NEW METHODS TO CONSTRUCT MODEL CATEGORIES

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MODEL CATEGORIES — MOTIVATION

- ▷ Nice framework to do homotopy theory
- ▷ Study objects up to a good notion of equivalences
- ▷ Approximate object with better behaved ones
- ▷ Compute derived functors, homotopy (co)limits, ...
- ▷ Define homotopy coherent structures (e.g. ∞-categories)

Examples:

- ▷ Categories up to equivalences
- ▷ 2-Categories up to biequivalences
- ▷ Spaces up to weak homotopy equivalences
- ▷ Chain complexes up to quasi-isomorphisms

MODEL CATEGORIES – DEFINITION

Let $\mathcal{M}$ be a complete and cocomplete category.

A weak factorization system (wfs) is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms in $\mathcal{M}$ closed under retracts such that:

(i)
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\mathcal{L}\ni"'] & \cdot \arrow[d, "\in\mathcal{R}"] \\ \cdot \arrow[r] \arrow[ur, dashed, "\exists"] & \cdot \end{tikzcd}$$

(ii)
$$\begin{tikzcd} \cdot \arrow[rr, "\forall f\in\mathcal{M}"] \arrow[dr, "\mathcal{L}\ni"'] & & \cdot \\ & \cdot \arrow[ur, "\in\mathcal{R}"'] & \end{tikzcd}$$

$\rightsquigarrow$ $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$

A model structure on $\mathcal{M}$ is a triple $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ of classes of morphisms in $\mathcal{M}$ with

- ▷ $\mathcal{W}$: weak equivalences $\xrightarrow{\sim}$
- ▷ $\mathcal{C}$: cofibrations $\hookrightarrow$
- ▷ $\mathcal{F}$: fibrations $\twoheadrightarrow$
- ▷ $\mathcal{C}\cap\mathcal{W}$: trivial cofibrations
- ▷ $\mathcal{F}\cap\mathcal{W}$: trivial fibrations

such that:

(i) $\mathcal{W}$ is closed under 2-of-3

(ii) $(\mathcal{C}\cap\mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F}\cap\mathcal{W})$ form wfs.

MODEL CATEGORIES – SOME FACTS

▷ The classes $(\mathcal{W},\mathcal{C})$, $(\mathcal{W},\mathcal{F})$, $(\mathcal{W},\mathcal{C}\cap\mathcal{W})$ or $(\mathcal{W},\mathcal{F}\cap\mathcal{W})$ determine the model structure

$\rightsquigarrow$ hard to construct

CONSTRUCTING MS ALONG ADJUNCTIONS

Let $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ be a model category and

$\mathcal{N} \underset{R}{\overset{L}{\underset{\perp}{\leftrightarrows}}} \mathcal{M}$ be an adjunction

The right-induced model structure $(\overline{\mathcal{W}}, \overline{\mathcal{C}}, \overline{\mathcal{F}})$ on $\mathcal{N}$ has if it exists

$$\overline{\mathcal{W}} = R^{-1}(\mathcal{W}) \quad \text{and} \quad \overline{\mathcal{F}} = R^{-1}(\mathcal{F})$$

$$\rightsquigarrow \quad \overline{\mathcal{F}} \cap \overline{\mathcal{W}} = R^{-1}(\mathcal{F} \cap \mathcal{W})$$

▷ There are results giving conditions under which the right-induced model structure exists

<u>Non-example</u> [Fiore–Paoli–Pronk]

$\mathrm{DblCat} := \mathrm{Cat}(\mathrm{Cat}) \underset{\perp}{\leftrightarrows} \mathrm{Cat}^{\Delta^{op}}$ ← Reedy model structure

internal categories to Cat: $X : \Delta^{op} \longrightarrow \mathrm{Cat}$ st $X_n \cong X_1 \times_{X_0} \dots \times_{X_0} X_1$

The right-induced MS on DblCat does not exist

MODEL CATEGORIES – EXAMPLES

▷ There is a model structure on $\mathcal{C}at$ with
   – $\mathcal{W}$: equivalences of categories
   – $\mathcal{F}\cap\mathcal{W}$: surjective on objects and fully faithful on morphisms functors

▷ [Lack] There is a model structure on $2\mathcal{C}at$ with
   – $\mathcal{W}$: biequivalences of 2-categories
   – $\mathcal{F}\cap\mathcal{W}$: surjective on objects, full on morphisms, fully faithful on 2-morphisms 2-functors

MODEL CATEGORIES – SOME FACTS

▷ The classes $(\mathcal{W},\mathcal{C})$, $(\mathcal{W},\mathcal{F})$, $(\mathcal{W},\mathcal{C}\cap\mathcal{W})$ or $(\mathcal{W},\mathcal{F}\cap\mathcal{W})$ determine the model structure

⇝ hard to construct

terminal object

An object $X\in\mathcal{M}$ is **fibrant** if $X \overset{!}{\twoheadrightarrow} * \in \mathcal{F}$

⇝ the fibrant objects are often the well-behaved ones

▷ [Joyal] The class $\mathcal{C}$ (or $\mathcal{F}\cap\mathcal{W}$) together with the fibrant objects determine the model structure

▷ $\forall X\in\mathcal{M}$, $\exists$

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, hook, "\sim", "\mathcal{C}\cap\mathcal{W} = {}^{\boxslash}\mathcal{F} \ni"'] & & * \\ & \hat{X} \arrow[ur, two heads, "\in \mathcal{F}"'] & \end{tikzcd}$$

fibrant

⇝ every object is equivalent to a fibrant one

⇝ what happens between fibrant objects determines the homotopy theory.

CONSTRUCTING MS ALONG ADJUNCTIONS

Let $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ and $\mathcal{N} \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathcal{M}$ as before

The fibrantly-induced model structure $(\overline{\mathcal{W}}, \overline{\mathcal{C}}, \overline{\mathcal{F}})$ on $\mathcal{N}$ has if it exists

$$\overline{\mathcal{F}} \cap \overline{\mathcal{W}} = R^{-1}(\mathcal{F} \cap \mathcal{W}) \; ; \; \text{fibrant} = R^{-1}(\text{fibrant})$$
$$\overline{\mathcal{W}} \text{ btw fibrant} = R^{-1}(\mathcal{W}) \text{ btw fibrant}$$

Thm [GMSV] The fibrantly-induced MS on $\mathcal{N}$ exists if

(i) $\forall X \xrightarrow{f} Y \in R^{-1}(\mathcal{F} \cap \mathcal{W})$, $\exists$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "{{}^{\boxslash}R^{-1}(\mathcal{F}) \ni}"'] & Y \arrow[d, "{\in {}^{\boxslash}R^{-1}(\mathcal{F})}"] \\ R^{-1}(\text{fibrant}) \ni \hat{X} \arrow[r, "\hat{f}"] & \hat{Y} \in R^{-1}(\text{fibrant}) \end{tikzcd}$$

with $\hat{f} \in R^{-1}(\mathcal{W})$

(ii) $\forall X \in R^{-1}(\text{fibrant})$, $\exists$

$$\begin{tikzcd} X \arrow[rr, "\text{diag}"] \arrow[dr, "R^{-1}(\mathcal{W}) \ni"'] & & X \times X \\ & PX \arrow[ur, "\in R^{-1}(\mathcal{F})"'] & \end{tikzcd}$$

Path object

trivial fibrations are w.e.

Example: [GMSV] $\mathrm{DblCat} \underset{\perp}{\rightleftarrows} \mathrm{Cat}^{\Delta^{op}}$ ← Reedy MS

The fibrantly-induced MS on DblCat exists

CONSTRUCTING MS – GENERAL VERSION

<u>Thm</u> [GMSV] Let $\mathcal{M}$ be a category and $(\overline{\mathcal{W}}, \mathcal{C}, \mathcal{NF})$ be classes of morphisms in $\mathcal{M}$

w.e. btw fibrant

naive fibrations

There is a model structure $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ on $\mathcal{M}$ with

▷ $\mathcal{C} = \mathcal{C}$ $\rightsquigarrow$ $\mathcal{F} \cap \mathcal{W} = \mathcal{C}^{\boxslash}$

naive fibrant

▷ fibrant $= \{ X \in \mathcal{M} \mid X \longrightarrow * \in \mathcal{NF} \}$

▷ $\mathcal{W}$ btw fibrant $= \overline{\mathcal{W}}$ btw fibrant

if the following conditions hold:

(i) $\forall X \xrightarrow{f} Y \in \mathcal{C}^{\boxslash}$, $\exists$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "{}^{\boxslash}\mathcal{NF} \ni"'] & Y \arrow[d, "\in {}^{\boxslash}\mathcal{NF}"] \\ \text{naive fibrant} \ni \hat{X} \arrow[r, "\hat{f}"] & \hat{Y} \in \text{naive fibrant} \end{tikzcd}$$

with $\hat{f} \in \overline{\mathcal{W}}$

trivial fibrations are w.e.

(ii) $\forall X \in$ naive fibrant, $\exists$

$$\begin{tikzcd} X \arrow[rr, "\mathrm{diag}"] \arrow[dr, "\overline{\mathcal{W}} \ni"'] & & X \times X \\ & PX \arrow[ur, "\in \mathcal{NF}"'] & \end{tikzcd}$$

Path object

(iii) $\forall X \xrightarrow{f} Y \in \overline{\mathcal{W}} \cap \mathcal{NF}$ with X, Y naive fibrant, $f \in \mathcal{C}^{\boxslash}$

fib + w.e. btw fibrant = trivial fib

MORE EXAMPLES

▷ [MSV] There is a MS on DblCat with

- fibrant: weakly horizontally invertible double cat's
  all horizontal equivalences have companions
- W btw fibrant: double (horizontal) biequivalences
- FNW: surj, full, full, fully faithful double functors

Lack MS

The adjunction $2\mathrm{Cat} \underset{\mathbb{H}}{\overset{\perp}{\leftrightarrows}} \mathrm{DblCat}$ is a Quillen pair

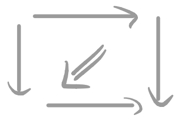
compatible with the homotopy theory

▷ [GMSV] There is a MS on DblCat with

- fibrant: transposable double cat's
  all horizontal & vertical mor have companions
- W btw fibrant & FNW as before

Lack MS

The adjunction $2\mathrm{Cat} \underset{\mathbb{S}\mathrm{q}}{\overset{\perp}{\leftrightarrows}} \mathrm{DblCat}$ is a Quillen equivalence



same homotopy theory