Radon-Nikodym derivatives and martingales

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3 July 2023, Louvain-la-Neuve

1 Radon-Nikodym derivatives

Radon-Nikodym theorem

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Let (X, Σ) be a measurable space. Let μ be a σ-finite measure and ν a finite signed measure on (X, Σ) such that $\nu \ll \mu$.[a] Then there exists a μ-almost surely unique integrable map $f : X \to \mathbb{R}$ such that

$$\nu(A) = \int_A f \mathrm{d}\mu,$$

for all A in Σ.

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The map f is called **the Radon-Nikodym derivative of ν with respect to μ** and is denoted as $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$.

Radon-Nikodym theorem

Consider de map $L^1(X,\Sigma,\mu) \to \{\nu \mid \nu \ll \mu\}$ that sends $f \in L^1(X,\Sigma,\mu)$ to the measure defined by

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for all $A \in \Sigma$.

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$$\nu(A) := \int_A f \mathrm{d}\mu,$$

for all $A \in \Sigma$.
The Radon-Nikodym theorem says that this is a bijection.

Radon-Nikodym theorem: examples

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- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra.

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- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra. An integrable $\mathcal{F}$-measurable map $X : \Omega \to \mathbb{R}$ defines a measure ν on $(\Omega, \mathcal{G})$ by:

$$\nu(A) := \int_A X \mathrm{d}\mathbb{P} \quad (= \mathbb{E}[X 1_A]),$$

  for all $A \in \mathcal{G}$.

Radon-Nikodym theorem: examples

Examples: For $A \in \mathcal{G}$ such that $\mathbb{P}\,|_{\mathcal{G}}\,(A) = 0$, we have

$$\nu(A) = \int_A X \mathrm{d}\mathbb{P} = 0.$$

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Examples: For $A \in \mathcal{G}$ such that $\mathbb{P}\mid_{\mathcal{G}}(A) = 0$, we have

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Therefore, $\nu \ll \mathbb{P}\mid_{\mathcal{G}}$ and there exists $\mathbb{P}$-almost surely unique $\mathcal{G}$-measurable integrable map $f : \Omega \to \mathbb{R}$ such that

$$\int_A X \mathrm{d}\mathbb{P} = \int_A f \mathrm{d}\mathbb{P}\mid_{\mathcal{G}},$$

or

$$(\mathbb{E}[X 1_A] = \mathbb{E}[f 1_A])$$

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for all $A \in \mathcal{G}$.The map f is called the **conditional expectation of X with respect to $\mathcal{G}$** and is denoted as $\mathbb{E}[X \mid \mathcal{G}]$.

Radon-Nikodym theorem: finite version

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$$a \mapsto \begin{cases} \frac{q_a}{p_a} & \text{if } p_a \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

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It can be checked that f is the Radon-Nikodym derivative of q with respect to p.

Categorically extending the finite version

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- Define $M_n(\Omega, \mathcal{F}, \mathbb{P})$ as the set

  $$\{\mu \mid \mu \leq n\mathbb{P}\},$$

  together with the *total variation metric*.
- Define $RV_n(\Omega, \mathcal{F}, \mathbb{P})$ as the set

  $$\mathbf{Mble}(\Omega, [0, n])/ =_{\mathbb{P}},$$

  together with the L^1-*metric* (multiplied by a factor $1/2$).

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These are *complete* metric spaces (Riesz-Fischer).

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We have two functors:

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By the finite Radon-Nikodym theorem, we see that

$$\begin{tikzcd} \mathbf{Prob}_f \arrow[r, bend left, "M_n^f"] \arrow[r, bend right, "RV_n^f"'] \arrow[r, phantom, "\cong"] & \mathbf{CMet}_1 \end{tikzcd}.$$

Categorically extending the finite version

It follows that also the right Kan extensions along $i : \mathbf{Prob}_f \to \mathbf{Prob}$ are isomorphic.

$$\begin{tikzcd}
\mathbf{Prob}_f \arrow[r, bend left, "M_n^f"] \arrow[r, bend right, "RV_n^f"'] \arrow[r, phantom, "\cong"] \arrow[d, "i"] & \mathbf{CMet}_1 \\
\mathbf{Prob} \arrow[ru, dashed, bend left] \arrow[ru, dashed, bend right] \arrow[ru, phantom, "\cong"] &
\end{tikzcd}$$

What do these Kan extensions look like?

Proposition

For a probability space $\mathbf{\Omega} := (\Omega, \mathcal{F}, \mathbb{P})$, we have for all $n \geq 1$ that

$$M_n(\mathbf{\Omega}) \to (\mathrm{Ran}_i M_n^f)(\mathbf{\Omega}),$$

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Proof (sketch): Let $\mathbf{\Omega} := (\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

$$\mathrm{Ran}_i M_n^f(\mathbf{\Omega}) \cong \int_{\mathbf{A} \in \mathbf{Prob}_f} [\mathbf{Prob}(\mathbf{\Omega}, i\mathbf{A}), M_n^f(\mathbf{A})]$$

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$$M_n(\mathbf{\Omega}) \to [\mathbf{Prob}(\mathbf{\Omega}, \mathbf{A}), M_n^f(\mathbf{A})],$$

defined by the assignment

$$\mu \mapsto (\mu \circ s^{-1})_{s \in \mathbf{Prob}[\mathbf{\Omega}, \mathbf{A}]}.$$

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This induces a morphism

$$M_n(\mathbf{\Omega}) \to \int_{\mathbf{A}} [\mathbf{Prob}(\mathbf{\Omega}, \mathbf{A}), M_n^f(\mathbf{A})] \cong (\mathrm{Ran}_i M_n^f)(\mathbf{\Omega}).$$

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For $E \in \mathcal{F}$, consider the finite probability space

$$\mathbf{2}_E := (\{0,1\}, \mathbb{P}(E^C)\delta_0 + \mathbb{P}(E)\delta_1),$$

and note that the indicator function 1_E becomes a measure-preserving map

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For $y \in Y$, define

$$\mu_y(E) := e_{\mathbf{2}_E}(y)(1_E)_1.$$

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It can be shown that $\mu_y \in M_n(\mathbf{\Omega})$. This gives a morphism $Y \to M_n(\mathbf{\Omega})$, making $M_n(\mathbf{\Omega})$ a universal wedge.

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Proposition

For a probability space $\mathbf{\Omega}$, we have for all $n \geq 1$ that

$$(\mathrm{Ran}_i RV_n^f)(\mathbf{\Omega}) \cong RV_n(\mathbf{\Omega}).$$

The proof for this results requires some measure theory.

Radon-Nikodym theorem

Combining everything gives a *bounded* Radon-Nikodym theorem, namely

$$\begin{aligned} \{\mu \mid \mu \leq n\mathbb{P}\} = M_n(\mathbf{\Omega}) &\cong \mathrm{Ran}_i M_n^f((\mathbf{\Omega}) \\ &\cong \mathrm{Ran}_i RV_n^f(\mathbf{\Omega}) \\ &\cong RV_n(\mathbf{\Omega}) = \mathbf{Mble}(\Omega, [0, n])/ =_{\mathbb{P}} \end{aligned}$$

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We can look at the colimit over all $n \geq 1$,

$$\begin{tikzcd} M_1\mathbf{\Omega} \arrow[r] \arrow[d, phantom, "\wr\|"] & M_2\mathbf{\Omega} \arrow[r] \arrow[d, phantom, "\wr\|"] & \dots \arrow[r] & M_n\mathbf{\Omega} \arrow[r] \arrow[d, phantom, "\wr\|"] & \dots \\ RV_1\mathbf{\Omega} \arrow[r] & RV_2\mathbf{\Omega} \arrow[r] & \dots \arrow[r] & RV_n\mathbf{\Omega} \arrow[r] & \dots \end{tikzcd}$$

This gives us

$$\{\mu \mid \mu \ll \mathbb{P}\} \cong \{f : \Omega \to [0, \infty) \mid f \text{ is integrable}\} / =_{\mathbb{P}} .$$

Remark on conditional expectation

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What can we say about $M_n(g) := (\mathrm{Ran}_i M_n^f)(g)$ and $RV_n(g) := (\mathrm{Ran}_i RV_n^f)(g)$ for $g : \mathbf{\Omega}_1 \to \mathbf{\Omega}_2$?

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What can we say about $M_n(g) := (\mathrm{Ran}_i M_n^f)(g)$ and $RV_n(g) := (\mathrm{Ran}_i RV_n^f)(g)$ for $g : \mathbf{\Omega}_1 \to \mathbf{\Omega}_2$?
They are the unique morphisms such that

$$\begin{tikzcd} M_n\mathbf{\Omega}_1 \arrow[rr, "M_n(g)"] \arrow[dr] & & M_n\mathbf{\Omega}_2 \arrow[dl] \\ & M_n^f\mathbf{A} & \end{tikzcd} \qquad \begin{tikzcd} RV_n\mathbf{\Omega}_1 \arrow[rr, "RV_n(g)"] \arrow[dr] & & RV_n\mathbf{\Omega}_2 \arrow[dl] \\ & RV_n^f\mathbf{A} & \end{tikzcd}$$

commute for morphisms $\mathbf{\Omega}_2 \to \mathbf{A}$.

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$$M_n(g)(\mu) \circ 1_E^{-1} = \mu \circ 1_{g^{-1}(E)}^{-1},$$

and

$$\int_E RV_n(g)(f) \mathrm{d}\mathbb{P}_2 = \int_{g^{-1}(E)} f \mathrm{d}\mathbb{P}_1$$

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This means that

$$M_n(g)(\mu) = \mu \circ g^{-1} \quad \text{and} \quad RV_n(g)(f) = \mathbb{E}[f \mid g].$$

Summary

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\begin{tikzcd}
\mathbf{Prob}_f \arrow[r, bend left, "M_n^f"] \arrow[r, bend right, "RV_n^f"'] \arrow[r, phantom, "\cong"] \arrow[d, "i"'] & \mathbf{CMet}_1 \\
\mathbf{Prob} \arrow[ur, dashed, bend left, "M_n"] \arrow[ur, dashed, bend right, "RV_n"'] \arrow[ur, phantom, "\cong"] &
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- (Bounded) Radon-Nikodym theorem:

$$M_n(\boldsymbol{\Omega}) = \{\mu \mid \mu \leq n\mathbb{P}\} \quad RV_n(\boldsymbol{\Omega}) = \mathbf{Mble}(\Omega, [0, n])/ =_{\mathbb{P}} .$$

- Conditional expectation:

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2 Martingales

Martingales

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A **martingale** is a collection of integrable random variables $X_i : (\Omega, \mathcal{F}_i) \to \mathbb{R}$ such that

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for all $i \leq j$.

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An L^1-*bounded martingale* $(X_n)_n$, converges $\mathbb{P}$-almost surely to a random variable $X : (\Omega, \mathcal{F}) \to \mathbb{R}$.

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Theorem

Let $p > 1$. An L^p-*bounded martingale* $(X_n)_n$ converges to a random variable $X : (\Omega, \mathcal{F}) \to \mathbb{R}$ in L^p and for all $n \geq 1$,

$$\mathbb{E}[X \mid \mathcal{F}_n] = X_n.$$

How does this translate categorically?

The space $\mathbf{\Omega}$ is the limit of

$$\mathbf{\Omega}_1 \xleftarrow[s_{21}]{} \mathbf{\Omega}_2 \xleftarrow[s_{32}]{} \mathbf{\Omega}_3 \longleftarrow \dots \longleftarrow \mathbf{\Omega}_m \longleftarrow \dots$$

in **Prob**, where $\mathbf{\Omega}_m := (\Omega, \mathcal{F}_m, \mathbb{P}\,|_{\mathcal{F}_m})$.

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in **Prob**, where $\mathbf{\Omega}_m := (\Omega, \mathcal{F}_m, \mathbb{P}\,|_{\mathcal{F}_m})$.
Suppose that $RV_n : \mathbf{Prob} \to \mathbf{CMet}_1$ preserves this limit, then

$$\begin{aligned} RV_n(\mathbf{\Omega}) &\cong \lim_m RV_n(\mathbf{\Omega}_m) \\ &\cong \{(X_m)_m \mid RV_n(s_{m_1 m_2})(X_{m_1}) = X_{m_2} \text{ for } m_2 \leq m_1\} \\ &\cong \{(X_m)_m \mid \mathbb{E}[X_{m_1} \mid \mathcal{F}_{n_2}] = X_{m_2} \text{ for } m_2 \leq m_1\} \\ &\cong \{(X_m)_m \mid \text{ martingale } \} \end{aligned}$$

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It follows that for every martingale $(X_m)_m$ such that $X_m \leq n$ for all m, there exists a random variable $X : (\Omega, \mathcal{F}) \to [0, n]$ such that for all m,

$$\mathbf{E}[X \mid \mathcal{F}_m] = X_m.$$

Enrichment over $\mathbf{CMet}_1$

Everything from the first part still works when everything is *enriched* over $\mathbf{CMet}_1$.

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\mathbf{Prob}_f \arrow[rr, bend left=10, "M_n^f"] \arrow[rr, bend right=10, "RV_n^f"'] \arrow[rr, phantom, "\cong"] \arrow[d, "i"'] & & \mathbf{CMet}_1 \\
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How is **Prob** enriched over $\mathbf{CMet}_1$?
Answer: $\mathbf{Prob}(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2)$ is the *completion* of

$$\{f : \boldsymbol{\Omega}_1 \to \boldsymbol{\Omega}_2 \mid \text{ measure preserving}\}$$

with the pseudometric

$$d(f_1, f_2) := \sup\left\{\mathbb{P}_1(f_1^{-1}(A)\Delta f_2^{-1}(A)) \mid A \in \mathcal{F}_2\right\}.$$

RV_n preserves cofiltered limits

For any finite probability space $\mathbf{A}$, we always have a map

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$$\begin{aligned} RV_n(\boldsymbol{\Omega}) &\cong \int_{\mathbf{A}} [\mathbf{Prob}(\boldsymbol{\Omega}, \mathbf{A}), RV_n^f(\mathbf{A})] \\ &\cong \int_{\mathbf{A}} [\mathrm{colim}_i \mathbf{Prob}(\boldsymbol{\Omega}_i, \mathbf{A}), RV_n^f(\mathbf{A})] \\ &\cong \int_{\mathbf{A}} \lim_i [\mathbf{Prob}(\boldsymbol{\Omega}_i, \mathbf{A}), RV_n^f(\mathbf{A})] \\ &\cong \lim_i \int_{\mathbf{A}} [\mathbf{Prob}(\boldsymbol{\Omega}_i, \mathbf{A}), RV_n^f(\mathbf{A})] \cong \lim_i RV_n(\boldsymbol{\Omega}_i) \end{aligned}$$

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\end{aligned}$$

<u>Remark</u>: We did not use anything about RV_n^f.

Summary

Enriched version of

$$
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\end{tikzcd}
$$

- (Bounded) Radon-Nikodym theorem:

$$M_n(\mathbf{\Omega}) = \{\mu \mid \mu \leq n\mathbb{P}\} \quad RV_n(\mathbf{\Omega}) = \mathbf{Mble}(\Omega, [0, n])/ =_{\mathbb{P}} .$$

- Conditional expectation:

$$RV_n(g)(X) = \mathbb{E}[X \mid f].$$

- Martingale convergence: RV_n preserves cofiltered limits.
- Weaker Kolmogorov extension theorem : M_n preserves cofiltered limits.

What about left Kan extensions?

Let $H : \mathbf{Prob}_f \to \mathbf{CMet}_1$ be a functor. Suppose that $\mathbf{\Omega}$ is a probability space that is **not** *essentially finite*.
Then $\mathbf{Prob}(\mathbf{A}, \mathbf{\Omega}) = \emptyset$ for all finite probability spaces $\mathbf{A}$ and

$$\mathrm{Lan}_i H(\mathbf{\Omega}) = \int^{\mathbf{A}} \mathbf{Prob}(\mathbf{A}, \mathbf{\Omega}) \times H\mathbf{A} = \emptyset.$$