Cartesian cubical model categories

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CT 2023
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Background

- There has recently been work on **cubical** homotopy theory.
- It is related to **homotopy type theory** which is being used for computerized proof checking.
- The cubes used for this are **closed under finite products**.
- This model of homotopy was also proposed by Lawvere who stressed the **tinyness of the geometric interval** $\mathbb{I}$.
- The tinyness of $\mathbb{I}$ is also used in the current theory.

Cartesian cubical sets

The **Cartesian cube category** $\Box$ is the opposite of the category $\mathbb{B}$ of finite, strictly bipointed sets,

$$\Box := \mathbb{B}^{\mathsf{op}}.$$

Thus $\Box$ is the **Lawvere theory of bipointed objects**: the free finite product category with a bipointed object $[0] \rightrightarrows [1]$.

The **Cartesian cubical sets** is the category of presheaves on $\Box$,

$$\mathsf{cSet} = \mathsf{Set}^{\Box^{\mathsf{op}}}.$$

Thus cSet consists of all **covariant** functors $\mathbb{B} \to \mathsf{Set}$.

The tiny interval $\mathbb{I}$

The 1-cube [1] represents the cubical set that **forgets the points**,

$$\mathbb{I} := \mathbb{B}([1], -) : \mathbb{B} \longrightarrow \mathsf{Set}\,.$$

It **generates** cSet under finite products and colimits.

The two points $1 \rightrightarrows \mathbb{I}$ have a trivial intersection.

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \big\downarrow & \lrcorner & \big\downarrow \\ 1 & \longrightarrow & \mathbb{I} \end{array}$$

This is the universal **interval** in a topos.

It provides a **good cylinder** $X + X \rightarrowtail \mathbb{I} \times X$ for every object X, and a **good path object** $X^{\mathbb{I}} \twoheadrightarrow X \times X$ for every **fibrant** object X.

The main result

Theorem (A. 2023)

There is a Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on cSet *where:*

- *the* **cofibrations** $\mathcal{C}$ *are an axiomatized class of monos,*
- *the* **fibrations** $\mathcal{F}$ *are those $f : X \to Y$ for which*

$$(f^{\mathbb{I}} \times \mathbb{I}, \mathrm{eval}) : X^{\mathbb{I}} \times \mathbb{I} \longrightarrow (Y^{\mathbb{I}} \times \mathbb{I}) \times_Y X$$

  lifts on the right against all cofibrations,
- *the* **weak equivalences** $\mathcal{W}$ *are those $f : X \to Y$ for which $K^f : K^Y \longrightarrow K^X$ is bijective under π_0 whenever K is fibrant.*

The construction of $(\mathcal{C}, \mathcal{W}, \mathcal{F})$

The **proof** of the theorem

- uses ideas from **type theory**,
- including the **univalence axiom** of Voevodsky,
- is **axiomatized** in terms of:
  1. a classifier $\Phi \hookrightarrow \Omega$ for the cofibrations,
  2. a tiny interval $1 \rightrightarrows \mathbb{I}$,
  3. a universal small map $\dot{\mathsf{V}} \to \mathsf{V}$,
- applies in several different cases.

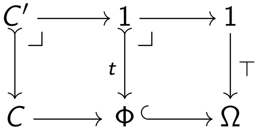
$(\mathcal{C}, \mathcal{W}, \mathcal{F})$ from $(\Phi, \mathbb{I}, \mathsf{V})$

The model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is constructed in 3 steps:

1. Φ is used to determine a wfs $(\mathcal{C}, \mathsf{TFib})$,
2. $\mathbb{I}$ is used to determine a wfs $(\mathsf{TCof}, \mathcal{F})$ with $\mathsf{TFib} \subseteq \mathcal{F}$,
3. V is used to show 3-for-2 for $\mathcal{W} := \mathsf{TFib} \circ \mathsf{TCof}$.

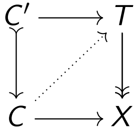
1. The cofibration wfs $(\mathcal{C}, \mathsf{TFib})$

The **cofibrations** $\mathcal{C}$ are the monos $C' \rightarrowtail C$ classified by $t : 1 \rightarrowtail \Phi$.



The **trivial fibrations** TFib are the maps $T \twoheadrightarrow X$ that lift against the cofibrations.

$$\mathcal{C}^{\pitchfork} =: \mathsf{TFib}$$



1. The cofibration wfs $(\mathcal{C}, \mathsf{TFib})$

Proposition

$(\mathcal{C}, \mathsf{TFib})$ *is an algebraic weak factorization system.*

Proof.

The classifier $t : 1 \rightarrowtail \Phi$ determines a **fibered polynomial monad**

$$P_t = \Phi_! t_* : \mathsf{cSet} \longrightarrow \mathsf{cSet}$$

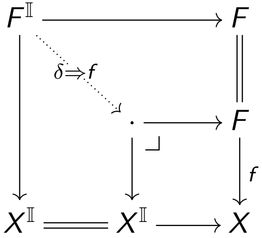
the algebras for which in $\mathsf{cSet}/_X$ are the trivial fibrations. □

2. The fibration wfs $(\mathsf{TCof}, \mathcal{F})$

The **fibrations** $\mathcal{F}$ are defined in terms of the trivial fibrations by

$$(f : F \to X) \in \mathcal{F} \qquad \text{iff} \qquad (\delta \Rightarrow f) \in \mathsf{TFib}$$

where $\delta \Rightarrow f$ is the **gap map** with $\delta : 1 \longrightarrow \mathbb{I}$ in $\mathsf{cSet}/_{\mathbb{I}}$.



The **trivial cofibrations** TCof are the maps that lift against $\mathcal{F}$.

$$\mathsf{TCof} := {}^{\pitchfork}\mathcal{F}$$

3. The weak equivalences $\mathcal{W}$

Let $\mathcal{W} := \mathsf{TFib} \circ \mathsf{TCof}$.

Proposition

$(\mathcal{C}, \mathsf{TFib})$ *and* $(\mathsf{TCof}, \mathcal{F})$ *form a Barton* **premodel structure***.*

$$\mathsf{TCof} = \mathcal{W} \cap \mathcal{C}$$
$$\mathsf{TFib} = \mathcal{W} \cap \mathcal{F}$$

Corollary

If $\mathcal{W}$ *satisfies 3-for-2, then* $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ *is a QMS.*

3. The weak equivalences $\mathcal{W}$

We use a **universal fibration** $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ to show 3-for-2 for $\mathcal{W}$.

(i) there is a **universal small map** $\dot{\mathsf{V}} \to \mathsf{V}$
(ii) U is the **classifying type** for fibration structures on $\dot{\mathsf{V}} \to \mathsf{V}$
(iii) $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ is **univalent**
(iv) U is **fibrant**
(v) fibrant U implies **3-for-2** for $\mathcal{W}$

The idea of getting a QMS **from** univalence is due to Sattler.

3(i). The universal small map $\dot{\mathsf{V}} \to \mathsf{V}$

The **category of elements** functor $\int_{\mathbb{C}}$

$$\textstyle\int_{\mathbb{C}} : \widehat{\mathbb{C}} \rightleftarrows \mathsf{Cat} : \nu_{\mathbb{C}}$$

always has a right adjoint **nerve** functor $\nu_{\mathbb{C}}$.

Proposition

For any small map $Y \to X$ in $\widehat{\mathbb{C}}$ there is a canonical pullback

$$\begin{array}{ccc} Y & \longrightarrow & \nu_{\mathbb{C}}\, \dot{\mathsf{set}}^{\mathsf{op}} \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & \nu_{\mathbb{C}}\, \mathsf{set}^{\mathsf{op}} \end{array}$$

since $\dot{\mathsf{set}}^{\mathsf{op}} \longrightarrow \mathsf{set}^{\mathsf{op}}$ *classifies small discrete fibrations in* Cat.

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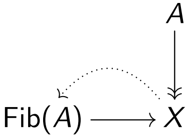
since $\dot{\mathsf{set}}^{\mathsf{op}} \to \mathsf{set}^{\mathsf{op}}$ *classifies small* **discrete fibrations** *in* Cat.

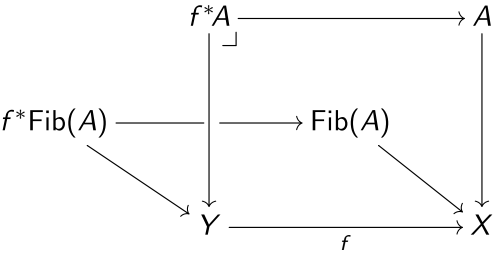
3(ii). The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

For any $A \to X$ in cSet there is a **classifying type** $\mathsf{Fib}(A) \to X$, the sections of which correspond to fibration structures.



3(ii). The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

The construction of $\mathrm{Fib}(A) \longrightarrow X$ is stable under pullback.



$$f^*\mathrm{Fib}(A) \;\cong\; \mathrm{Fib}(f^*A)$$

This uses the **root** functor $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$.

3(ii). The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

Let U be the type of fibration structures on $\dot{\mathsf{V}} \to \mathsf{V}$

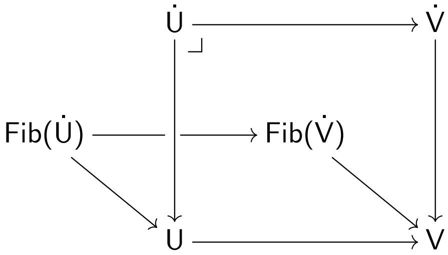
$$\begin{array}{ccc} & & \dot{\mathsf{V}} \\ & & \downarrow \\ \mathsf{U} := \mathrm{Fib}(\dot{\mathsf{V}}) & \longrightarrow & \mathsf{V} \end{array}$$

then define $\dot{\mathsf{U}} \to \mathsf{U}$ by pulling back.

$$\begin{array}{ccc} \dot{\mathsf{U}} & \longrightarrow & \dot{\mathsf{V}} \\ \downarrow & \lrcorner & \downarrow \\ \mathsf{U} & \longrightarrow & \mathsf{V} \end{array}$$

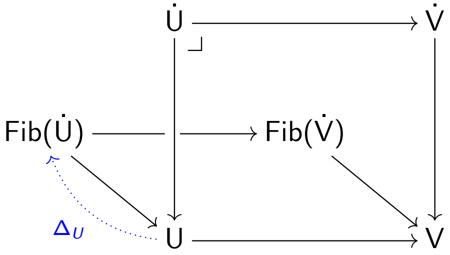
3(ii). The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

Since Fib(−) is stable, the lower square is a pullback.



3(ii). The universal fibration $\dot{U} \twoheadrightarrow U$

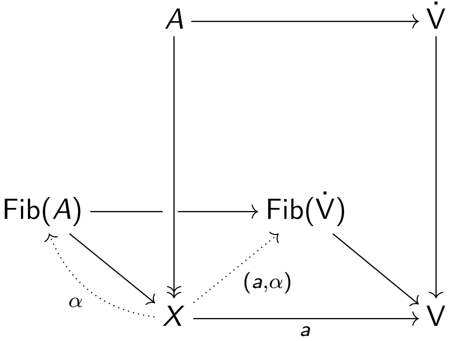
Since Fib(−) is stable the lower square is also a pullback.



But since U = Fib($\dot{V}$) there is a section of Fib($\dot{U}$).
So $\dot{U} \to U$ is a fibration.

3(ii). The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

A fibration structure α on a small map $A \to X$ determines a factorization (a, α) of its classifying map $a : X \to \mathsf{V}$.



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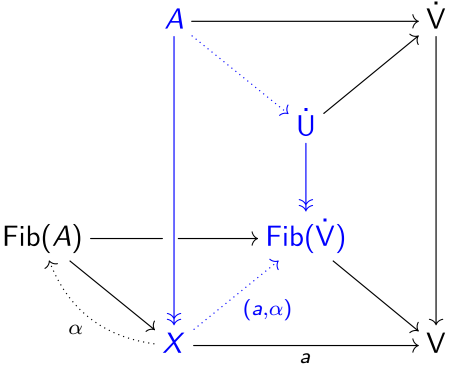


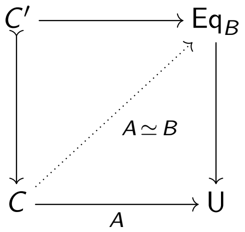
which classifies $A \twoheadrightarrow X$ **as a fibration** since $\mathrm{Fib}(\dot{\mathsf{V}}) = \mathsf{U}$.

3(iii). $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ is univalent

The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ is **univalent** if the type

$$\mathsf{Eq}_B \;=\; \Sigma_B \mathsf{Eq}(-, B) \longrightarrow \mathsf{U}$$

of **based equivalences** is always a trivial fibration.



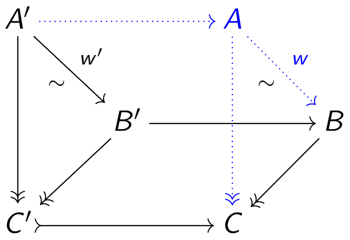
(*)

Remark

In HoTT this implies $(A = B) \simeq (A \simeq B)$.

3(iii). $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ is univalent

Unwinding $(*)$ gives the **equivalence extension property**:
weak equivalences extend along cofibrations $C' \rightarrowtail C$.



3(iii). $\dot{\mathrm{U}} \twoheadrightarrow \mathrm{U}$ is univalent

Proposition

The universal fibration $\dot{\mathrm{U}} \twoheadrightarrow \mathrm{U}$ *is univalent.*

Voevodsky proved this **classically** for Kan fibrations in sSet.

Coquand gave a constructive proof in **type theory** for cSet.

We have generalized Coquand's proof to cartesian cubical sets.

3(iv). U is fibrant

Univalence of $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ implies that U is fibrant.

Proposition

The universe U *is fibrant.*

Voevodsky proved this for Kan sSets using **minimal fibrations**.

Shulman proved it using **3-for-2** for $\mathcal{W}$.

Coquand proved it from univalence without 3-for-2 using **Kan composition** for cSets in type theory.

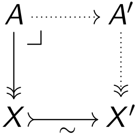
We give a general proof from univalence without using 3-for-2.

3(v). From fibrant U to 3-for-2

Finally, we can apply the following.

Proposition (Sattler)

$\mathcal{W}$ satisfies 3-for-2 if fibrations extend along trivial cofibrations.

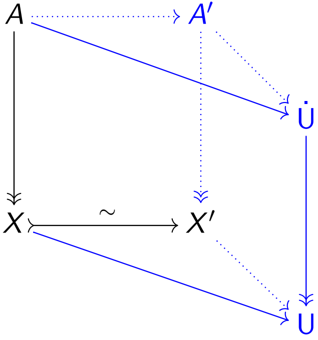


This is called the **fibration extension property**.

3(v). From fibrant U to 3-for-2 for $\mathcal{W}$

Lemma

Given a universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ *the FEP holds if* U *is fibrant.*



References

- S. Awodey, Cartesian cubical model categories, 2023.
- C. Cohen, et al., Cubical type theory: A constructive interpretation of the univalence axiom, 2016.
- C. Kapulkin and P. LeFanu Lumsdaine, The simplicial model of univalent foundations (after Voevodsky), 2018.
- C. Sattler, The equivalence extension property and model structures, 2017.
- M. Shulman, All $(\infty, 1)$-toposes have strict univalent universes, 2019.

Appendix: U is fibrant (sketch)

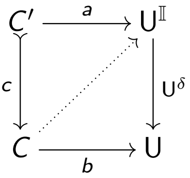
It suffices to show the following.

Proposition

Evaluation at the **generic point** $\mathsf{U}^{\mathbb{I}} \longrightarrow \mathsf{U}$ *is a trivial fibration.*

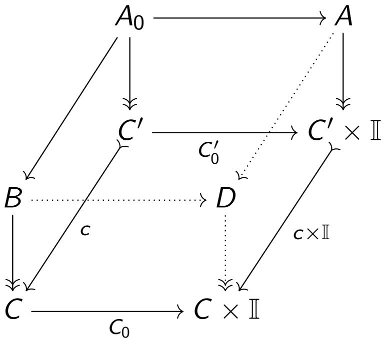
Proof.

We need a diagonal filler for any cofibration c.



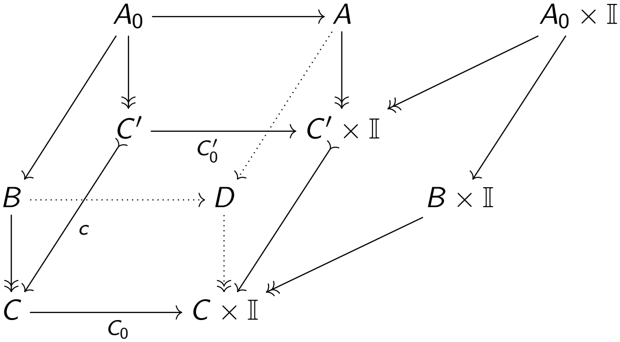
Appendix: U is fibrant (sketch)

Transposing by $\mathbb{I}$ and using the classifying property of U gives the following equivalent problem.



Appendix: U is fibrant (sketch)

Apply the functor $(-) \times \mathbb{I}$ to the left face to get:



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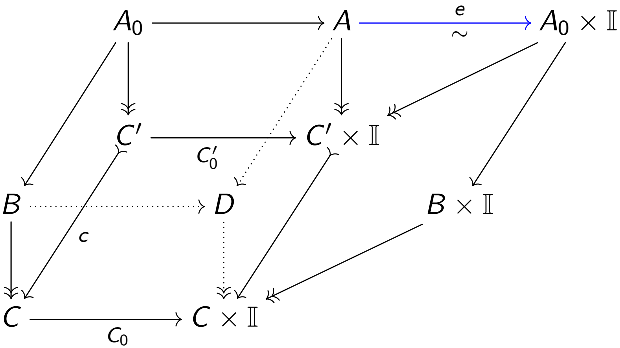

There is a weak equivalence $e : A \xrightarrow{\sim} A_0 \times \mathbb{I}$ to which we can apply the EEP.

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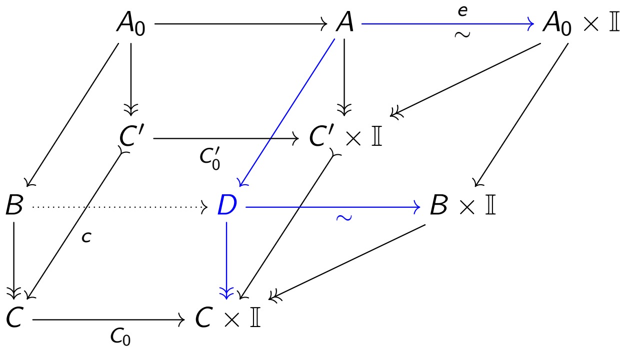

There is a weak equivalence $e : A \simeq A_0 \times \mathbb{I}$ to which we can apply the EEP. This produces the required fibration $D \twoheadrightarrow Z \times \mathbb{I}$. □