Jónsson categories

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- Link: Jónsson cats, $\boldsymbol{n}$-permutability, distributivity, Trapezoid Lemma

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(different names)

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2 types of matrices
(n odd or even)

The extended matrices $\mathrm{J}_{\boldsymbol{n}}$

- Jónsson terms and equations give rise to *extended matrices* of 2 types (we call) $\mathrm{J}_{\boldsymbol{n}}$

The extended matrices J_n

- Jónsson terms and equations give rise to *extended matrices* of 2 types (we call) J_n

$$\mathrm{J}_n = \left[\begin{array}{ccc|ccccccc} x_1 & x_1 & y_1 & x_1 & z_1 & z_1 & z_3 & \cdots & z_{n-4} & z_{n-2} & z_{n-2} \\ x_2 & y_2 & x_2 & x_2 & x_2 & x_2 & x_2 & \cdots & x_2 & x_2 & x_2 \\ y_3 & x_3 & x_3 & z_2 & z_2 & z_4 & z_4 & \cdots & z_{n-1} & z_{n-1} & x_3 \end{array}\right] \qquad (n \text{ odd})$$

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- $\mathbb{V}$ var ualgs *satisfies the matrix condition* $\mathsf{J}_{\boldsymbol{n}}$ (or, *has* $\mathsf{J}_{\boldsymbol{n}}$*-closed relations*)

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when: any ternary homomorphic relation $\boldsymbol{R} \rightarrowtail \boldsymbol{X}^3$ in $\mathbb{V}$ is sth

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$$\boxed{(x_1, x_2, y_3), (x_1, y_2, x_3), (y_1, x_2, x_3) \in R} \quad \Rightarrow$$

$$\exists z_1, \cdots, z_n \in X \text{ sth } \boxed{(x_1, x_2, z_2), (z_1, x_2, z_2), (z_1, x_2, z_4), \cdots \in R}$$

- Prop1: $\mathbb{V}$ var ualgs satisfies the matrix condition J_n iff it admits $n \geqslant 1$ ternary terms of equation **(JT)** (iff $\mathbb{V}$ is congruence distributive)

[Z. Janelidze, Closedness properties and internal relations I-V (2006-2008)]

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- $\mathbb{C}$ regular cat is called a Jónsson category of order n when it satisfies the matrix condition J_n; $\mathbb{C}$ is called a Jónsson category when it is a Jónsson category of some order n

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(some Jónsson terms are like majority terms)

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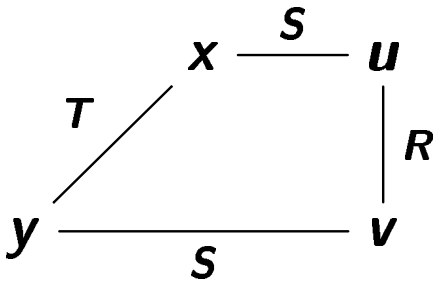

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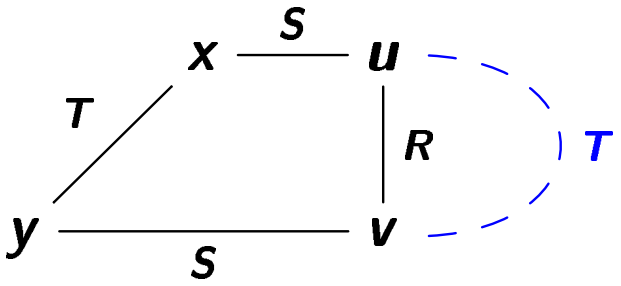

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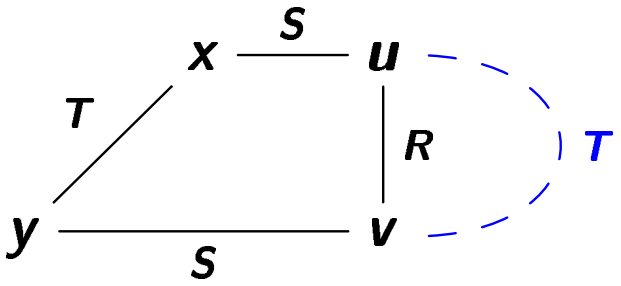

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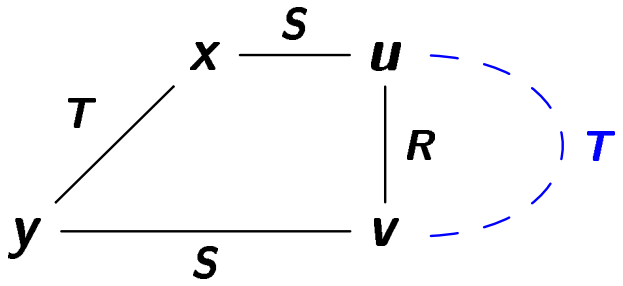

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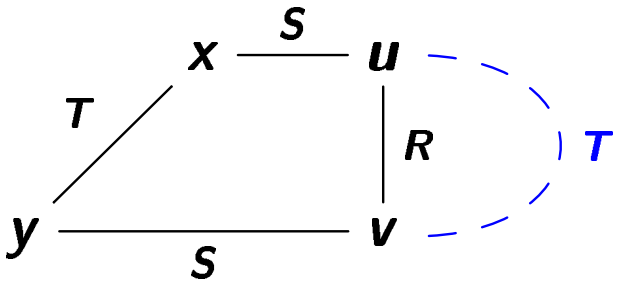

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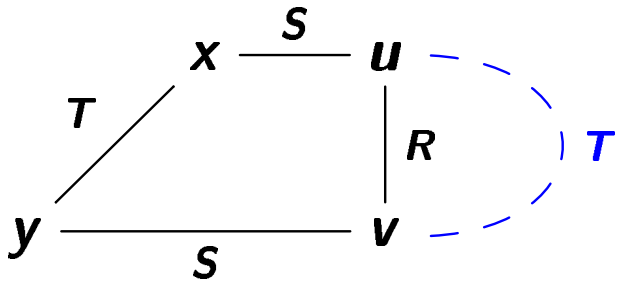

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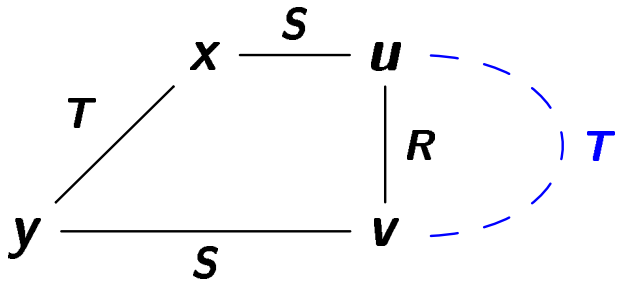

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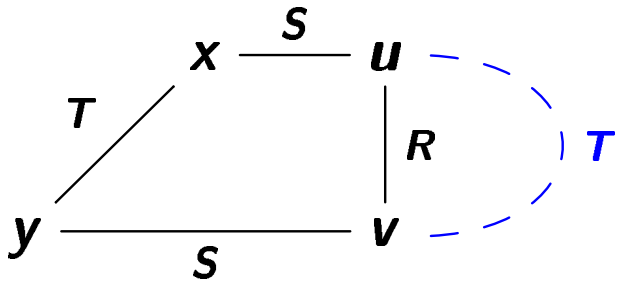

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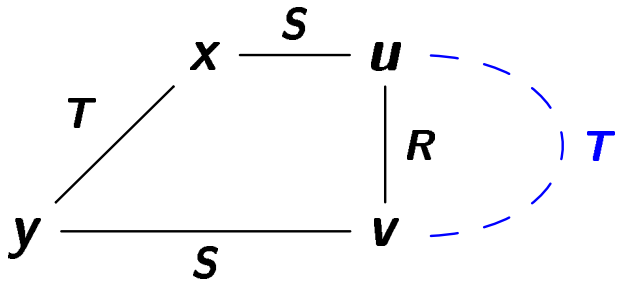

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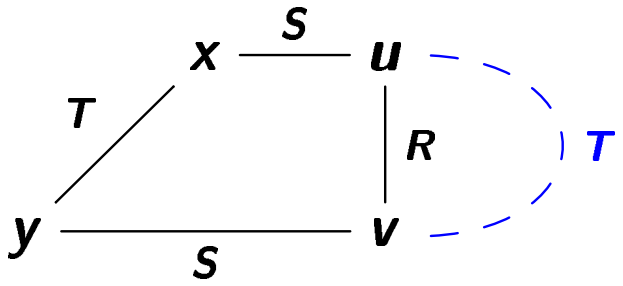

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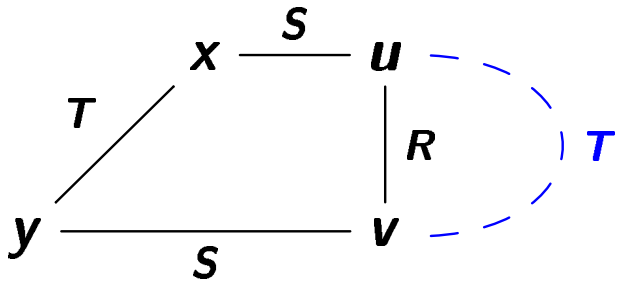

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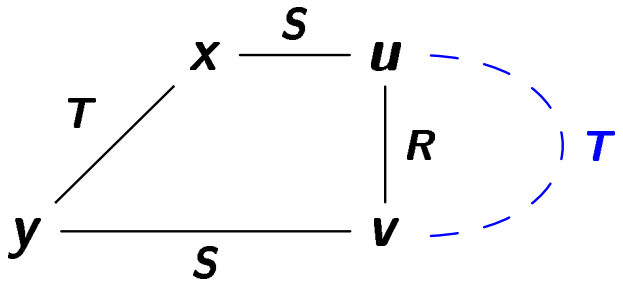

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Examples of Jónsson categories

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variety of distributive lattices with an additional operation of countable arity [GJ, A history ... (2016)]	×	×	×

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quasivarieties of $\mathbb{V}$ or constructions on $\mathbb{C}$ ⋆	×		

⋆ certain "constructions" wrt Jónsson categories $\mathbb{C}$ of order $n = 1$

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variety of distributive lattices with an additional operation of countable arity [GJ, A history ... (2016)]	×	×	×
quasivarieties of $\mathbb{V}$ or constructions on $\mathbb{C}$ ⋆	×	depends on $\mathbb{V}/\mathbb{C}$	depends on $\mathbb{V}/\mathbb{C}$

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variety of implication algebras	✓	✓	✓

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variety of implication algebras	✓	✓	✓
variety w/ $3\frac{1}{2}$-near-unanimity term	✓	?	✓

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quasivariety of $\mathbb{V}$	×	depends on $\mathbb{V}$	depends on $\mathbb{V}$

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variety w/ $3\frac{1}{2}$-near-unanimity term	✓	?	✓
quasivariety of $\mathbb{V}$	×	depends on $\mathbb{V}$	depends on $\mathbb{V}$
constructions on $\mathbb{C}$	×	depends on $\mathbb{C}$	depends on $\mathbb{C}$

Examples of Jónsson categories

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- order $n = 3$

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not of order 2	variety	distributive

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not of order 2	variety	distributive
variety w/ 4-near-unanimity term	✓	✓

Examples of Jónsson categories

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not of order 2	variety	distributive
variety w/ 4-near-unanimity term	✓	✓
quasivariety of $\mathbb{V}$	×	depends on $\mathbb{V}$

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variety w/ 4-near-unanimity term	✓	✓
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not of order 2	variety	distributive
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- $\mathbb{V}$ with a $k\frac{1}{2}$-near-unanimity term, $k \geqslant 3$

  $\rightsquigarrow$ Jónsson category of order $n = 2k - 4$

Examples of Jónsson categories

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not of order 2	variety	distributive
variety w/ 4-near-unanimity term	✓	✓
quasivariety of $\mathbb{V}$	×	depends on $\mathbb{V}$
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