Non-pointed abelian categories

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Category Theory 2023

Joint work with Arnaud Duvieusart and Sandra Mantovani

Abelian and modular categories

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Definition (Carboni)

A finitely complete category $\mathcal{C}$ with coproducts is modular if it satisfies the following conditions:

1. *the category of pointed objects* $\mathrm{Pt}_1(\mathcal{C}) = 1\backslash\mathcal{C}$ *is additive with kernels;*
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A pointed category $\mathcal{C}$ is modular if and only if it is additive.

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Definition

$\mathcal{C}$ with pullbacks is

- *naturally Mal'tsev if all fibers $\mathrm{Pt}_B(\mathcal{C})$ are additive;*
- *essentially affine if all change-of-base functors of the fibration of points are equivalences.*

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Proposition

Let $\mathcal{C}$ be a regular category satisfying condition (P), and let

$$\begin{array}{ccc} X & \xrightarrow{m} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle u} \\ Z & \xrightarrow[v]{} & T \end{array}$$

be a pushout square, where p is a regular epimorphism and m is a monomorphism. Then v is a monomorphism.

Normal monomorphisms

Definition (Bourn)

A morphism $m \colon M \to X$ is normal to an equivalence relation R on X if m induces a discrete fibration between equivalence relations

$$\begin{array}{ccc} M \times M & \xrightarrow{\tilde{m}} & R \\ \pi_1 \downarrow\downarrow \pi_2 & & r_1 \downarrow\downarrow r_2 \\ M & \xrightarrow[m]{} & X, \end{array}$$

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Proposition

If $\mathcal{C}$ is regular and satisfies condition (P), then every monomorphism $m\colon M \to X$ is Bourn-normal to some effective equivalence relation.

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$$\begin{array}{ccc} M & \xrightarrow{m} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ Z & \xrightarrow[m']{} & T \end{array}$$

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$$\begin{array}{ccc} M & \xrightarrow{m} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ Z & \xrightarrow[m']{} & T \end{array}$$

is a pullback. Taking kernel pairs of p and p' then gives a discrete fibration

$$\begin{array}{ccc} M \times M & \xrightarrow{\tilde{m}} & Eq(p') \\ {\scriptstyle \pi_1}\downarrow\downarrow{\scriptstyle \pi_2} & & {\scriptstyle p'_1}\downarrow\downarrow{\scriptstyle p'_2} \\ M & \xrightarrow[m]{} & X. \end{array}$$

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Involving Barr-exactness

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Proof.

If R is an equivalence relation on an object X and $M \leq X$ is a subobject which is Bourn-normal with respect to R, M is Bourn-normal with respect to an effective equivalence relation R'.

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If R is an equivalence relation on an object X and $M \leq X$ is a subobject which is Bourn-normal with respect to R, M is Bourn-normal with respect to an effective equivalence relation R'. Since $\mathcal{C}$ is protomodular, this implies that $R \cong R'$, so that R is effective. $\square$

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Theorem

A category $\mathcal{C}$ is abelian if and only if it is homological and satisfies condition (P).

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Proof.

Any homological category satisfying condition (P) is pointed and naturally Mal'tsev, thus additive, and exact; hence it is abelian by Tierney's equation. □

(P) and penessentially affine categories

A functor $G\colon \mathcal{C} \to \mathcal{D}$ creates subobjects if, for every monomorphism $n\colon N \to G(X)$, there exist a monomorphism $m\colon M \to X$ and an isomorphism $\varphi\colon N \to G(M)$ such that $n = G(m)\varphi$.

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Theorem

If $\mathcal{C}$ is Barr-exact, then $\mathcal{C}$ is penessentially affine if and only if it is protomodular and satisfies condition (P).

Examples

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The category $\mathrm{AbExt}_B(\mathrm{Grp})$ of abelian extensions over a fixed group B (i.e. the full subcategory of Grp/B containing the surjective morphisms with abelian kernels) is essentially affine, and thus satisfies condition (P).

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The category $\mathsf{AbExt}_B(\mathsf{Grp})$ of abelian extensions over a fixed group B (i.e. the full subcategory of Grp/B containing the surjective morphisms with abelian kernels) is essentially affine, and thus satisfies condition (P). The same is true for $\mathsf{AbExt}_B(\mathcal{C})$ if $\mathcal{C}$ is a semi-abelian category satisfying a weak form of the axiom of *normality of unions*.

Definition (Borceux, Janelidze, Kelly)

A semi-abelian category $\mathcal{C}$ satisfies the axiom of normality of unions if, whenever a subobject N of X is normal in two subobjects A, B of X, N is normal in the join $A \vee B$.

Strongly semi-abelian categories

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If $\mathcal{C}$ satisfies the axiom of the normality of unions, it is strongly semi-abelian. The converse is false: non-associative rings.

Theorem

A semi-abelian category is strongly semi-abelian if and only if, whenever a subobject N of X is normal in two subobjects A, B with A normal in X, N is normal in the join $A \vee B$.