A skew approach to enrichment for Gray-categories

CT2023

(Joint work with Gabriele Lobbia)

Plan

- Classical framework for enriched cat. theory: symmetric monoidal closed cats.
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- Classical framework for enriched cat. theory : symmetric monoidal closed cats.
- Problem : too rigid to handle certain higher dimensional structures.
- Context : semi-strict higher cats
- Goal : solve problem using skew monoidal cats (Szlachanyi 2012)

BACKGROUND STORY

- (Set, x, 1) cartesian closed (CCC)

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so can repeat:

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$$\mapsto \cdots \longmapsto n\text{-}\mathrm{Cat} \longmapsto \cdots$$

strict n-categories

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strict n-categories

PROBLEM 1

- Not every weak 3-cat equivalent to strict 3-cat.

BACKGROUND - GRAY T.P.

- Symmetric monoidal closed str (SMCC) $(2\text{-Cat}, \otimes, 1)$

  $\otimes$: Gray tensor product

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- Symmetric monoidal closed str (SMCC) (2-Cat, $\otimes$, 1)
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- Internal hom $Ps(A,B)$:
  - 2-functors $F : A \longrightarrow B$
  - pseudonatural transformations

$$\begin{array}{ccc} Fa & \xrightarrow{\eta_a} & Ga \\ {\scriptstyle Ff}\downarrow & \eta_f \; \cong & \downarrow{\scriptstyle Gf} \\ Fb & \xrightarrow[\eta_b]{} & Gb \end{array}$$

  pseudonatural: v.i.p. in 2-cat theory

  - modifications.

Background - Gray t.p. very nice

① Captures pseudonatural transformations

② Monoidal model cat

(Lack model **structure**)

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③ Enriched cats := Gray-cats $\mathcal{C}$

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③ Enriched cats := Gray-cats $\mathcal{C}$

- Each weak 3-cat equivalent to Gray-cat (semi-strict 3-cat)
- Almost as strict as 3-cats,

except @ $a \underset{f'}{\overset{f}{\rightrightarrows}} b \underset{g'}{\overset{g}{\rightrightarrows}} c$ with $\alpha \Downarrow$, $\alpha' \Downarrow$

middle 4-interchange fails.

- Replaced by

$$\begin{array}{ccc} gf & \overset{g\alpha}{\Longrightarrow} & gf' \\ \beta f \Downarrow & \beta_\alpha \cong & \Downarrow \beta f' \\ g'f & \underset{g'\alpha}{\Longrightarrow} & g'f' \end{array}$$

satisfying coherence equations

Background - Gray cats continued

Can we find a symmetric monoidal closed structure on Gray-Cat capturing

- higher pseudo-transformations
- good homotopical behaviour
- semi-strict higher cats ?

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PROBLEM 2: NO!!

The obstruction (Crans)

- Consider SMCC on Gray-Cat, $[A,B]$ capturing "pseudo-transformations"

ie.

$$\begin{array}{ccc} Fa & \xrightarrow{\eta_a} & Ga \\ {\scriptstyle Ff}\downarrow & \eta_f \swarrow\!\!\swarrow & \downarrow{\scriptstyle Gf} \\ Fb & \xrightarrow[\eta_b]{} & Gb \end{array} \rightsquigarrow \text{adjoint equiv sat. higher coherence laws}$$

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η_f ~ adjoint equiv sat. higher coherence laws

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- $2 = \{0 \to 1\} \xrightarrow{\eta} [A,B] \in$ Gray-Cat corresponds by biclosedness to $A \xrightarrow{\bar{\eta}} [2,B] \in$ Gray-Cat.

- Functoriality in A forces equality:

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- In fact, impossible - transformations satisfying (*) can't be composed!

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- MUST MOVE BEYOND SMCC'S!

Seeking a resolution

- <u>Real</u> pseudotransformations involve <u>iso</u> <u>3-cell</u>

$$
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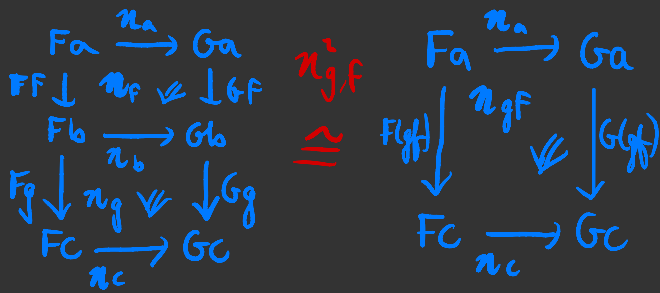
Seeking a resolution

- **Real** pseudotransformations

involve **iso** **3-cell**



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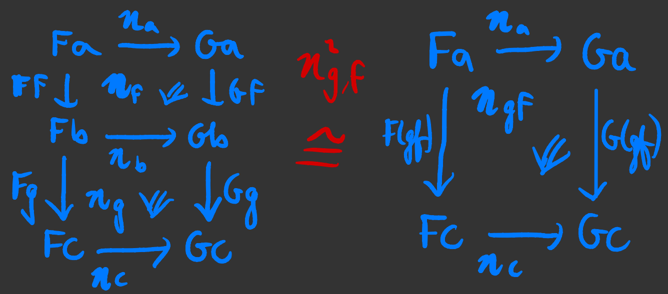

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- HOW???

Skew monoidal cats (Szl' 2012)

- $(\mathcal{C}, \otimes, I)$ +

$$(AB)C \xrightarrow{\alpha} A(B, C)$$
$$IA \xrightarrow{\ell} A$$
$$A \xrightarrow{r} AI$$

satisfying 5 equations.

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- Closed if $- \otimes A \dashv [A, -]$.
- Then weak maps $\equiv$ "elements" $I \to [A,B]$

Mapping spaces for Gray-cats

- Pseudomap $F : A \rightsquigarrow B$ (Gohla 2012)
- Very strict:
- @ $a \xrightarrow{f} b \xrightarrow{g} c$ have $Fa \xrightarrow{Ff} Fb \xrightarrow{Fg} Fc$ and $Fa \xrightarrow{F(gf)} Fc$, with $\cong$ iso 2-cell
- like normal pseudofunctors in 2-cat theory.

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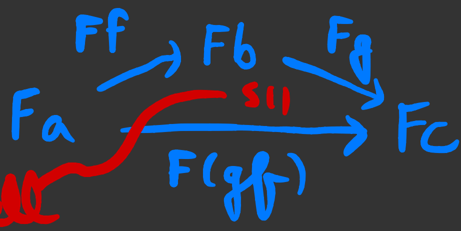
e.g. a $\xrightarrow{f}$ b $\xrightarrow{g}$ c have 

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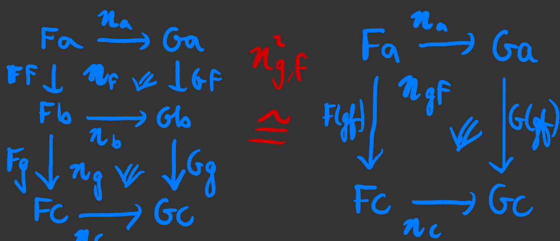
- Lax transformations



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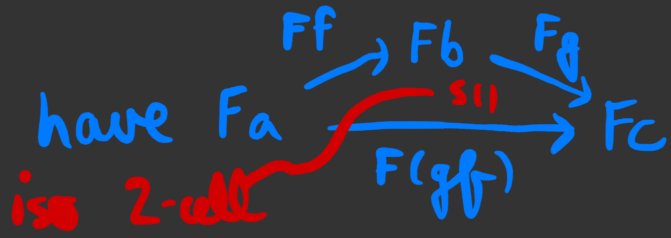

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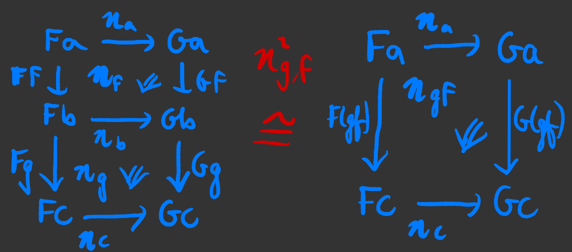

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Gohla 2012

• Lax(A,B) - pseudomaps, lax transf...., ...

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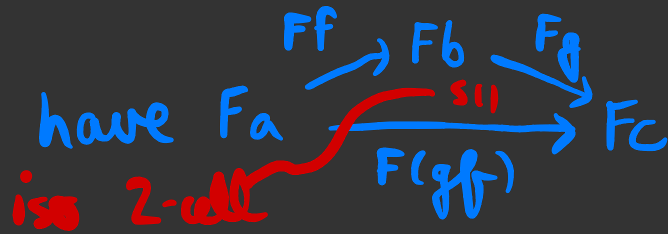

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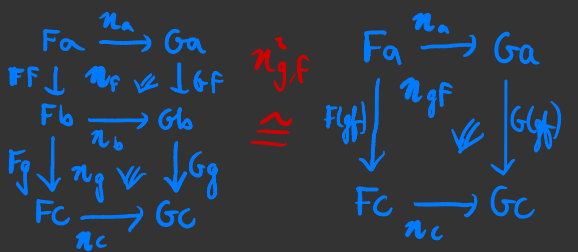

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really normal lax transformations

- Lax(A,B) - pseudomaps, lax transf ...., ... (Gohla 2012)
- Ps(A,B) - pseudomaps, ~~pseudo~~ transf ...., ...
  η_f adjoint equiv. etc

Theorem (Bourke-Lobbia '22)

There is a closed skew monoidal structure $(\text{Gray-Cat}, \otimes, 1)$ with internal hom $Ps(A,B)$.

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- Let's call this the flexible skew structure.

- How does it behave?

① Categorical behaviour

- right normal $A \overset{r}{\cong} A \otimes I$

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$Q = I \otimes -$ $\equiv$ pseudomaps of Gohla.

- Symmetric skew monoidal

$\Rightarrow$ symmetric closed multicat of pseudomaps $\rightarrow$ captures

$$\frac{\mathcal{Z} \rightsquigarrow Ps(A,B)}{A \rightsquigarrow Ps(\mathcal{Z},B)}$$

overcoming the obstruction!

① Categorical behaviour ☺

- right normal $A \overset{r}{\cong} A \otimes I$
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- Symmetric skew monoidal

⇒ symmetric closed multicat of pseudomaps → captures

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② Homotopical behaviour ☹

- ⊗ does not preserve cofibrant obs (Lack model str) ✗

Theorem (Bourke-Lobbia '22)

There is closed skew mon. structure $(\text{Gray-Cat}, \otimes_\#, 1)$ with internal hom $Ps_s(A, B)$.

sharp r.p.

strict Gray-Functors
pseudotransformations

Call this the sharp structure.

Theorem (Bourke-Lobbia '22)

There is closed skew mon. structure $(\text{Gray-Cat}, \otimes_{\#}, 1)$ with internal hom $Ps_s(A,B)$.

$\otimes_{\#}$: sharp t.p.

$Ps_s(A,B)$: strict Gray-functors, pseudotransformations

Call this the <u>sharp structure</u>.

Categorical behaviour

- <u>left</u> & <u>right</u> normal
- <u>Not</u> symmetric.

② Homotopical behaviour ☺

- $\otimes_{\#}$ pres cofib. obs & weak equivs between, unit cofibrant.
- $- \otimes_{*} A \dashv \mathrm{Ps}(A, -)_s$ Quillen adj.

⇒ derived skew mon closed str. on ho(Gray-Cat).

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$\Rightarrow$ derived skew mon closed str. on $ho(\text{Gray-Cat})$.

- Derived structure is symmetric monoidal closed

- $\text{skew} \xmapsto[\text{cat}]{\text{homotopy}} \text{classical}$

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$\Rightarrow$ <u>derived</u> skew mon closed str. on <u>ho(Gray-Cat)</u>.

- Derived structure is symmetric monoidal closed

- skew $\xrightarrow[\text{cat}]{\text{homotopy}}$ classical
- E.g. $(A \otimes_\# B) \otimes_\# C \xrightarrow{\alpha} A \otimes_\# (B \otimes_\# C)$ a <u>weak equiv</u> if A, B, C cofibrant.

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Could not make it any stricter and keep this homotopical behavior.

Semistrict 4-cats?

Cats enriched in sharp tensor product:

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  - strict whiskering by 1-cells,

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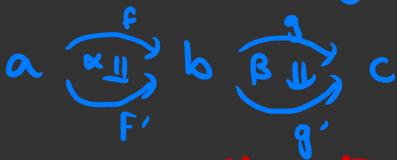
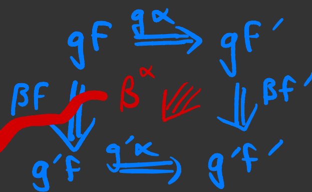
① @ 



adjoint equivalence

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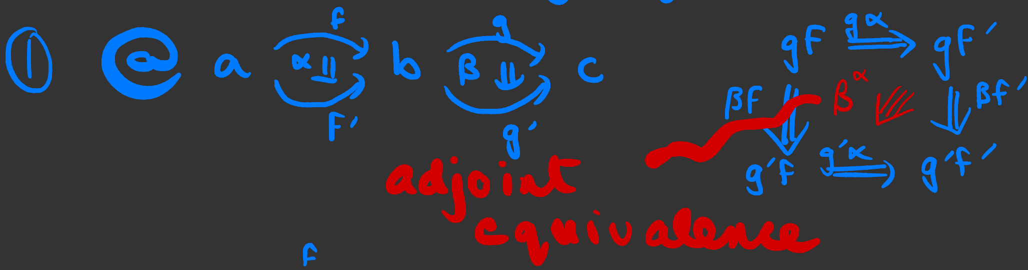

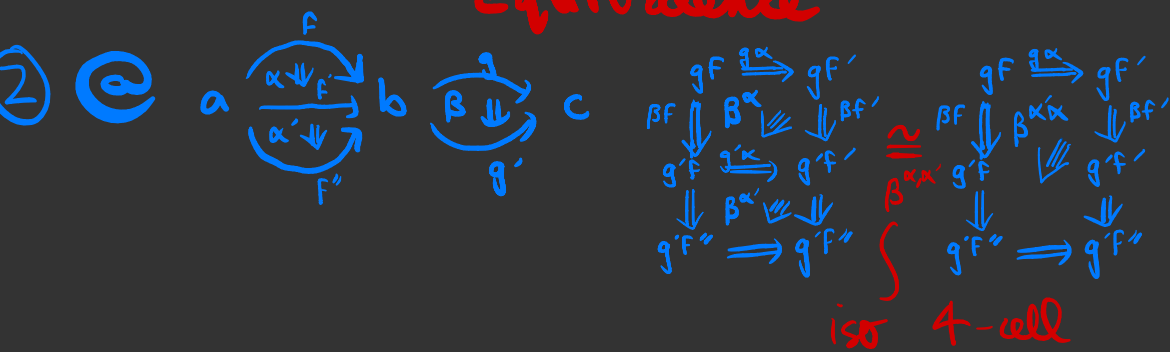

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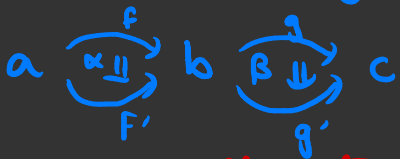
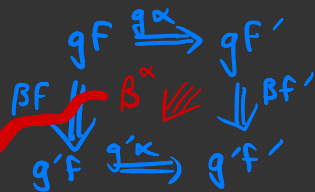
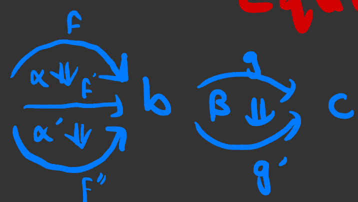
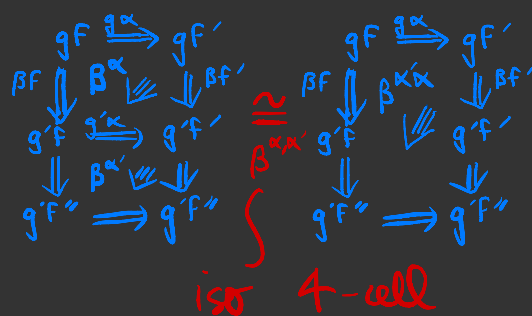




adjoint equivalence

② @





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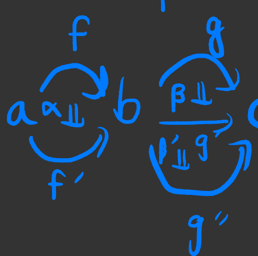
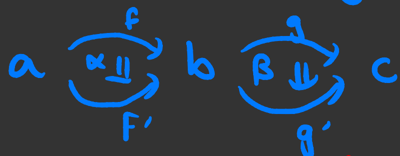
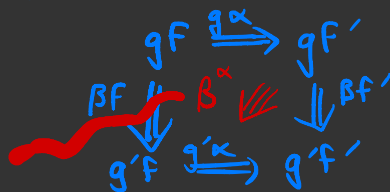


have strict equality

Semistrict 4-cats?

- Cats enriched in sharp tensor product:
  - hom Gray-cat $A(x,y)$,
  - strict whiskering by 1-cells,

① @ 



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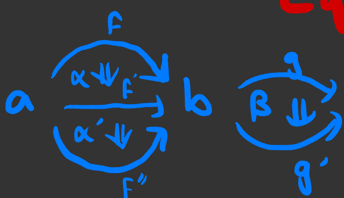
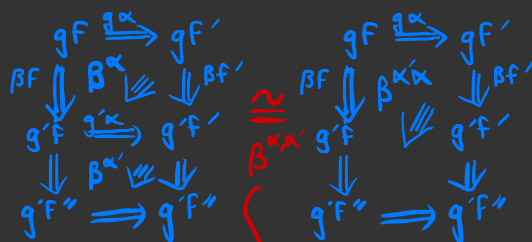
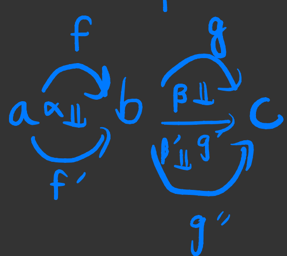
② @ 



iso 4-cell

③ At  have

strict equality

④, ⑤  → iso 4-cells

satisfying various equations.

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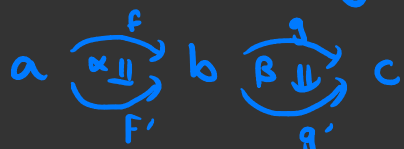
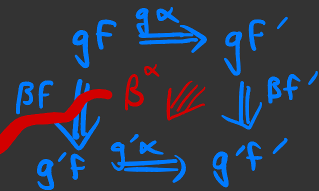
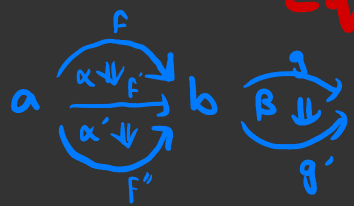
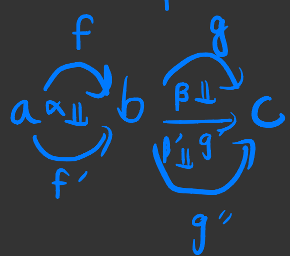
① @ 



adjoint equivalence

② @ 

iso 4-cell

③ At 

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④,⑤ 
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- Eg Gray-Cat with homs Ps(A,B)s.

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  - hom Gray-cat $A(x,y)$,
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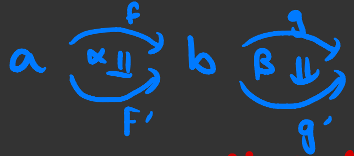
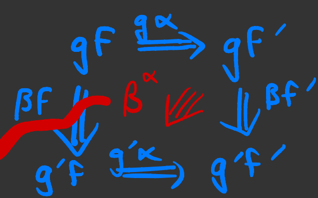
① @ a b c





adjoint equivalence

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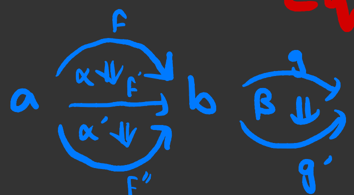

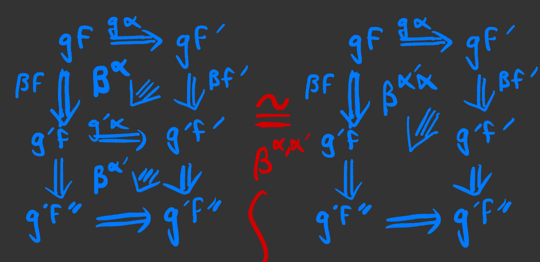

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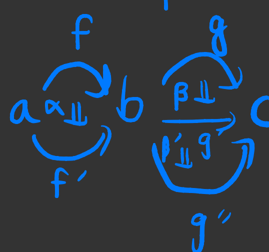

have strict equality

④,⑤

→ iso 4-cells

satisfying various equations.

- E.g Gray-Cat with homs $Ps(A,B)_s$.
- A. Miranda's Thesis (studies subclass of these).

Final remarks

- If $\mathcal{C}$ sharply enriched, $\mathcal{C}^{\phi}$ only flexibly enriched.

Final remarks

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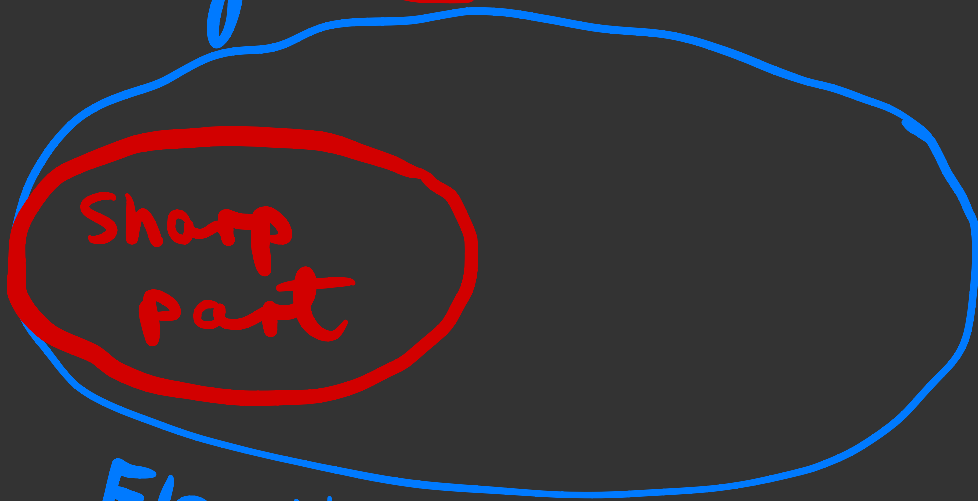
- Really one structure



Flexible structure
(skew multicat)

Final remarks

- If $\mathcal{C}$ sharply enriched, $\mathcal{C}^{\Phi}$ only flexibly enriched.
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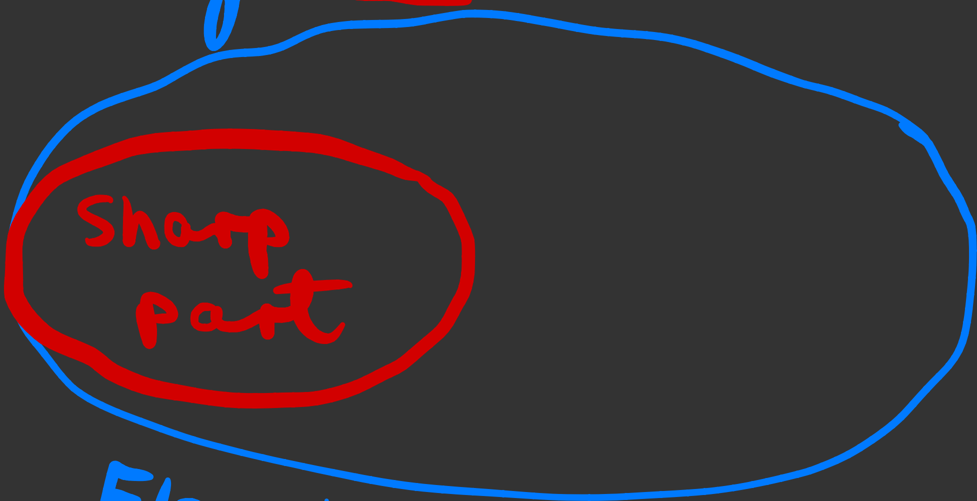

Flexible structure (skew multicat)

- Thanks for listening!