Category Theory 2023
Université catholique de Louvain

Categorical incarnations of infinite games

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Joint work (in progress) with P. Szeptycki and W. Tholen

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- are between two players (ALICE and BOB);
- are "turn-based" (ALICE starts);
- two players compete;
- with no draws;
- of "perfect information";
- infinite (countable) runs.

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For a set M, a set $T \subseteq \bigcup_{n\in\mathbb{N}} M^n$ of finite sequences in M is a *decision tree over* M if

(I) If $t \in T$, then $t \restriction k \in T$ for all $k \leq |t|$;

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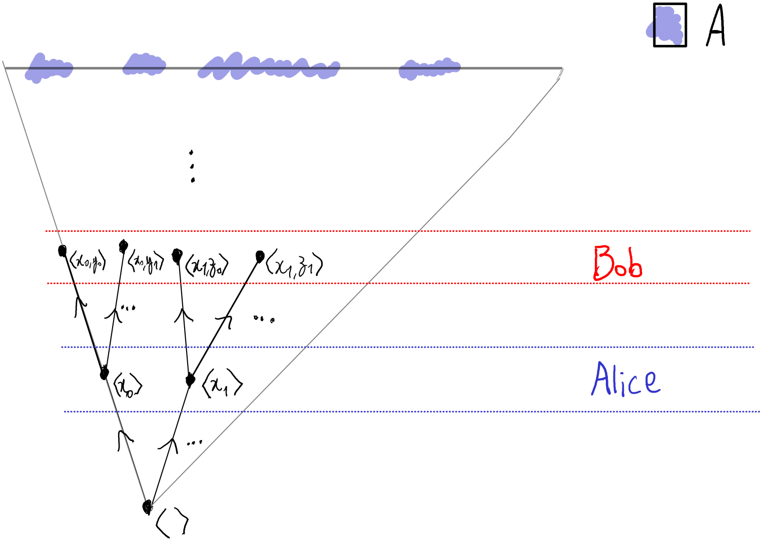
Definition

A pair $G = (T, A)$ is an *infinite game* if T is a decision tree over a set M and

$$A \subseteq \mathrm{Run}(T) = \left\{ R \in M^{\mathbb{N}} : R \restriction n \in T \text{ for every } n \in \mathbb{N} \right\}.$$

The set A is called the *payoff set* of G.

All of our games will be infinite in this talk, so we will omit the word “infinite” from now on.


A
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Given a non-empty space (X, τ), consider the following game:

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  - Alice chooses a non-empty open set U_0;
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- At the following nth innings:
  - Alice chooses a non-empty open set U_n contained in the open set V_{n-1} chosen by Bob in the previous inning;
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$$T = \{ t \in (\tau \setminus \{\emptyset\})^n : n \in \mathbb{N}, \forall i \leq j < n \, (t(i) \subseteq t(j)) \} .$$

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Then BOB wins the run $(U_0, V_0, \ldots, U_n, V_n, \ldots)$ if $\bigcap_{n\in\omega} V_n \neq \emptyset$ (and ALICE wins otherwise).

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$$A = \left\{ R \in (\tau \setminus \{\emptyset\})^{\mathbb{N}} : \bigcap_{n\in\mathbb{N}} R(2n+1) = \emptyset \right\}.$$

Example (World's most boring game)

$$(\emptyset, \emptyset)$$

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(by vacuity, with $M = \emptyset$)

Motivation

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"It is possible to construct a sequence $s = (a_n : n \in \mathbb{N})$ such that $P(s)$."

The concepts of games and strategies are then used to define stronger properties of the kind:

"It is possible to construct a sequence $s = (a_n : n \in \mathbb{N})$ such that $P(s)$, **even with someone trying to hinder that process along the way**."

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Theorem (J. Oxtoby, 1957)

A space X is Baire if, and only if, ALICE *has no winning strategy in the Banach-Mazur game over X.*

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Then, if $f\colon T \to T'$ is chronological and $R \in \operatorname{Run} T$, there is a unique $\overline{f}(R) \in \operatorname{Run} T'$ such that $f(R \restriction n) = \overline{f}(R) \restriction n$ for all $n \in \mathbb{N}$.

Definition (A-morphism)

An A-morphism $G \xrightarrow{f} G$ between games $G = (T, A)$ and $G' = (T', A')$ is a chronological map $f\colon T \to T'$ such that for every run $R \in A$ in the game G, $\overline{f}(R) \in A'$.

Definition ($\mathbb{B}$-morphism)

A $\mathbb{B}$-morphism $G \xrightarrow{f} G$ between games $G = (T, A)$ and $G' = (T', A')$ is a chronological map $f\colon T \to T'$ such that for every run $R \notin A$ in the game G, $\overline{f}(R) \notin A'$.

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Proposition

The categories $\textbf{Game}_{\mathrm{A}}$ *and* $\textbf{Game}_{\mathrm{B}}$ *are isomorphic.*

Why are these categorical frameworks appropriate?

Properties

Theorem (D., P. Szeptycki, W. Tholen – 202?)

Suppose **C** *is either* **Gme**, $\mathbf{Game}_{\mathrm{A}}$, $\mathbf{Game}_{\mathrm{B}}$. *Then:*

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- **C** *is extensive.*
- **C** *has no classifier of strong subobjects.*
- **C** *has a weak classifier of strong partial maps.*

Topological games as functors

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Example ($G_1(\Omega_x, \Omega_x)$)

Given a space X and a fixed $x \in X$, consider the game: in each inning $n \in \omega$,

- ALICE chooses $A_n \subset X$ such that $x \in \overline{A_n}$;
- BOB responds with $a_n \in A_n$.

BOB wins the run $(A_0, a_0, \ldots, A_n, a_n, \ldots)$ if, for every $k \in \mathbb{N}$, $x \in \overline{\{ a_n : n \geq k \}}$ (ALICE wins otherwise).

Example ($G_1(\Omega, \Omega)$)

Given a space X, consider the game: in each inning $n \in \omega$,

- ALICE chooses an ω-cover $\mathcal{U}_n$, that is, an open cover $\mathcal{U}_n$ such that

$$\forall F \in [X]^{<\omega} \exists U \in \mathcal{U}_n (F \subset U),$$

- BOB responds with $U_n \in \mathcal{U}_n$.

BOB wins the run $(\mathcal{U}_0, U_0, \ldots, \mathcal{U}_n, U_n, \ldots)$ if, for every $k \in \omega$, $\{ U_n : n \geq k \}$ is an ω-cover (ALICE wins otherwise).

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Example ($\mathsf{G}_1(\Omega_x, \Omega_x)$)

Let $\mathrm{Tight}\colon \mathsf{Top}_* \to \mathbf{Game}_{\mathrm{B}}$ be such that

- On objects, $\mathrm{Tight}(X, x) = \mathsf{G}_1(\Omega_x, \Omega_x)$ over X.
- On morphisms, given a continuous $f\colon X \to Y$ such that $f(x) = y$, let

$$\begin{aligned} \mathrm{Tight}(X, x) &\xrightarrow{\mathrm{Tight} f} \mathrm{Tight}(Y, y) \\ (A_0, a_0, \ldots, A_n, a_n) &\longmapsto (f[A_0], f(a_0), \ldots, f[A_n], f(a_n)). \end{aligned}$$

Example ($\mathsf{G}_1(\Omega, \Omega)$)

Let $\mathrm{Cover}\colon \mathsf{Top}^{\mathsf{op}} \to \mathbf{Game}_{\mathrm{B}}$ be such that:

- On objects, $\mathrm{Cover} X = \mathsf{G}_1(\Omega, \Omega)$ over X.
- On morphisms, for $Y \xrightarrow{f} X$ in $\mathsf{Top}^{\mathsf{op}}$, let

$$\begin{aligned} \mathrm{Cover} Y &\xrightarrow{\mathrm{Cover} f} \mathrm{Cover} X \\ (\mathcal{U}_0, U_0, \ldots, \mathcal{U}_n, U_n) &\longmapsto (f^{-1}[\mathcal{U}_0], f^{-1}(U_0), \ldots, f^{-1}[\mathcal{U}_n], f^{-1}(U_n)), \end{aligned}$$

Topological games as functors

Theorem

If X is a $T_{3\frac{1}{2}}$ space, then

- $\mathrm{A}\uparrow \mathsf{G}_1(\Omega, \Omega)$ *over* $X \iff \mathrm{A}\uparrow \mathsf{G}_1(\Omega_{\bar{0}}, \Omega_{\bar{0}})$ *over* $\mathsf{C}_\mathsf{p}(X)$ *(*M. Scheepers, 1997*)*
- $\mathrm{B}\uparrow \mathsf{G}_1(\Omega, \Omega)$ *over* $X \iff \mathrm{B}\uparrow \mathsf{G}_1(\Omega_{\bar{0}}, \Omega_{\bar{0}})$ *over* $\mathsf{C}_\mathsf{p}(X)$ *(*M. Scheepers, 2014*)*

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There are two natural transformations from $\mathrm{Tight} \circ \mathrm{C}$ *to* Cover *that, together, translate winning strategies of* ALICE *and* BOB *in both directions.*

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Where $\mathrm{C} \colon \mathsf{Top}^{\mathsf{op}} \to \mathsf{Top}_*$ is s.t.

- on objects, $\mathrm{C}X = (\mathsf{C_p}(X), \bar{0})$,
- on morphisms, for $f \colon X \to Y$ continuous,

$$(\mathsf{C_p}(Y), \bar{0}) \xrightarrow{\mathrm{C}f} (\mathsf{C_p}(X), \bar{0})$$
$$\varphi \longmapsto \varphi \circ f.$$

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Let $\mathrm{Fun}\colon \mathbf{Gme} \to \mathrm{Epi}(\mathbf{Set})^{\mathbb{N}^{\mathrm{op}}}$ be the functor such that $(\mathrm{Fun}\, T)_n = \{\, t \in T : |t| = n+1 \,\}$ and $\Gamma^n_m(t) = t \restriction m+1$.

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Theorem

$\mathrm{Epi}(\mathbf{Set})^{\mathbb{N}^{\mathrm{op}}}$ *is a full coreflective subcategory of* $\mathbf{Set}^{\mathbb{N}^{\mathrm{op}}}$ *and* $\mathbf{Gme} \xrightarrow{\mathrm{Fun}} \mathrm{Epi}(\mathbf{Set})^{\mathbb{N}^{\mathrm{op}}}$ *is an equivalence between categories.*

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The full inclusion functor $\mathrm{Epi}(\mathbf{Set})^{\mathbb{N}^{\mathrm{op}}} \hookrightarrow \mathbf{Set}^{\mathbb{N}^{\mathrm{op}}}$ has an easily described right adjoint: it sends $T \in \mathrm{Obj}(\mathbf{Set}^{\mathbb{N}^{\mathrm{op}}})$ to the system T^* with

$$T^*_n = \pi_n[\mathrm{Lim}\, T],$$

where $\mathrm{Lim}\, T$ is T's (projective) limit in **Set**:

$$\begin{aligned}\mathrm{Lim}\, T &= \{\, (t_n : n \in \mathbb{N}) : t_n \in T_n,\, \Gamma^n_m(t_n) = t_m \quad \forall n \geq m \in \mathbb{N} \,\} \\ &\subseteq \prod_{n \in \mathbb{N}} T_n\end{aligned}$$

and $\pi_n \colon \mathrm{Lim}\, T \to T_n$ is the projection of the nth coordinate.

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The forgetful functor $\mathbf{Game}_{\mathrm{A}} \to \mathbf{Gme}$ (which forgets the payoff set) is topological.
Let $\mathrm{Sub}\left(\mathbf{Set}^{\mathbb{N}^{\mathrm{op}}}\right)$ be such that:

- objects are pairs (T, A) with $T \in \mathbf{Set}^{\mathbb{N}^{\mathrm{op}}}$ and $A \subseteq \mathrm{Lim}\, T$,
- morphisms $(T, A) \to (T', A')$ are natural transformations $T \xrightarrow{f} T'$ such that $\mathrm{Lim} f[A] \subseteq A'$.

Then the forgetful functor $\mathrm{Sub}\left(\mathbf{Set}^{\mathbb{N}^{\mathrm{op}}}\right) \to \mathbf{Set}^{\mathbb{N}^{\mathrm{op}}}$ is also topological.
If we trade $\mathbf{Set}^{\mathbb{N}^{\mathrm{op}}}$ for $\mathrm{Epi}(\mathbf{Set})^{\mathbb{N}^{\mathrm{op}}}$, so that only surjective systems are being considered, we obtain the category

$$\mathbf{FunGame} := \mathrm{Sub}\left(\mathrm{Epi}(\mathbf{Set})^{\mathbb{N}^{\mathrm{op}}}\right)$$

whose objects we consider as "games in their *functorial* description".

Indeed, the previous theorem has a “Sub-lifting”’, as in

$$
\begin{array}{ccccc}
\mathbf{Game}_{\mathrm{A}} & \xrightarrow{\simeq} & \mathbf{FunGame} & \qquad & (T^*, A) \\
 & & \mathrm{inc}\downarrow \dashv \uparrow & & \uparrow \\
 & & \mathrm{Sub}\left(\mathbf{Set}^{\mathbb{N}^{\mathrm{op}}}\right) & & (T, A),
\end{array}
$$

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Let **SeqSpace** be the full subcategory of $\mathbf{CUMet}_1$ whose objects are spaces (X, d) such that the image of d is contained in $\{0\} \cup \left\{ \frac{1}{n+1} : n \in \mathbb{N} \right\}$.

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Theorem

SeqSpace *is a full coreflective subcategory of* $\mathbf{CUMet}_1$ *and* $\mathbf{Gme} \overset{\mathrm{Run}}{\to} \mathbf{SeqSpace}$ *is an equivalence between categories.*

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Then the forgetful functor $\mathrm{Sub}(\mathbf{CUMet}_1) \to \mathbf{CUMet}_1$ is, again, topological.

If we trade $\mathbf{CUMet}_1$ for $\mathbf{SeqSpace}$, so that only spaces with distance function ranging over $\{0\} \cup \left\{ \frac{1}{n+1} : n \in \mathbb{N} \right\}$, we obtain the category

$$\mathbf{MetGame} := \mathrm{Sub}(\mathbf{SeqSpace})$$

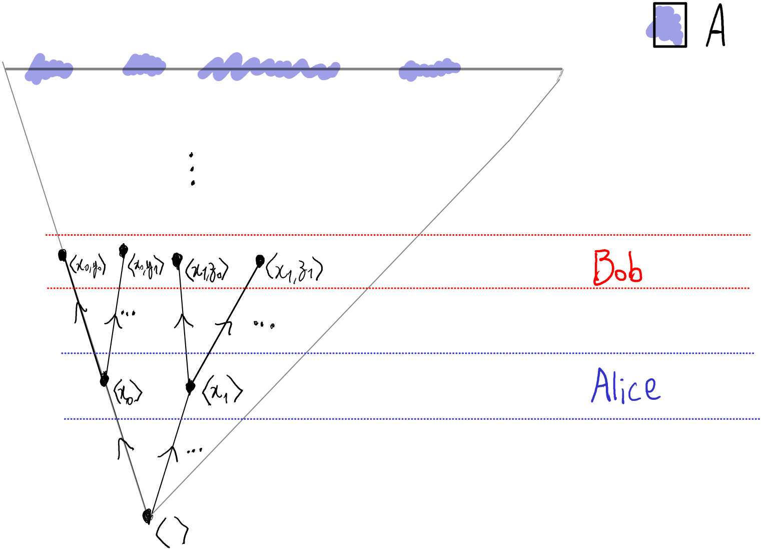
whose objects we consider as "games in their *metrical* description".

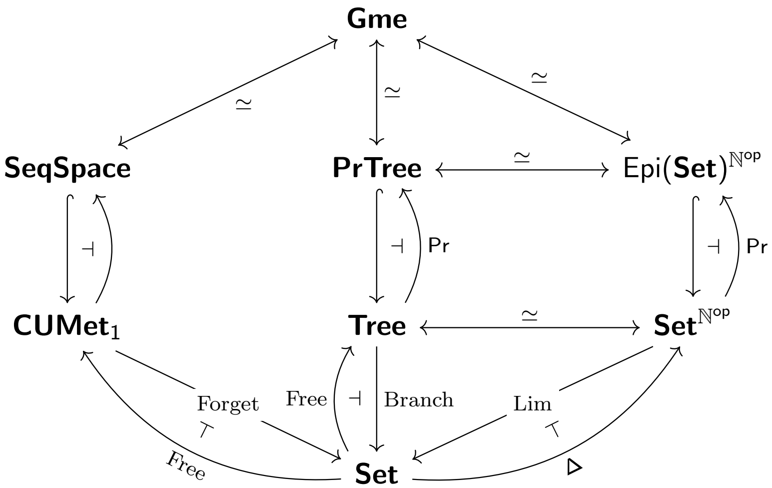
Indeed, the previous theorem also has a “Sub-lifting”’, as in

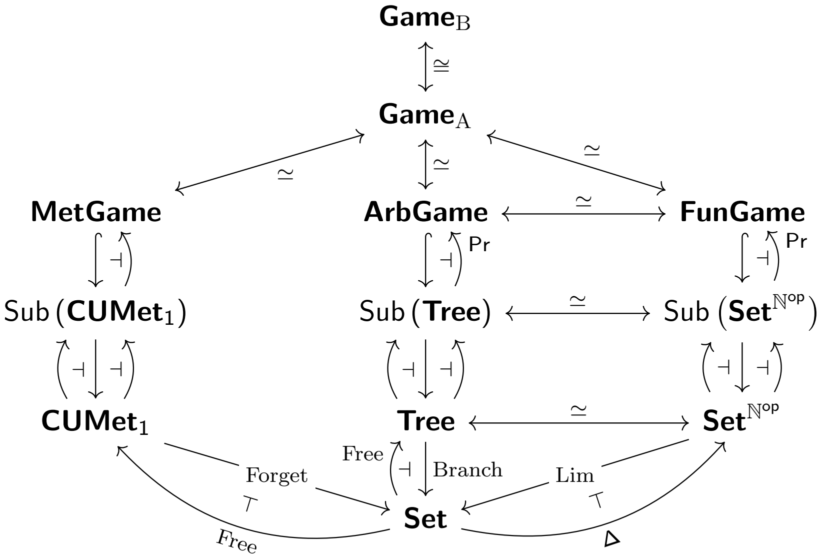
$$\begin{array}{ccc} \mathbf{Game}_{\mathrm{A}} & \xrightarrow{\simeq} & \mathbf{MetGame} \\ & & \mathrm{inc} \downarrow \dashv \uparrow \\ & & \mathrm{Sub}\left(\mathbf{CUMet}_1\right) \end{array}$$

Arboreal games

Arboreal games




Gme
≃
≃
≃
SeqSpace
PrTree
≃
Epi(Set)^ℕop
⊣
⊣
Pr
⊣
Pr
CUMet1
Tree
≃
Set^ℕop
Forget
⊤
Free
Free
⊣
Branch
Lim
⊤
▷
Set


Game_B
Game_A
MetGame
ArbGame
FunGame
Pr
Pr
Sub (CUMet_1)
Sub (Tree)
Sub (Set^{N^op})
CUMet_1
Tree
Set^{N^op}
Free
Branch
Forget
Lim
Free
Set

Theorem (D., P. Szeptycki, W. Tholen – 202?)

Suppose **C** *is either* **Gme**, $\mathbf{Game}_{\mathrm{A}}$ *or* $\mathbf{Game}_{\mathrm{B}}$. *Then:*

- **C** *is complete and co-complete.*
- **C** *is cartesian closed.*
- **C** *is not locally cartesian closed.*
- **C** *has orthogonal factorization systems.*
- **C** *is regular.*
- **C** *is extensive.*
- **C** *has no classifier of strong subobjects.*
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Merci!