

Radon-Nikodym derivatives and martingales

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3 July 2023, Louvain-la-Neuve

1 Radon-Nikodym derivatives

Radon-Nikodym theorem

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$$\nu(A) = \int_A f d\mu,$$

for all A in Σ .

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The map f is called **the Radon-Nikodym derivative of ν with respect to μ** and is denoted as $\frac{d\nu}{d\mu}$.

Radon-Nikodym theorem

Consider the map $L^1(X, \Sigma, \mu) \rightarrow \{\nu \mid \nu \ll \mu\}$ that sends $f \in L^1(X, \Sigma, \mu)$ to the measure defined by

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for all $A \in \Sigma$.

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The Radon-Nikodym theorem says that this is a bijection.

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- Let N be the standard normal distribution and λ the Lebesgue measure on \mathbb{R} ,

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- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub- σ -algebra.

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- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub- σ -algebra. An integrable \mathcal{F} -measurable map $X : \Omega \rightarrow \mathbb{R}$ defines a measure ν on (Ω, \mathcal{G}) by:

$$\nu(A) := \int_A X d\mathbb{P} \quad (= \mathbb{E}[X1_A]),$$

for all $A \in \mathcal{G}$.

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Therefore, $\nu \ll \mathbb{P} \llcorner_{\mathcal{G}}$ and there exists \mathbb{P} -almost surely unique \mathcal{G} -measurable integrable map $f : \Omega \rightarrow \mathbb{R}$ such that

$$\int_A X d\mathbb{P} = \int_A f d\mathbb{P} \llcorner_{\mathcal{G}},$$

or

$$(\mathbb{E}[X1_A] = \mathbb{E}[f1_A])$$

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$$(\mathbb{E}[X1_A] = \mathbb{E}[f1_A])$$

for all $A \in \mathcal{G}$. The map f is called the **conditional expectation of X with respect to \mathcal{G}** and is denoted as $\mathbb{E}[X \mid \mathcal{G}]$.

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$$a \mapsto \begin{cases} \frac{q_a}{p_a} & \text{if } p_a \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

It can be checked that f is the Radon-Nikodym derivative of q with respect to p .

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$$\{\mu \mid \mu \leq n\mathbb{P}\},$$

together with the *total variation metric*.

- Define $RV_n(\Omega, \mathcal{F}, \mathbb{P})$ as the set

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together with the L^1 -metric (multiplied by a factor 1/2).

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These are *complete* metric spaces (Riesz-Fischer).

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By the finite Radon-Nikodym theorem, we see that

$$\mathbf{Prob}_f \begin{array}{c} \xrightarrow{M_n^f} \\ \xrightarrow{\cong} \\ \xrightarrow{RV_n^f} \end{array} \mathbf{CMet}_1.$$

Categorically extending the finite version

It follows that also the right Kan extensions along $i : \mathbf{Prob}_f \rightarrow \mathbf{Prob}$ are isomorphic.

$$\begin{array}{ccc} \mathbf{Prob}_f & \begin{array}{c} \xrightarrow{M_n^f} \\ \xrightarrow{R} \end{array} & \mathbf{CMet}_1 \\ \downarrow i & \begin{array}{c} \xrightarrow{RV_n^f} \\ \xrightarrow{R} \end{array} & \nearrow \end{array}$$

What do these Kan extensions look like?

Proposition

For a probability space $\Omega := (\Omega, \mathcal{F}, \mathbb{P})$, we have for all $n \geq 1$ that

$$M_n(\Omega) \rightarrow (\text{Ran}_i M_n^f)(\Omega),$$

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Proof (sketch): Let $\Omega := (\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

$$\text{Ran}_i M_n^f(\Omega) \cong \int_{\mathbf{A} \in \text{Prob}_f} [\mathbf{Prob}(\Omega, i\mathbf{A}), M_n^f(\mathbf{A})]$$

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defined by the assignment

$$\mu \mapsto (\mu \circ s^{-1})_{s \in \mathbf{Prob}[\mathbf{\Omega}, \mathbf{A}]}$$

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This induces a morphism

$$M_n(\mathbf{\Omega}) \rightarrow \int_{\mathbf{A}} [\mathbf{Prob}(\mathbf{\Omega}, \mathbf{A}), M_n^f(\mathbf{A})] \cong (\text{Ran}_i M_n^f)(\mathbf{\Omega}).$$

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For $E \in \mathcal{F}$, consider the finite probability space

$$\mathbf{2}_E := (\{0, 1\}, \mathbb{P}(E^C)\delta_0 + \mathbb{P}(E)\delta_1),$$

and note that the indicator function 1_E becomes a measure-preserving map

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For $y \in Y$, define

$$\mu_y(E) := e_{\mathbf{2}_E}(y)(1_E)_1.$$

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It can be shown that $\mu_y \in M_n(\Omega)$. This gives a morphism $Y \rightarrow M_n(\Omega)$, making $M_n(\Omega)$ a universal wedge.

What do these Kan extensions look like?

Proposition

For a probability space Ω , we have for all $n \geq 1$ that

$$(\text{Ran}_i RV_n^f)(\Omega) \cong RV_n(\Omega).$$

The proof for this results requires some measure theory.

Radon-Nikodym theorem

Combining everything gives a *bounded* Radon-Nikodym theorem, namely

$$\begin{aligned}\{\mu \mid \mu \leq n\mathbb{P}\} &= M_n(\Omega) \cong \text{Ran}_i M_n^f(\Omega) \\ &\cong \text{Ran}_i RV_n^f(\Omega) \\ &\cong RV_n(\Omega) = \mathbf{Mble}(\Omega, [0, n]) / \approx_{\mathbb{P}}\end{aligned}$$

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We can look at the colimit over all $n \geq 1$,

$$\begin{array}{ccccccc}M_1\Omega & \longrightarrow & M_2\Omega & \longrightarrow & \dots & \longrightarrow & M_n\Omega & \longrightarrow & \dots \\ \parallel & & \parallel & & & & \parallel & & \\ RV_1\Omega & \longrightarrow & RV_2\Omega & \longrightarrow & \dots & \longrightarrow & RV_n\Omega & \longrightarrow & \dots\end{array}$$

This gives us

$$\{\mu \mid \mu \ll \mathbb{P}\} \cong \{f : \Omega \rightarrow [0, \infty) \mid f \text{ is integrable}\} / \cong_{\mathbb{P}}.$$

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What can we say about $M_n(g) := (\text{Ran}_i M_n^f)(g)$ and $RV_n(g) := (\text{Ran}_i RV_n^f)(g)$ for $g : \Omega_1 \rightarrow \Omega_2$?

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They are the unique morphisms such that

$$\begin{array}{ccc} M_n \Omega_1 & \xrightarrow{M_n(g)} & M_n \Omega_2 \\ & \searrow & \swarrow \\ & M_n^f \mathbf{A} & \end{array}$$

$$\begin{array}{ccc} RV_n \Omega_1 & \xrightarrow{RV_n(g)} & RV_n \Omega_2 \\ & \searrow & \swarrow \\ & RV_n^f \mathbf{A} & \end{array}$$

commute for morphisms $\Omega_2 \rightarrow \mathbf{A}$.

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$$M_n(g)(\mu) \circ 1_E^{-1} = \mu \circ 1_{g^{-1}(E)}^{-1},$$

and

$$\int_E RV_n(g)(f) d\mathbb{P}_2 = \int_{g^{-1}(E)} f d\mathbb{P}_1$$

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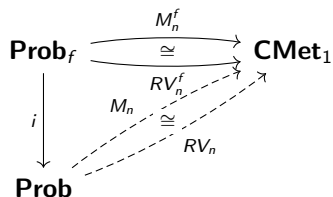
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This means that

$$M_n(g)(\mu) = \mu \circ g^{-1} \quad \text{and} \quad RV_n(g)(f) = \mathbb{E}[f \mid g].$$

Summary



- (Bounded) Radon-Nikodym theorem:

$$M_n(\Omega) = \{\mu \mid \mu \leq n\mathbb{P}\} \quad RV_n(\Omega) = \mathbf{Mble}(\Omega, [0, n]) / \cong_{\mathbb{P}} .$$

- Conditional expectation:

$$RV_n(g)(X) = \mathbb{E}[X \mid f].$$

2 Martingales

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A **martingale** is a collection of integrable random variables $X_i : (\Omega, \mathcal{F}_i) \rightarrow \mathbb{R}$ such that

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for all $i \leq j$.

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for all $i \leq j$. Example: Brownian motion.

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Theorem

An L^1 -bounded martingale $(X_n)_n$, converges \mathbb{P} -almost surely to a random variable $X : (\Omega, \mathcal{F}) \rightarrow \mathbb{R}$.

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Theorem

Let $p > 1$. An L^p -bounded martingale $(X_n)_n$ converges to a random variable $X : (\Omega, \mathcal{F}) \rightarrow \mathbb{R}$ in L^p and for all $n \geq 1$,

$$\mathbb{E}[X \mid \mathcal{F}_n] = X_n.$$

How does this translate categorically?

The space Ω is the limit of

$$\Omega_1 \xleftarrow{s_{21}} \Omega_2 \xleftarrow{s_{32}} \Omega_3 \xleftarrow{\quad} \dots \xleftarrow{\quad} \Omega_m \xleftarrow{\quad} \dots$$

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Suppose that $RV_n : \mathbf{Prob} \rightarrow \mathbf{CMet}_1$ preserves this limit, then

$$\begin{aligned} RV_n(\Omega) &\cong \lim_m RV_n(\Omega_m) \\ &\cong \{(X_m)_m \mid RV_n(s_{m_1 m_2})(X_{m_1}) = X_{m_2} \text{ for } m_2 \leq m_1\} \\ &\cong \{(X_m)_m \mid \mathbb{E}[X_{m_1} \mid \mathcal{F}_{n_2}] = X_{m_2} \text{ for } m_2 \leq m_1\} \\ &\cong \{(X_m)_m \mid \text{martingale}\} \end{aligned}$$

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It follows that for every martingale $(X_m)_m$ such that $X_m \leq n$ for all m , there exists a random variable $X : (\Omega, \mathcal{F}) \rightarrow [0, n]$ such that for all m ,

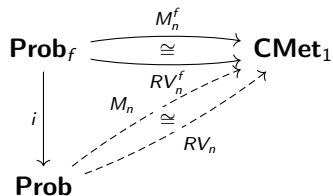
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Enrichment over \mathbf{CMet}_1

Everything from the first part still works when everything is *enriched* over \mathbf{CMet}_1 .

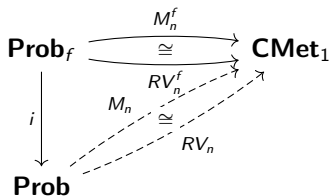
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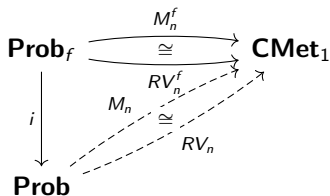
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Everything from the first part still works when everything is *enriched* over \mathbf{CMet}_1 .



How is \mathbf{Prob} enriched over \mathbf{CMet}_1 ?

Answer: $\mathbf{Prob}(\Omega_1, \Omega_2)$ is the *completion* of

$$\{f : \Omega_1 \rightarrow \Omega_2 \mid \text{measure preserving}\}$$

with the pseudometric

$$d(f_1, f_2) := \sup \{ \mathbb{P}_1(f_1^{-1}(A) \Delta f_2^{-1}(A)) \mid A \in \mathcal{F}_2 \}.$$

RV_n preserves cofiltered limits

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$$\begin{aligned} RV_n(\Omega) &\cong \int_{\mathbf{A}} [\mathbf{Prob}(\Omega, \mathbf{A}), RV_n^f(\mathbf{A})] \\ &\cong \int_{\mathbf{A}} [\operatorname{colim}_i \mathbf{Prob}(\Omega_i, \mathbf{A}), RV_n^f(\mathbf{A})] \\ &\cong \int_{\mathbf{A}} \lim_i [\mathbf{Prob}(\Omega_i, \mathbf{A}), RV_n^f(\mathbf{A})] \\ &\cong \lim_i \int_{\mathbf{A}} [\mathbf{Prob}(\Omega_i, \mathbf{A}), RV_n^f(\mathbf{A})] \cong \lim_i RV_n(\Omega_i) \end{aligned}$$

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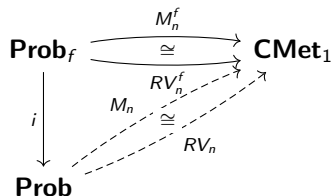
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Remark: We did not use anything about RV_n^f .

Summary

Enriched version of



- (Bounded) Radon-Nikodym theorem:

$$M_n(\Omega) = \{\mu \mid \mu \leq n\mathbb{P}\} \quad RV_n(\Omega) = \mathbf{Mble}(\Omega, [0, n]) / \cong_{\mathbb{P}}.$$

- Conditional expectation:

$$RV_n(g)(X) = \mathbb{E}[X \mid f].$$

- Martingale convergence: RV_n preserves cofiltered limits.
- Weaker Kolmogorov extension theorem : M_n preserves cofiltered limits.

What about left Kan extensions?

Let $H : \mathbf{Prob}_f \rightarrow \mathbf{CMet}_1$ be a functor. Suppose that Ω is a probability space that is **not essentially finite**.

Then $\mathbf{Prob}(\mathbf{A}, \Omega) = \emptyset$ for all finite probability spaces \mathbf{A} and

$$\mathrm{Lan}_i H(\Omega) = \int^{\mathbf{A}} \mathbf{Prob}(\mathbf{A}, \Omega) \times H\mathbf{A} = \emptyset.$$