

GALOIS STRUCTURES IN PREORDERED GROUPS

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Joint work with Marino Gran



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- 1 The category PreOrdGrp of preordered groups
- 2 An absolute Galois structure in PreOrdGrp
- 3 Generalization of the Galois theory induced by the abelianization functor

Preordered group

Definition

A **preordered group** (G, \leq) is a group $(G, +, 0)$ endowed with a preorder relation \leq on G which is compatible with $+$:

$$a \leq c \text{ and } b \leq d \Rightarrow a + b \leq c + d \quad \text{for } a, b, c, d \in G.$$

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Example

The group \mathbb{Z} of integers with the usual order \leq : (\mathbb{Z}, \leq)
 (\mathbb{Z}, \leq) is a **partially ordered abelian group**.

Morphism of preordered groups

Definition

A **morphism of preordered groups** $f: (G, \leq_G) \rightarrow (H, \leq_H)$ is a group morphism $f: G \rightarrow H$ which preserves the preorder:

$$a \leq_G b \Rightarrow f(a) \leq_H f(b).$$

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All preordered groups and morphisms between them define a category denoted by **PreOrdGrp**.

Alternative definition of PreOrdGrp

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- **arrows:** $f: (G, P_G) \rightarrow (H, P_H)$ such that $f(P_G) \subseteq P_H$:

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- **arrows:** $f: (G, P_G) \rightarrow (H, P_H)$ such that $f(P_G) \subseteq P_H$:

$$\begin{array}{ccc} P_G & \xrightarrow{\bar{f}} & P_H \\ \downarrow & & \downarrow \\ G & \xrightarrow{f} & H \end{array}$$

Some properties

Proposition [M. M. Clementino, N. Martins-Ferreira, A. Montoli (2019)]

PreOrdGrp is complete, cocomplete and a **normal** category in the sense of Z. Janelidze (2010), that is,

- it is pointed;
- it is regular;
- any regular epimorphism is normal (i.e. a cokernel).

Some properties

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Regular epimorphisms in PreOrdGrp :

$$\begin{array}{ccc}
 P_G & \xrightarrow{\bar{f}} \twoheadrightarrow & P_H \\
 \downarrow & & \downarrow \\
 G & \xrightarrow{f} \twoheadrightarrow & H
 \end{array}$$

Some properties

Proposition

PreOrdGrp is neither protomodular nor subtractive.

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The full subcategory of partially ordered groups

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Objects of ParOrdGrp

= preordered groups (G, \leq) such that the preorder \leq is **antisymmetric**

= preordered groups (G, P_G) such that the positive cone P_G is a **reduced monoid**

A functor to ParOrdGrp

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$$I(G, P_G) = ?$$

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Consider then the functor $I: \text{PreOrdGrp} \rightarrow \text{ParOrdGrp}$:

$$N_G = \{x \in G \mid x \in P_G \text{ and } -x \in P_G\}$$

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$$\begin{array}{ccc}
 P_G & & \\
 \downarrow & & \\
 G & \xrightarrow{\eta_G} & G/N_G
 \end{array}$$

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$$\begin{array}{ccc}
 P_G & \overset{\bar{\eta}_G}{\dashrightarrow} & \eta_G(P_G) \\
 \downarrow & & \downarrow \\
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 \end{array}$$

$$I(G, P_G) = (G/N_G, \eta_G(P_G))$$

An admissible Galois structure Γ_{abs} .

This gives rise to a reflection:

$$\text{PreOrdGrp} \begin{array}{c} \xrightarrow{I} \\ \perp \\ \xleftarrow{H} \end{array} \text{ParOrdGrp}$$

An admissible Galois structure $\Gamma_{abs.}$

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Proposition

$\Gamma_{abs.} = (\text{PreOrdGrp}, \text{ParOrdGrp}, I, H, \mathcal{E}_{abs.}, \mathcal{Z}_{abs.})$ is a Galois structure, where

- $\mathcal{E}_{abs.}$ is the class of **all morphisms** in PreOrdGrp;
- $\mathcal{Z}_{abs.}$ is the class of **all morphisms** in ParOrdGrp.

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Proposition

$\Gamma_{abs.}$ is admissible.

Admissibility of a Galois structure

Reminder [G. Janelidze (1990)]

A Galois structure $\Gamma = (\mathcal{C}, \mathcal{F}, F, U, \mathcal{E}, \mathcal{Z})$ is **admissible** when F preserves all pullbacks of the form

$$\begin{array}{ccc}
 B \times_{UF(B)} U(X) & \xrightarrow{\pi_2} & U(X) \\
 \pi_1 \downarrow & & \downarrow U(\phi) \\
 B & \xrightarrow{\eta_B} & UF(B)
 \end{array}$$

where $\phi \in \mathcal{Z}$.

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where $\phi \in \mathcal{Z}$.

Reminder

If \mathcal{C} and \mathcal{F} admit all pullbacks, $F: \mathcal{C} \rightarrow \mathcal{F}$ is a reflector, and \mathcal{E} and \mathcal{Z} are the classes of all morphisms, then

Γ is admissible $\Leftrightarrow F$ is **semi-left-exact**.

Trivial extensions

Reminder

An arrow $f : A \rightarrow B$ in \mathcal{E} is a $(\Gamma\text{-})$ trivial extension when the square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & UF(A) \\ f \downarrow & & \downarrow UF(f) \\ B & \xrightarrow{\eta_B} & UF(B) \end{array}$$

is a pullback.

Central extensions

Reminder

An arrow $f: A \rightarrow B$ in \mathcal{E} is a $(\Gamma-)$ central extension when there exists an effective descent morphism $p: E \rightarrow B$ in \mathcal{E} such that $p^*(f)$ is a $(\Gamma-)$ trivial extension,

Central extensions

Reminder

An arrow $f: A \rightarrow B$ in \mathcal{E} is a (Γ) -central extension when there exists an effective descent morphism $p: E \rightarrow B$ in \mathcal{E} such that $p^*(f)$ is a (Γ) -trivial extension, that is, the left-hand square below is a pullback, where $\pi_1 = p^*(f)$ is the pullback of f along p :

$$\begin{array}{ccccc}
 UF(E \times_B A) & \xleftarrow{\eta_{E \times_B A}} & E \times_B A & \xrightarrow{\pi_2} & A \\
 UF(\pi_1) \downarrow & & \pi_1 \downarrow & & \downarrow f \\
 UF(E) & \xleftarrow{\eta_E} & E & \xrightarrow{p} & B.
 \end{array}$$

Normal extensions

Reminder

An arrow $f: A \rightarrow B$ in \mathcal{E} is a (Γ) -normal extension when f is an effective descent morphism and $f^(f)$ is a (Γ) -trivial extension.*

Characterization of $\Gamma_{abs.}$ -trivial and $\Gamma_{abs.}$ -central extensions

Theorem [M. Gran, A. Michel (2021)]

Let $(f, \bar{f}): (G, P_G) \rightarrow (H, P_H)$ be a morphism in PreOrdGrp.

Then:

- (f, \bar{f}) is a $\Gamma_{abs.}$ -**trivial extension** if and only if the restriction $\phi: N_G \rightarrow N_H$ of f to $N_G = \{x \in G \mid x \in P_G \text{ and } -x \in P_G\}$ is a group isomorphism.

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- (f, \bar{f}) is a $\Gamma_{abs.}$ -**central extension** if and only if its kernel $\text{Ker}(f, \bar{f})$ lies in ParOrdGrp.

From preordered groups to V -groups

Remark

All these results can be generalized to \mathbf{V} -groups (for V a commutative, unital and integral quantale).

→ See the following article for more details:

A. Michel, *Torsion theories and coverings of V -groups*, Appl. Categ. Struct. 30 (2022), 659-684.

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Goal

Generalize the Galois structure Γ_{ab} to the context of **preordered groups**:

	Grp	PreOrdGrp
“Suitable” adjunction	$\text{Grp} \begin{array}{c} \xrightarrow{ab} \\ \perp \\ \xleftarrow{u} \end{array} \text{Ab}$	$\text{PreOrdGrp} \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} ?$
Admissible Galois structure	Γ_{ab}	$\Gamma = ?$
Characterization of central and normal extensions	Central = normal = reg. epis $f: A \twoheadrightarrow B$ s.t. $\text{Ker}(f) \subseteq Z(A)$	Central $\stackrel{?}{=} \text{Normal}$ Description of both classes ?

Commutative objects

Consider the full subcategory of **commutative objects** in PreOrdGrp .

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Consider the full subcategory of **commutative objects** in PreOrdGrp .

It is the full subcategory PreOrdAb of **preordered abelian groups**,
i.e. of preordered groups (G, P_G) such that $G \in \text{Ab}$.

The functor to the subcategory of commutative objects

Consider then the functor $C: \text{PreOrdGrp} \rightarrow \text{PreOrdAb}$:

The functor to the subcategory of commutative objects

Consider then the functor $C: \text{PreOrdGrp} \rightarrow \text{PreOrdAb}$:

$$\begin{array}{c} P_G \\ \downarrow \\ G \end{array}$$

$$C(G, P_G) = ?$$

The functor to the subcategory of commutative objects

Consider then the functor $C : \text{PreOrdGrp} \rightarrow \text{PreOrdAb}$:

$$\begin{array}{ccc} P_G & & \\ \downarrow & & \\ G & \xrightarrow{\eta_G} & ab(G) = G/[G, G] \end{array} \quad C(G, P_G) = ?$$

The functor to the subcategory of commutative objects

Consider then the functor $C: \text{PreOrdGrp} \rightarrow \text{PreOrdAb}$:

$$\begin{array}{ccc}
 P_G & \xrightarrow{\bar{\eta}_G} & \eta_G(P_G) \\
 \downarrow & & \downarrow \\
 G & \xrightarrow{\eta_G} & ab(G) = G/[G, G]
 \end{array}
 \qquad C(G, P_G) = ?$$

The functor to the subcategory of commutative objects

Consider then the functor $C: \text{PreOrdGrp} \rightarrow \text{PreOrdAb}$:

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 P_G & \xrightarrow{\bar{\eta}_G} & \eta_G(P_G) \\
 \downarrow & & \downarrow \\
 G & \xrightarrow{\eta_G} & ab(G) = G/[G, G]
 \end{array}$$

$$C(G, P_G) = (ab(G), \eta_G(P_G))$$

Galois structure Γ_C

This gives rise to an adjunction:

$$\text{PreOrdGrp} \begin{array}{c} \xrightarrow{\quad C \quad} \\ \perp \\ \xleftarrow{\quad V \quad} \end{array} \text{PreOrdAb.}$$

Galois structure Γ_C

This gives rise to an adjunction:

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Proposition

$\Gamma_C = (\text{PreOrdGrp}, \text{PreOrdAb}, C, V, \mathcal{E}_C, \mathcal{Z}_C)$ is an admissible Galois structure, where

- \mathcal{E}_C is the class of **regular epimorphisms** in PreOrdGrp ;
- \mathcal{Z}_C is the class of **regular epimorphisms** in PreOrdAb .

Abelian objects

Consider the full subcategory of **abelian objects** in PreOrdGrp .

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It is the full subcategory of preordered groups (G, P_G) such that $G \in \text{Ab}$ and $P_G \in \text{Ab}$


Abelian objects

Consider the full subcategory of **abelian objects** in PreOrdGrp .

It is the full subcategory of preordered groups (G, P_G) such that $G \in \text{Ab}$ and $P_G \in \text{Ab}$, i.e. it is the category $\mathbf{Mono}(\mathbf{Ab})$ of monomorphisms in the category of abelian groups.

The functor to the subcategory of abelian objects

Consider the composite

$$\text{PreOrdGrp} \xrightarrow{C} \text{PreOrdAb} \xrightarrow{A} \text{Mono}(\text{Ab})$$


The functor to the subcategory of abelian objects

Consider the composite

$$\begin{array}{ccccc}
 & & & F & \\
 & & & \curvearrowright & \\
 \text{PreOrdGrp} & \xrightarrow{C} & \text{PreOrdAb} & \xrightarrow{A} & \text{Mono}(\text{Ab})
 \end{array}$$

with $A(G, P_G) = (G, \text{grp}(P_G))$ for any $(G, P_G) \in \text{PreOrdAb}$, where $\text{grp}(P_G)$ is the **group completion** of P_G .

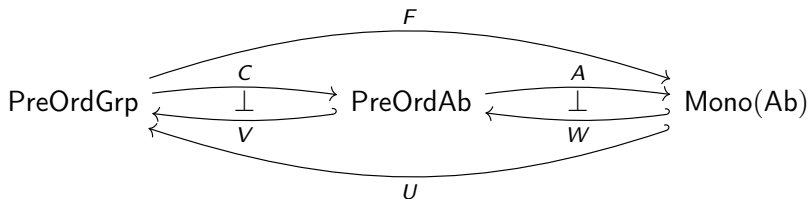
The functor to the subcategory of abelian objects

$$\Rightarrow F(G, P_G) = (ab(G), grp(\eta_G(P_G))) \quad \text{for any } (G, P_G) \in \text{PreOrdGrp}$$

$$\begin{array}{ccccc}
 & & \hat{\eta}_G & & \\
 & \swarrow & \text{---} & \searrow & \\
 P_G & \xrightarrow{\bar{\eta}_G} & \eta_G(P_G) & \xrightarrow{j_G} & grp(\eta_G(P_G)) \\
 \downarrow & & \downarrow & & \downarrow \\
 G & \xrightarrow{\eta_G} & ab(G) & \xlongequal{\quad} & ab(G) \\
 & \searrow & \text{---} & \swarrow & \\
 & & \eta_G & &
 \end{array}$$

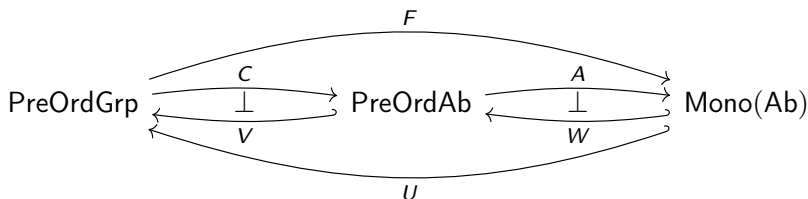
Galois structure Γ

This gives rise to an adjunction: $F \dashv U$.

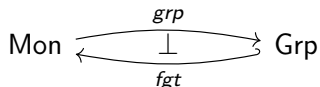


Galois structure Γ

This gives rise to an adjunction: $F \dashv U$.



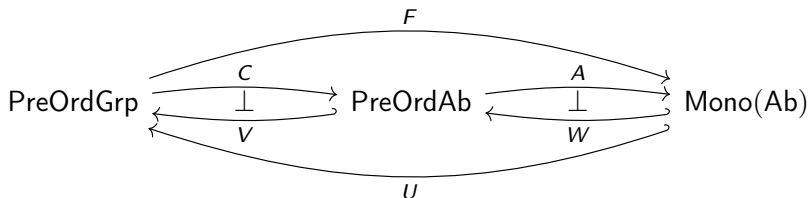
The adjunction $A \dashv W$ is induced by the adjunction



(studied in the paper “*A Galois theory for monoids*” by A. Montoli, D. Rodelo and T. Van der Linden (2014)).

Galois structure Γ

This gives rise to an adjunction: $F \dashv U$.



Proposition

$\Gamma = (\text{PreOrdGrp}, \text{Mono}(\text{Ab}), F, U, \mathcal{E}, \mathcal{Z})$ is a Galois structure, where

- \mathcal{E} is the class of **regular epimorphisms** in PreOrdGrp ;
- \mathcal{Z} is the class of **regular epimorphisms** in $\text{Mono}(\text{Ab})$.

Characterization of Γ -central and Γ -normal extensions

Proposition

Γ is *admissible*.

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Proposition

Γ is admissible.

Theorem [M. Gran, A. Michel (2023)]

Let $(f, \bar{f}): (G, P_G) \twoheadrightarrow (H, P_H)$ be a regular epimorphism in PreOrdGrp . Then, the following conditions are equivalent:

Characterization of Γ -central and Γ -normal extensions

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Γ is admissible.

Theorem [M. Gran, A. Michel (2023)]










Let $(f, \bar{f}): (G, P_G) \twoheadrightarrow (H, P_H)$ be a regular epimorphism in PreOrdGrp. Then, the following conditions are equivalent:

- 1 (i) $\text{Ker}(f) \subseteq Z(G)$;
 (ii) for any $(x, y) \in \text{Eq}(\bar{f})$, $y - x \in P_G$
 (\bar{f} is a **special Schreier surjection** in Mon).
- 2 (f, \bar{f}) is a Γ -normal extension.
- 3 (f, \bar{f}) is a Γ -central extension.

Conclusion

	Grp	PreOrdGrp
“Suitable” adjunction	$\text{Grp} \begin{array}{c} \xrightarrow{ab} \\ \perp \\ \xleftarrow{u} \end{array} \text{Ab}$	$\text{PreOrdGrp} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{U} \end{array} \text{Mono}(\text{Ab})$
Admissible Galois structure	Γ_{ab}	Γ (extensions = reg. epis)
Central = normal	Reg. epis $f: A \twoheadrightarrow B$ s.t. $\text{Ker}(f) \subseteq Z(A)$ ($f = \text{alg. central ext.}$)	Reg. epis $(f, \bar{f}): (G, P_G) \twoheadrightarrow (H, P_H)$ s.t. f is an algebraically central ext. and \bar{f} is a special Schreier surjection

References

-  M.M. Clementino, N. Martins-Ferreira, and A. Montoli, *On the categorical behaviour of preordered groups*, J. Pure Appl. Algebra 223 (2019), 4226-4245.
-  M.M. Clementino, and A. Montoli, *On the categorical behaviour of V-groups*, J. Pure Appl. Algebra 225 (2021), 106550.
-  M. Gran, and A. Michel, *Torsion theories and coverings of preordered groups*, Algebra Univers. 82 (22) (2021).
-  M. Gran, and A. Michel, *Central extensions of preordered groups*, Preprint (2023).
-  G. Janelidze, *Pure Galois theory in categories*, J. Algebra 132, no. 2 (1990), 270-286.
-  G. Janelidze, and G.M. Kelly, *Galois theory and a general notion of central extension*, J. Pure Appl. Algebra 97 (1994), 135-161.
-  Z. Janelidze, *The pointed subobject functor, 3×3 lemmas, and subtractivity of spans*, Theory Appl. Categ. 23, no. 11 (2010), 221-242.
-  A. Michel, *Torsion theories and coverings of V-groups*, Appl. Categ. Struct. 30 (2022), 659-684.
-  A. Montoli, D. Rodelo, and T. Van der Linden, *A Galois theory for monoids*, Theory Appl. Categ. 29, no. 7 (2014), 198-214.