Norms on Categories

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Motivation

Motivations:

- to find analogs of the Cantor-Schröder-Bernstein theorem (CSB theorem) from set theory in other categories,
- for systematic and convenient metrization of families of equivalence classes of spaces, like the Gromov-Hausdorff space, moduli spaces, and representation spaces,
- to prove general theorems in the developed categorical framework that give insights and concrete useful applications in many different areas of mathematics,
- to work with categories with large classes of morphisms.

Areas of application:

- metric spaces,
- topological spaces, especially representation spaces
- metric measure spaces
- Riemannian manifolds, moduli spaces, manifolds with currents, etc.
- new perspective on many other areas

Motivation: Example of Metrization

Definition 26
A *pointed measure* on a topological space Z is a pair (μ, p) where $p \in Z$ and μ is a Borel measure on Z. A *pointed subset* of Z is a pair (X, p) where $X \subset Z$ and $p \in Z$.

Definition 27
Given a subset F of a metric space Z, let $N^o_Z(F, \epsilon)$ denote the open ϵ-neighborhood of F in Z.

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Definition 28
We say that two pointed measures $(\mu_1, p_1), (\mu_2, p_2)$ on a metric space Z are (ϵ, R)-*related* if, for every closed $F_i \subset B_Z(p_i, R)$,

$$\mu_1(F_1) < \mu_2(N^o_Z(F_1, \epsilon)) + \epsilon, \qquad \mu_2(F_2) < \mu_1(N^o_Z(F_2, \epsilon)) + \epsilon,$$

and $\mathrm{dist}_Z(p_1, p_2) < \epsilon$. We say two pointed subsets $(X_1, p_1), (X_2, p_2)$ of Z are (ϵ, R)-*related* if $\mathrm{dist}_Z(p_1, p_2) < \epsilon$ and

$$B_Z(p_1, R) \cap X_1 \subset N^o_Z(X_2, \epsilon), \qquad B_Z(p_2, R) \cap X_2 \subset N^o_Z(X_1, \epsilon).$$

A sequence $\{(X_i, p_i)\}_{i=1}^\infty$ of pointed closed subsets of Z converges to (X_∞, p_∞) in the *pointed Hausdorff topology* if, for every $\epsilon, R > 0$, there is an I such that $i > I$ implies that (X_i, p_i) and (X_∞, p_∞) are (ϵ, R)-related.

Definition 29 ((ϵ, R)*-related mm*^{n}*-spaces*)
We say that mm^n-spaces $(M_1, p_1), (M_2, p_2)$ are (ϵ, R)-related if there exist a metric space Z and isometric embeddings $\varphi_i : M_i \to Z$ such that

- $(\varphi_1(M_1), \varphi_1(p_1)), (\varphi_2(M_2), \varphi_2(p_2))$ are (ϵ, R)-related as pointed subsets of Z;
- for every $k = 1, \ldots, n$, $((\varphi_1)_* \mathrm{vol}^{(k)}_{M_1}, \varphi_1(p_1))$ and $((\varphi_2)_* \mathrm{vol}^{(k)}_{M_2}, \varphi_2(p_2))$ are (ϵ, R)-related as pointed measures of Z.

Let $N_{\epsilon,R}(M, p)$ denote the set of all $[M', p'] \in \mathbb{M}^n$ such that (M', p') is (ϵ', R')-related to (M, p) for some $\epsilon' < \epsilon$ and $R' > R$. We show below that this is an open set.

Definition 30
A *pseudometric* d on a set X is a function $d : X \times X \to [0, \infty)$ satisfying all the properties of a metric with one exception: it may happen that $d(x, y) = 0$ even if $x \neq y$.

LEMMA B.1
Let Z be a set equal to a disjoint union $Z = \bigsqcup_{i=1}^\infty M_i$ of its subsets M_i. Suppose that for each i there is a metric dist_{M_i} on M_i, suppose that there is a collection $\{L_j\}_{j \in J}$ of subsets $L_j \subset Z$, and suppose that for each j there is a pseudometric dist_{L_j} on L_j. Suppose as well that if $x, y \in L_j \cap M_i$ for some i, j, then $\mathrm{dist}_{M_i}(x, y) = \mathrm{dist}_{L_j}(x, y)$. Lastly, we assume that for any $x, y \in Z$ there is a sequence $x = x_1, x_2, \ldots, x_n = y$ such that for each i either $x_i, x_{i+1} \in M_k$ for some k or $x_i, x_{i+1} \in L_j$ for some j. Then there is a pseudometric dist_Z on Z such that

- *$\mathrm{dist}_Z(x, y) = \mathrm{dist}_{M_i}(x, y)$ for any $x, y \in M_i$, for any i;*
- *$\mathrm{dist}_Z(x, y) \leq \mathrm{dist}_{L_j}(x, y)$ for any $x, y \in L_j$ for any j.*

Definitions I

A **seminorm** on a category $\underline{C}$ is a function $\|\,.\,\|\colon \underline{C}_1 \to [0,\infty]$ such that

(N1) $\|\mathrm{id}_X\| = 0$ for every object $X \in \underline{C}_0$,

(N2) $\|f\,;\,g\| \leq \|f\| + \|g\|$.

An isomorphism $f\colon X \to Y$ with inverse $g\colon Y \to X$ is called a **norm isomorphism** if $\|f\| = \|g\| = 0$.

A seminorm is called a **norm** if for all objects X, Y the following holds

(N3) if there are modulators $f\colon X \to Y$ and $g\colon Y \to X$, then X and Y are norm isomorphic; and

(N4) if for all $\varepsilon > 0$ there is $f\colon X \to Y$ with $\|f\| \leq \varepsilon$, then there is $f\colon X \to Y$ with $\|f\| = 0$.

Induced metric:

$$d^{+}_{\|\,.\,\|}(X,Y) := \frac{1}{2}\left(d_{\|\,.\,\|}(X,Y) + d_{\|\,.\,\|}(Y,X)\right)$$

Left dual:

$$\|f\|^{*\mathrm{L}} := \sup_{f'}{}^{0}\left(\|f'\| - \|f'\,;\,f\|\right)$$

(where $\sup^a_{x\in X} f(x) = \sup\{a\} \cup \{\, f(x) \mid x \in X \,\}$)

Alternative approaches:

- Kubiś (2017)
- Perrone (2023)

Definitions II

Norms from capacities

By a **concrete category with generalized subobjects** $(\underline{C}; \mathrm{GS})$ we understand a concrete category $(\underline{C}, F)$ additionally endowed with an extension

$$\begin{array}{ccc} \underline{SC} & & \\ {\scriptstyle S}\uparrow & \searrow^{F_S} & \\ \underline{C} & \xrightarrow{F} & \underline{\mathrm{SET}} \end{array}$$

and a selection function GS assigning to each $X \in \underline{C}_0$ a subset of $\mathrm{Sub}(SX)$, called **(generalized) subobjects** , such that

1. for each $X \in \underline{C}_0$ the order preserving induced functor

$$|\,\mathrm{GS}\,|(X)\colon\ \mathrm{GS}(X) \to \mathrm{Sub}(F(X)), \quad C \mapsto (F_S)_1(C)$$

   is well-defined.,
2. if $f\colon X \to Y$ and $C \in \mathrm{GS}(Y)$, then there is a $B \in \mathrm{GS}(X)$ (written $B = f^*(C)$) with $|B| = (Ff)^*(|C|)$, that is maximal in $\mathrm{GS}(Y)$ with this property.

Note: $|C| := (F_S C)(\mathrm{source}\ C) \subseteq F(X)$.

A **precapacity** w on a concrete category with subobjects $(\underline{C}; \mathrm{GS})$ is a function

$$c\colon \bigsqcup_{X \in \underline{SC}_0} \mathrm{GS}(X) \to [-\infty, \infty]$$

and it is called a **capacity** if it is monotone or antimonotone, i.e. for $B, C \in \mathrm{GS}(X)$ with $B \subseteq C$ we have $c(\mathrm{source}\ B) \leq c(\mathrm{source}\ C)$.

Each precapacity gives rise to a norm by the assignment

$$\|f\|_c := \sup^0_{\substack{C \in \mathrm{GS}(Y),\\ c(C) < \infty}} cf^*C - c(C).$$

METRIC SPACES

<u>MET</u> = compact metric spaces $\mathcal{M} = (M, d_{\mathcal{M}})$ and multivalued maps.

$$\begin{array}{ccc} \underline{\text{SMET}} & & \\ S \uparrow & \searrow^{F_S} & \\ \underline{\text{MET}} & \xrightarrow{F} & \underline{\text{SET}} \end{array} ,$$

$F(\mathcal{M}) = \mathcal{P}(M \times M)$
$F(f) = \lambda P.\{\, (y, y') \mid y \in f[x], y' \in f[x'], (x, x') \in P \,\}$,
$\underline{\text{SMET}} = \underline{\text{SET}}$
$\mathrm{GS}(\mathcal{M}) = \mathcal{P}(F(\mathcal{M}))$

$$c_{\mathrm{diam}} := \lambda A. \sup_{P \in A}{}^{0} \inf_{p \in P} d_{\mathcal{M}}(p).$$

$$\|f\|_{\mathrm{diam}} = \sup_{\substack{x, x' \in M, \\ y \in f[x], y' \in f[x']}}{}^{0} \; |x\, x'| - |y\, y'|.$$

Generalizing a classical theorem of Feudenthal and Hurewicz we show that $\|f\|_{\mathrm{diam}}$ is a norm on <u>MET</u>. Moreover the metric $\mathrm{d}^{+}_{\mathrm{diam}}$ induced by this norm is almost the Gromov-Hausdorff distance d_{GH}; to be precise the identity map

$$(\{ {}^{\text{isometry classes of}}_{\text{compact metric spaces}} \}, \mathrm{d}_{\mathrm{GH}}) \to (\{ {}^{\text{isometry classes of}}_{\text{compact metric spaces}} \}, \mathrm{d}^{+}_{\mathrm{diam}})$$

is 2-Lipschitz with Cauchy continuous inverse.

Lemma behind

Let $\mathcal{M}, \mathcal{M}'$ be compact metric spaces. For all L, l with $l > L \geq 0$ it holds for sufficiently small $\delta > 0$ that for every $h \colon \mathcal{M}' \to \mathcal{M}'$ with $\|h\|_{\mathrm{diam}} < \delta$ and $\mathrm{d}_{\mathrm{diam}}(\mathcal{M}', \mathcal{M}) \leq L$ we have that

- $h_*(M')$ is l-dense, and
- $\|h\|^{*\mathrm{L}}_{\mathrm{diam}} \leq 4l + C\delta$ where $C = C(l - L, \mathcal{M})$.

Core of the Proof

$$\mathbb{N} \xrightarrow{\mathrm{pack}_{\mathcal{M}}} \mathrm{GS}(\mathcal{M}) \xrightarrow{c_{\mathrm{diam}}} [0, \infty]$$

Future work

Current state: Insall and Luckhardt (2021).
New version under way.

- Generalize lemma from last slide. Apply to other categories (approximation theorems in its own rights).
- starting with a normed category $\underline{C}$ define a normed completion of the category: Objects are defined by some directed index category $\underline{I}$ and a morphism $\vec{X}\colon \underline{I} \to T/\underline{C}$ satisfying the Cauchy condition (compare Kubiś, 2017, Def. 3.3):

  $$\forall \varepsilon > 0\colon \exists i_\varepsilon\colon \forall i \to i' \text{ with } i \geq i_\varepsilon\colon \|X_{i\to i'}\|, \varepsilon.$$

  The set of morphisms between $\vec{X}$ and $\vec{Y}$ is given by all $f^i_j \in \underline{C}[\vec{X}, \vec{Y}] = \lim_i \operatorname{colim}_j \underline{C}[X_i, Y_j]$ such that $\limsup_{ij}\|f^i_j\|^{*\mathrm{L}} \leq \limsup\|f^i_j\| < \infty$.

- Define a norm on this category by means of a Choquet style integral: For a directed set $I = (I, \leq)$ and an order preserving function $F\colon I \to [0, 1]$, thought of as the distribution of a probability measure, set for $f \in \underline{C}[\vec{X}, \vec{Y}]$

  $$\int f(i)\,\mathrm{d}\dot{F} := \int 1 - F(\sup\{\, i \mid f(i) \leq t \,\})\,\mathrm{d}t$$

  where $f(i) := \inf\{\, \|g\| \mid g \in \underline{C}[X_i, Y_j] \text{ with } \iota_{ij}(g) = \mathrm{pr}_i f \,\}$, where ι_{ij} is the universal map $\underline{C}[X_i, Y_j] \to \operatorname{colim}_{j\in I} \underline{C}[X_i, Y_j]$.
- Generalize the notion of a normed category to 2-categories. This generalization should for instance capture coarse structure.

References I

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- Kubiś, W. (2017, May 29). *Categories with norms*. arXiv: 1705.10189 [math.CT].
- Perrone, P. (2023). *Lifting couplings in wasserstein spaces*. arXiv: 2110.06591 [math.CT].