections from $\mathbb{N}$ to a set X, we use a locale of partial surjections from $2^{\mathbb{N}}$ to a compact Hausdorff locale X.

Analogous constructions

Our construction can be understood as giving universal étale bundles (with certain extra properties and structure).

This suggests other similar constructions. For instance, we might replace local homeomorphisms with proper separated locale maps.

- Then instead of using (open) partial equivalence relations on $\mathbb{N}$, we use *closed* partial equivalence relations on $2^{\mathbb{N}}$.
- Instead of the locale of partial surjections from $\mathbb{N}$ to a set X, we use a locale of partial surjections from $2^{\mathbb{N}}$ to a compact Hausdorff locale X. (This requires a constructive version of the Hausdorff–Alexandroff theorem, which holds for not-necessarily second countable X.)

Analogous constructions

Our construction can be understood as giving universal étale bundles (with certain extra properties and structure).

This suggests other similar constructions. For instance, we might replace local homeomorphisms with proper separated locale maps.

- Then instead of using (open) partial equivalence relations on $\mathbb{N}$, we use *closed* partial equivalence relations on $2^{\mathbb{N}}$.
- Instead of the locale of partial surjections from $\mathbb{N}$ to a set X, we use a locale of partial surjections from $2^{\mathbb{N}}$ to a compact Hausdorff locale X. (This requires a constructive version of the Hausdorff–Alexandroff theorem, which holds for not-necessarily second countable X.)
- We should then be able to classify compact Hausdorff models for a 'dual geometric' logic by a similar construction.

References

[1] Graham Manuell and Joshua Wrigley. **The representing localic groupoid for a geometric theory.** arXiv:2305.15209. 2023.

[2] André Joyal and Myles Tierney. **An extension of the Galois theory of Grothendieck.** In: *Mem. Amer. Math. Soc.* 51.309 (1984).

[3] Ieke Moerdijk. **The Classifying Topos of a Continuous Groupoid I.** In: *Trans. Amer. Math. Soc.* 310.2 (1988), pp. 629–668.