GALOIS STRUCTURES IN PREORDERED GROUPS

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Preordered group

Definition

A **preordered group** $(G, \leqslant)$ is a group $(G, +, 0)$ endowed with a preorder relation $\leqslant$ on G which is compatible with $+$:

$$a \leqslant c \text{ and } b \leqslant d \;\Rightarrow\; a + b \leqslant c + d \qquad \text{for } a, b, c, d \in G.$$

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Example

The group $\mathbb{Z}$ of integers with the usual order $\leqslant$: $(\mathbb{Z}, \leqslant)$
*$(\mathbb{Z}, \leqslant)$ is a **partially ordered abelian group**.*

Morphism of preordered groups

Definition

A **morphism of preordered groups** $f\colon (G, \leqslant_G) \to (H, \leqslant_H)$ is a group morphism $f\colon G \to H$ which preserves the preorder:

$$a \leqslant_G b \;\Rightarrow\; f(a) \leqslant_H f(b).$$

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All preordered groups and morphisms between them define a category denoted by **PreOrdGrp**.

Alternative definition of PreOrdGrp

Proposition

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- ***arrows***: $f : (G, P_G) \to (H, P_H)$ *such that* $f(P_G) \subseteq P_H$:

$$\begin{tikzcd} P_G \arrow[r, "\bar{f}"] \arrow[d, tail] & P_H \arrow[d, tail] \\ G \arrow[r, "f"'] & H \end{tikzcd}$$

Some properties

Proposition [M. M. Clementino, N. Martins-Ferreira, A. Montoli (2019)]

PreOrdGrp *is complete, cocomplete and a* ***normal*** *category in the sense of Z. Janelidze (2010), that is,*

- *it is pointed;*
- *it is regular;*
- *any regular epimorphism is normal (i.e. a cokernel).*

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Regular epimorphisms in PreOrdGrp:

$$\begin{tikzcd} P_G \arrow[r, two heads, "\bar{f}"] \arrow[d, tail] & P_H \arrow[d, tail] \\ G \arrow[r, two heads, "f"'] & H \end{tikzcd}$$

Some properties

Proposition

PreOrdGrp *is neither protomodular nor subtractive.*

The full subcategory of partially ordered groups

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Objects of ParOrdGrp
= preordered groups $(G, \leqslant)$ such that the preorder $\leqslant$ is **antisymmetric**
= preordered groups (G, P_G) such that the positive cone P_G is a **reduced** monoid

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$$N_G = \{x \in G \mid x \in P_G \text{ and } -x \in P_G\}$$

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$$\begin{tikzcd} P_G \arrow[d, tail] & \\ G \arrow[r, two heads, "\eta_G"'] & G/N_G \end{tikzcd}$$

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$$\begin{tikzcd} P_G \arrow[r, dotted, two heads, "\bar{\eta}_G"] \arrow[d, tail] & \eta_G(P_G) \arrow[d, dotted, tail] \\ G \arrow[r, two heads, "\eta_G"'] & G/N_G \end{tikzcd}$$

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$$I(G, P_G) = (G/N_G, \eta_G(P_G))$$

An admissible Galois structure Γ_{abs}.

This gives rise to a reflection:

$$\begin{tikzcd} \mathsf{PreOrdGrp} \arrow[r, bend left, "I"] \arrow[r, phantom, "\perp"] & \mathsf{ParOrdGrp} \arrow[l, bend left, hook, "H"] \end{tikzcd}$$

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Proposition

$\Gamma_{abs.} = (\mathsf{PreOrdGrp}, \mathsf{ParOrdGrp}, I, H, \mathcal{E}_{abs.}, \mathcal{Z}_{abs.})$ *is a Galois structure, where*

- $\mathcal{E}_{abs.}$ *is the class of* ***all morphisms*** *in* PreOrdGrp*;*
- $\mathcal{Z}_{abs.}$ *is the class of* ***all morphisms*** *in* ParOrdGrp*.*

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Proposition

$\Gamma_{abs.}$ *is admissible.*

Admissibility of a Galois structure

Reminder [G. Janelidze (1990)]

A Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathcal{E}, \mathcal{Z})$ *is* ***admissible*** *when* F *preserves all pullbacks of the form*

$$\begin{tikzcd} B \times_{UF(B)} U(X) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & U(X) \arrow[d, "U(\phi)"] \\ B \arrow[r, "\eta_B"'] & UF(B) \end{tikzcd}$$

where $\phi \in \mathcal{Z}$.

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where $\phi \in \mathcal{Z}$.

Reminder

If $\mathscr{C}$ and $\mathscr{F}$ admit all pullbacks, $F \colon \mathscr{C} \to \mathscr{F}$ is a reflector, and $\mathcal{E}$ and $\mathcal{Z}$ are the classes of all morphisms, then

$$\Gamma \text{ is admissible} \quad \Leftrightarrow \quad F \text{ is } \textbf{semi-left-exact}.$$

Trivial extensions

Reminder

*An arrow $f : A \to B$ in $\mathcal{E}$ is a **(Γ-)trivial extension** when the square*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & UF(A) \arrow[d, "UF(f)"] \\ B \arrow[r, "\eta_B"'] & UF(B) \end{tikzcd}$$

is a pullback.

Central extensions

Reminder

An arrow $f\colon A \to B$ in $\mathcal{E}$ is a ***(Γ-)central extension*** *when there exists an effective descent morphism $p\colon E \to B$ in $\mathcal{E}$ such that $p^*(f)$ is a (Γ-)trivial extension,*

Central extensions

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An arrow $f\colon A \to B$ in $\mathcal{E}$ is a **(Γ-)central extension** *when there exists an effective descent morphism $p\colon E \to B$ in $\mathcal{E}$ such that $p^*(f)$ is a (Γ-)trivial extension, that is, the left-hand square below is a pullback, where $\pi_1 = p^*(f)$ is the pullback of f along p:*

$$\begin{tikzcd}
UF(E \times_B A) \arrow[d, "UF(\pi_1)"'] & E \times_B A \arrow[l, "\eta_{E \times_B A}"'] \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "f"] \\
UF(E) & E \arrow[l, "\eta_E"] \arrow[r, "p"'] & B.
\end{tikzcd}$$

Normal extensions

Reminder

An arrow $f\colon A \to B$ in $\mathcal{E}$ is a ***(Γ-)normal extension*** *when f is an effective descent morphism and $f^*(f)$ is a (Γ-)trivial extension.*

Characterization of $\Gamma_{abs.}$-trivial and $\Gamma_{abs.}$-central extensions

Theorem [M. Gran, A. Michel (2021)]

Let $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ *be a morphism in* PreOrdGrp*. Then:*

- $(f, \bar{f})$ *is a* $\Gamma_{abs.}$*-**trivial extension** if and only if the restriction* $\phi\colon N_G \to N_H$ *of* f *to* $N_G = \{x \in G \mid x \in P_G$ *and* $-x \in P_G\}$ *is a group isomorphism.*

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- $(f, \bar{f})$ *is a* $\Gamma_{abs.}$*-**central extension** if and only if its kernel* $\mathrm{Ker}(f, \bar{f})$ *lies in* ParOrdGrp*.*

From preordered groups to V-groups

Remark

All these results can be generalized to **V-groups** *(for V a commutative, unital and integral quantale).*

→ See the following article for more details:
A. Michel, *Torsion theories and coverings of V-groups*, Appl. Categ. Struct. 30 (2022), 659-684.

Goal

Generalize the Galois structure Γ_{ab} to the context of **preordered groups**:

	Grp	PreOrdGrp
"Suitable" adjunction	$\mathrm{Grp} \underset{u}{\overset{ab}{\rightleftarrows}} \mathrm{Ab}$, $\perp$	$\mathrm{PreOrdGrp} \rightleftarrows ?$, $\perp$
Admissible Galois structure	Γ_{ab}	$\Gamma = ?$
Characterization of central and normal extensions	Central = normal = reg. epis $f\colon A \twoheadrightarrow B$ s.t. $\mathrm{Ker}(f) \subseteq Z(A)$	Central $\overset{?}{=}$ Normal Description of both classes ?

Commutative objects

Consider the full subcategory of **commutative objects** in PreOrdGrp.

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Consider the full subcategory of **commutative objects** in PreOrdGrp.

It is the full subcategory PreOrdAb of **preordered abelian groups**, i.e. of preordered groups (G, P_G) such that $G \in \mathrm{Ab}$.

The functor to the subcategory of commutative objects

Consider then the functor $C\colon \mathsf{PreOrdGrp} \to \mathsf{PreOrdAb}$:

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Consider then the functor $C\colon \mathsf{PreOrdGrp} \to \mathsf{PreOrdAb}$:

$$\begin{tikzcd} P_G \arrow[d, tail] \\ G \end{tikzcd} \qquad\qquad C(G, P_G) = \,?$$

The functor to the subcategory of commutative objects

Consider then the functor $C\colon \mathsf{PreOrdGrp} \to \mathsf{PreOrdAb}$:

$$\begin{tikzcd} P_G \arrow[d, tail] & \\ G \arrow[r, two heads, "\eta_G"'] & ab(G) = G/[G,G] \end{tikzcd}$$

$$C(G, P_G) = \,?$$

The functor to the subcategory of commutative objects

Consider then the functor $C\colon \mathsf{PreOrdGrp} \to \mathsf{PreOrdAb}$:

$$\begin{tikzcd} P_G \arrow[r, dotted, two heads, "\bar{\eta}_G"] \arrow[d, tail] & \eta_G(P_G) \arrow[d, dotted, tail] \\ G \arrow[r, two heads, "\eta_G"'] & ab(G) = G/[G,G] \end{tikzcd}$$

$$C(G, P_G) = ?$$

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$$C(G, P_G) = (ab(G), \eta_G(P_G))$$

Galois structure Γ_C

This gives rise to an adjunction:

$$\begin{tikzcd} \mathsf{PreOrdGrp} \arrow[r, bend left, "C"] \arrow[r, phantom, "\perp"] & \mathsf{PreOrdAb}. \arrow[l, bend left, hook, "V"] \end{tikzcd}$$

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$$\begin{tikzcd} \mathrm{PreOrdGrp} \arrow[r, bend left, "C"] \arrow[r, phantom, "\perp"] & \mathrm{PreOrdAb}. \arrow[l, bend left, hook, "V"] \end{tikzcd}$$

Proposition

$\Gamma_C = (\mathrm{PreOrdGrp}, \mathrm{PreOrdAb}, C, V, \mathcal{E}_C, \mathcal{Z}_C)$ *is an admissible Galois structure, where*

- $\mathcal{E}_C$ *is the class of* ***regular epimorphisms*** *in* PreOrdGrp;
- $\mathcal{Z}_C$ *is the class of* ***regular epimorphisms*** *in* PreOrdAb.

Abelian objects

Consider the full subcategory of **abelian objects** in PreOrdGrp.

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Abelian objects

Consider the full subcategory of **abelian objects** in PreOrdGrp.

It is the full subcategory of preordered groups (G, P_G) such that $G \in \mathsf{Ab}$ and $P_G \in \mathsf{Ab}$, i.e. it is the category $\mathbf{Mono}(\mathbf{Ab})$ of monomorphisms in the category of abelian groups.

The functor to the subcategory of abelian objects

Consider the composite

$$\begin{tikzcd} \mathsf{PreOrdGrp} \arrow[r, "C"] \arrow[rr, bend left, "F"] & \mathsf{PreOrdAb} \arrow[r, "A"] & \mathsf{Mono}(\mathsf{Ab}) \end{tikzcd}$$

The functor to the subcategory of abelian objects

Consider the composite

$$\begin{tikzcd} \mathrm{PreOrdGrp} \arrow[r, "C"] \arrow[rr, bend left, "F"] & \mathrm{PreOrdAb} \arrow[r, "A"] & \mathrm{Mono(Ab)} \end{tikzcd}$$

with $A(G, P_G) = (G, grp(P_G))$ for any $(G, P_G) \in \mathrm{PreOrdAb}$, where $grp(P_G)$ is the **group completion** of P_G.

The functor to the subcategory of abelian objects

$$\Rightarrow F(G, P_G) = (ab(G), grp(\eta_G(P_G))) \qquad \text{for any } (G, P_G) \in \mathsf{PreOrdGrp}$$

$$\begin{tikzcd}
P_G \arrow[r, two heads, "\bar{\eta}_G"] \arrow[d, tail] \arrow[rr, bend left, "\hat{\eta}_G"] & \eta_G(P_G) \arrow[r, "j_G"] \arrow[d, tail] & grp(\eta_G(P_G)) \arrow[d, tail] \\
G \arrow[r, two heads, "\eta_G"'] \arrow[rr, two heads, bend right, "\eta_G"'] & ab(G) \arrow[r, equal] & ab(G)
\end{tikzcd}$$

Galois structure Γ

This gives rise to an adjunction: $F \dashv U$.

$$
\begin{tikzcd}
\mathrm{PreOrdGrp} \arrow[r, shift left, "C"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left, "F"] & \mathrm{PreOrdAb} \arrow[l, hook, shift left, "V"] \arrow[r, shift left, "A"] \arrow[r, phantom, "\perp"] & \mathrm{Mono(Ab)} \arrow[l, hook, shift left, "W"] \arrow[ll, hook, bend left, "U"]
\end{tikzcd}
$$

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This gives rise to an adjunction: $F \dashv U$.

$$\begin{tikzcd}
\mathrm{PreOrdGrp} \arrow[r, bend left, "C"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left=40, "F"] & \mathrm{PreOrdAb} \arrow[l, bend left, "V"] \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \mathrm{Mono(Ab)} \arrow[l, bend left, "W"] \arrow[ll, bend left=40, "U"]
\end{tikzcd}$$

The adjunction $A \dashv W$ is induced by the adjunction

$$\begin{tikzcd}
\mathrm{Mon} \arrow[r, bend left, "grp"] \arrow[r, phantom, "\perp"] & \mathrm{Grp} \arrow[l, bend left, "fgt"]
\end{tikzcd}$$

(studied in the paper "*A Galois theory for monoids*" by A. Montoli, D. Rodelo and T. Van der Linden (2014)).

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\mathsf{PreOrdGrp} \arrow[r, bend left, "C"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left=40, "F"] & \mathsf{PreOrdAb} \arrow[l, bend left, "V"] \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \mathsf{Mono}(\mathsf{Ab}) \arrow[l, bend left, "W"] \arrow[ll, bend left=40, "U"]
\end{tikzcd}
$$

Proposition

$\Gamma = (\mathsf{PreOrdGrp}, \mathsf{Mono}(\mathsf{Ab}), F, U, \mathcal{E}, \mathcal{Z})$ is a Galois structure, where

- *$\mathcal{E}$ is the class of* ***regular epimorphisms*** *in* PreOrdGrp*;*
- *$\mathcal{Z}$ is the class of* ***regular epimorphisms*** *in* Mono(Ab)*.*

Characterization of Γ-central and Γ-normal extensions

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Γ *is admissible.*

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Theorem [M. Gran, A. Michel (2023)]

Let $(f, \bar{f})\colon (G, P_G) \twoheadrightarrow (H, P_H)$ *be a regular epimorphism in* PreOrdGrp*. Then, the following conditions are equivalent:*

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Γ is admissible.

Theorem [M. Gran, A. Michel (2023)]

Let $(f, \bar{f})\colon (G, P_G) \twoheadrightarrow (H, P_H)$ be a regular epimorphism in PreOrdGrp. *Then, the following conditions are equivalent:*

1. (i) $\mathrm{Ker}(f) \subseteq Z(G)$;
   (ii) *for any* $(x, y) \in \mathrm{Eq}(\bar{f})$, $y - x \in P_G$
   ($\bar{f}$ is a ***special Schreier surjection*** *in* Mon*).*
2. *$(f, \bar{f})$ is a Γ-normal extension.*
3. *$(f, \bar{f})$ is a Γ-central extension.*

Conclusion

	Grp	PreOrdGrp
"Suitable" adjunction	$\mathsf{Grp} \underset{u}{\overset{ab}{\rightleftarrows}} \mathsf{Ab}$	$\mathsf{PreOrdGrp} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{Mono(Ab)}$
Admissible Galois structure	Γ_{ab}	Γ (extensions = reg. epis)
Central = normal	Reg. epis $f \colon A \twoheadrightarrow B$ s.t. $\mathrm{Ker}(f) \subseteq Z(A)$ (f = alg. central ext.)	Reg. epis $(f, \bar{f}) \colon (G, P_G) \twoheadrightarrow (H, P_H)$ s.t. f is an algebraically central ext. and $\bar{f}$ is a special Schreier surjection

References

- M.M. Clementino, N. Martins-Ferreira, and A. Montoli, *On the categorical behaviour of preordered groups*, J. Pure Appl. Algebra 223 (2019), 4226-4245.
- M.M. Clementino, and A. Montoli, *On the categorical behaviour of V-groups*, J. Pure Appl. Algebra 225 (2021), 106550.
- M. Gran, and A. Michel, *Torsion theories and coverings of preordered groups*, Algebra Univers. 82 (22) (2021).
- M. Gran, and A. Michel, *Central extensions of preordered groups*, Preprint (2023).
- G. Janelidze, *Pure Galois theory in categories*, J. Algebra 132, no. 2 (1990), 270-286.
- G. Janelidze, and G.M. Kelly, *Galois theory and a general notion of central extension*, J. Pure Appl. Algebra 97 (1994), 135-161.
- Z. Janelidze, *The pointed subobject functor, 3×3 lemmas, and subtractivity of spans*, Theory Appl. Categ. 23, no. 11 (2010), 221-242.
- A. Michel, *Torsion theories and coverings of V-groups*, Appl. Categ. Struct. 30 (2022), 659-684.
- A. Montoli, D. Rodelo, and T. Van der Linden, *A Galois theory for monoids*, Theory Appl. Categ. 29, no. 7 (2014), 198-214.