# NORMS ON CATEGORIES

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# MOTIVATION

#### Motivations:

- to find analogs of the Cantor-Schröder-Bernstein theorem (CSB theorem) from set theory in other categories,
- for systematic and convenient metrization of families of equivalence classes of spaces, like the Gromov-Hausdorff space, moduli spaces, and representation spaces,
- to prove general theorems in the developed categorical framework that give insights and concrete useful applications in many different areas of mathematics,
- to work with categories with large classes of morphisms.

Areas of application:

- metric spaces,
- topological spaces, especially representation spaces
- metric measure spaces
- Riemannian manifolds, moduli spaces, manifolds with currents, etc.
- new perspective on many other areas

# MOTIVATION: EXAMPLE OF METRIZATION

CHERRICONSTANTS AND L <sup>2</sup> -BETTI NO	MILES 605							
Let $S=\{s_1,\ldots,s_r\}\subset F_r$ be a f	iree generating set. Let $E_i \in \mathcal{G}(\Lambda)$ be the set	on Links sowin	CHERCER CONSTANTS AND L <sup>2</sup> ARTEI NUMBER	s 607	608	LEWS BOWEN	s	
containing $((f,m), (fs_1,m))$ for every $f \in F_r, 1 \le j \le r$ , and $m \in \mathbb{Z}$ with $p \mid (m-i)$ : We say that two pointed measures $(\mu_1, p_1), (\mu_2, p_2)$ on a metric space Z are $(r, R)$ -		be a finite set such that for every compact	where $F \subset B_Z(p_N, R + \epsilon)$ there exists	for $\alpha < \beta$ let			CHILDRECONSIANS AND 2-4070 MORES 40	
$\{(f, m), (f, m + 1)\}$ for every $f \in F$ , and $m \in \mathbb{Z}$ with $p \nmid (m - i - 1)$ . related if, for every closed $F_i \subset B_{\mathbb{Z}}(p_i, \mathbb{R})$ .			g ∈ F such that g = 1 on F, g = 0 on the co all of Z. To see that such a set exists, let Ø in	inglement of $N_Z^{\alpha}(F, \epsilon)$ , and $0 \le g \le 1$ on	Fla.8	$0 = lx \in \mathbb{Z}$ , $\alpha < f(x) < \beta$ $\cap S$ .		We say that mm <sup>4</sup> -spaces $(M_1, p_1), (M_2, p_2)$ are $(\epsilon, R)$ -related if there exist a metric
Lenserve time tree grapes with vertex set, is an origin set $L_1$ is a freed. Moreover, the law $\mu_1(F_1) < \mu_2(W_2(F_1, t)) + \epsilon$ , $\mu_2(F_2) < \mu_1(W_2^2(F_2, t)) + \epsilon$ , of $E_1$ is an invariant resolution in the massive $\lambda_1$ one of $(A_1)$ . Finally, $(A_2) = 0$ , $(A_1) = 0$ , $(A_2) = 0$ .			by open balls of radius less than c. Let 3 subsections to 40 Juni 37 he for out of all	by open bills of radius less than c. Let 3				space Z and isometric embeddings φ <sub>1</sub> : M <sub>1</sub> → Z such that (φ <sub>1</sub> (M <sub>1</sub> ), φ <sub>2</sub> (p <sub>2</sub> )), (φ <sub>2</sub> (M <sub>2</sub> ), φ <sub>3</sub> (p <sub>2</sub> )) are (ε, R)-related as pointed subsets of
A with $n \le m$ , $(f, n), (g, w)$ are in the same connected component of $(\Lambda, E_i)$ if and mlated if $dit_Z(p_1, p_2) < c$ . We say two pointed subsets $(X_1, p_1), (X_2, p_2)$ of Z are $(c, R)$ - mlated if $dit_Z(p_1, p_2) < c$ and			subsets $\mathcal{O}' \subset \mathcal{O}$ . If $F \subset \mathcal{B}_Z(p_{\infty}, R + \epsilon)$	subsets $0^{\prime} \subset 0$ . If $F \subset B_{2}(p_{m}, R+\epsilon)$ Definition 29 (( $\epsilon, R$ )-related mm <sup>-</sup> -spaces)				Z; for every hole of the house
soly if there does not coist an integer q with $n \le q < m$ such that $p \nmid (q - i - 1)$ . This secures with could be a solubility count to $(n -  m - m ) < n$ if $m = n \le n$ . In matrixing this $B_{2}(p_{1}, R) \cap X_{1} \subset N_{2}^{2}(X_{2}, t)$ , $B_{2}(p_{1}, R) \cap X_{2} \subset N_{2}^{2}(X_{2}, t)$ .			$F \neq \emptyset$ , then $g = 1$ on $F, 0 \le g \le 1$ , and required.	$r_{\text{prime}} = 1007.05\pm51.44$ We say that mm <sup>-</sup> -spaces $(M_1, p_1), (M_2, p_2)$ are $(\epsilon, R)$ -related if there exist a metric				(c, R) related as pointed measures of Z.
probability tends to 1 as $p \rightarrow \infty$ . This implies that A is almost treesble. A sequence $((X_{i_1}, p_i))^{(2)}_{i_1}$ , of pointed closed subsets of Z converges to $(X_{i_2}, p_{i_2})$ in		Let <i>I</i> be large enough so that $i > I$ in $m_i(\alpha)   < i$ for all $\alpha \in Y$ . Let $F \subset B_0(\alpha)$	Let I be toget ensures so that $> i$ in $w_i(x) \leq i x \in \mathbb{R}$ and isometric embeddings $\varphi_i : M_i \to Z$ such that				Let $N_{c,R}(M, p)$ denote the set of all $[M', p'] \in M^n$ such that $(M', p')$ is $(c', R')$ - related to $(M, p)$ for some $c' < c$ and $R' > R$ . We show below that this is an open	
By Lemma 8.6, it follows that eve	ry lattice in Isom(IP) is almost treeable.	the accisted Handorff topology if for every c. R > 0, there is an L such that c > L included (V = 1) and (V = 1) and (C. B) added	So there exists $g \in \mathcal{F}$ as above. Observe the	<ul> <li>(φ<sub>1</sub>(M<sub>1</sub>), φ<sub>1</sub></li> </ul>	$(p_1)), (\varphi_2(M_2), \varphi_2(p_2))$ are	$(\epsilon, R)$ -related as pointed subs	sets of	set.
LEMMA 8.9		implete that (A1, J1) and (A2, J22) are (1, A) tender.	$\mu_i(F) \leq \int g d\mu_i < \epsilon + \int g$	Z;				Definition 30
is a subgroup of the fundamental g			Solute & F.c. R.(n. P) the	$ \text{for every } k = 1, \dots, n, ((\varphi_1), \operatorname{vol}_{M_1}^k, \varphi_1(p_1)) \text{ and } ((\varphi_2), \operatorname{vol}_{M_2}^k, \varphi_2(p_2)) \text{ are } $				A pseudometric d on a set X is a function $d : X \times X \rightarrow [0, \infty)$ satisfying all the properties of a metric with one exception: it may happen that $d(x, y) = 0$ even if
reaction, and of (1) - of a con-	A pointed measure on a 1		$(\epsilon, R)$ -related as pointed measures of Z.				x≠3.	
Proof Dús is tone because I' is alsocut teachi	Boral massure on Z A n	consisted subset of Z is a pair $(Y, p)$ where $Y \subset Z$ and $p \in Z$	$\mu_{\infty}(F) \leq \int g  d\mu_{\infty} < \epsilon + j$	$\lim_{\mu \to 0} \sum_{j=1}^{n} \lim_{\mu \to 0} \sum_{j=1}^{n} \sum_{j=1}^$			( R').	LIMMA B.1
recable group is almost trecable by Le	Borel measure on Z. A pointed subset of Z is a pair $(A, p)$ where $A \subseteq Z$ and $p \in Z$ .		This shows that $\mu_i$ , $\mu_{\infty}$ are $(\epsilon, R)$ -related. Now suppose that for every $\epsilon, R > 0$	This does that $\mu_{L,R}(\alpha, r_{L})$ exclusion $\mu_{L,R}(\alpha, r_{L})$ and $\mu_{L,R}(\alpha, r_{L})$ are the order of an $[M, r_{L,R}(\alpha, r_{L})] = 0$ . We show below that this is an one of the second state of $M$ and $M = M$ . We show below that this is an one of the second state of $M$ and $M = M$ .			n open	Let Z be a set equal to a disjoint accou $Z = \bigcup_{i=1}^{n} M_i$ of its satisfies $M_i$ . Suppose that for each i there is a metric dist <sub>M</sub> , on $M_i$ , suppose that there is a collection $\{L_j\}_{j \in J}$
$b_d^{(l)}(\Lambda) = 0$ for every $d \ge 2$ by Lemm	D. C. M		$(\mu_i, p_i)$ and $(\mu_{\infty}, p_{\infty})$ are $(\epsilon, R)$ -related.	sot	or some $e < e$ and $K > K$ .	the show below that this is an	open	of subsets $L_j \subset \mathbb{Z}$ , and suppose that for each $j$ there is a pseudometric dist <sub>L_j</sub> on $L_j$ . Suppose as well that if $x, y \in L_j \cap M_j$ for some $i, j$ , then dist <sub>L_j</sub> $(x, y) =$
Print of Condury 1.3 Definition 27		related.	set.				$dist_{L_2}(x, y)$ . Lastly, we assume that for any $x, y \in \mathbb{Z}$ there is a sequence $x = x_1, x_2, \dots = x_n$ that for each 1 other $x, y \in M$ , for some $k$ or $x, y \in L$ . For	
his follows from Theorem 1.2 and Pri Given a subset F of a metric space Z, let $N_Z^{\sigma}(F, \epsilon)$ denote the open $\epsilon$ -neighborhood		CLAIM 1	<b>D</b> 0 11 00				some j. Then there is a pseudometric distg on Z such that	
Appendices	of $F$ in $Z$ .		For any compact $S \subset \mathbb{Z}$ ,	Definition 30				<ul> <li>dist<sub>Z</sub>(x, y) = dist<sub>M</sub>(x, y) for any x, y ∈ M<sub>i</sub>, for any i:</li> <li>dist<sub>Z</sub>(x, y) ≤ dist<sub>L</sub>(x, y) for any x, y ∈ L<sub>i</sub> for any j.</li> </ul>
A Printed subsets and measures of a			$\lim_{t\to\infty} \mu_t(N^n_Z(S,$	$\lim_{t\to\infty}\mu(N_{2}(s)  \text{A pseudometric } d \text{ on a set } X \text{ is a function } d: X \times X \to [0,\infty) \text{ satisfying all the}$				Pred
<ol> <li>Pointed subsets and measures of a The purpose of this appendix is to prov</li> </ol>			Proof	properties of a metric with one exception: it may happen that $d(x, y) = 0$ even if				For each $x,y\in Z$ we define $dist_Z(x,y)=inf\sum_{k=1}^rdist_{N_k}(x_k,x_{k+1})$ where the infi-
measures and pointed subspaces of a g	i i		For all sufficiently large $i, S \subset B_Z(p_i, R_i)$	$x \neq y$ .				mum is over all sequences $x = x_1,, x_r = y$ and choices $N_k \in \{M_i\}_{i=1}^n \cup \{L_j\}_{j \in J}$ such that $x_k, x_{k+1} \in N_k$ for all $1 \le k < r$ . It is easy to check that the conclusions
Definition 26			$\mu_{\infty}(S) \le \mu_1(N_Z^{\varphi}(S, \epsilon_t)) + \cdot$					hold.
A pointed measure on a topological sp			By taking the limit as t→∞, the claim 1 20.11 is finite for all sufficiently base t with	LEMMA B.1				LEMMA B.2
Borel measure on Z. A power subset	·		proper.	Let Z be a set equa	It to a disjoint union $Z = \prod_{i=1}^{\infty}$	. M: of its subsets M: Suppos	se that	Suppose that $\{(M_i, p_i)\}_{i=1}^{n}$ is a sequence of mm <sup>4</sup> spaces such that $(M_i, p_i)$ and $(M_j, p_j)$ are $(c_{ij}, R_{ij})$ -related for all $i, j$ (where $c_{ij}, R_{ij}$ are positive real numbers).
Definition 27			Now let f be a real-valued compact	for each i there is a	metric dist an Mi support	e that there is a collection {]	Acres	Then there exist a complete separable metric space Z and isometric embeddings $q_1: M_i \rightarrow Z$ such that for all $i, i, k$
of F in Z.			suffices to show that $\lim_{i\to\infty} \mu_i(f) = \mu_0$	of subsets L : C 7	and suppose that for each	i there is a preudometric d	diet	
	101	L TRUE DOWN		on $L$ . Suppose of	, and suppose that for each $M_{1}$	for some i i then dist.	unst_j	
	606	LEWIS BOWEN	·	in Lj. Suppose as	went that $y x, y \in L_j \cap M_i$	for some $i, j$ , then $\operatorname{dist}_{M_i}(x, y)$	, y) =	
	Definition 28			$dist_{L_j}(x, y)$ . Lastly	y, we assume that for any x, y	$\in \mathbb{Z}$ there is a sequence $x = x$	$x_1, x_2,$	
We want that two pointed means		= (1, 2, 2)		$\dots, x_n = y$ such that for each <i>i</i> either $x_i, x_{i+1} \in M_k$ for some <i>k</i> or $x_i, x_{i+1} \in L_j$ for			L <sub>j</sub> for	
	we say that two pointed measures $(\mu_1, p_1), (\mu_2, p_2)$ on a metric space Z are $(\epsilon, R)$ -			some j. Then there is a pseudometric $dist_Z$ on Z such that				
related if, for every closed $F_i \subset B_Z(p_i, R)$ ,			• dist <sub>Z</sub> (x, y) = dist <sub>M<sub>i</sub></sub> (x, y) for any x, y $\in$ M <sub>i</sub> , for any i;					
	$\mu_{1}(E_{1}) < \mu_{2}(N)$	$ \mathcal{O}(\mathbf{F}_{1,c})\rangle + c \qquad \mu_{1}(\mathbf{F}_{2}) < \mu_{2}(N^{O}(\mathbf{F}_{1,c})) + c$		<ul> <li>dist<sub>Z</sub>(x, y)</li> </ul>	$\leq \text{dist}_{L_i}(x, y)$ for any $x, y \in$	$L_j$ for any $j$ .		
<ul> <li>(n(M) n(n)) (n(M))</li> </ul>	$\mu_1(r_1) < \mu_2(n_Z(r_1,\epsilon)) + \epsilon,  \mu_2(r_2) < \mu_1(n_Z(r_2,\epsilon)) + \epsilon,$		datate as a data (t. y.) + data(t. y.)	direct a location via diante via				
efZ:	$ \begin{array}{l} \substack{(0,1,0),(0,1$		<ul> <li>Decreme Maxie proper this implies that (x, 1<sup>10</sup>)</li> </ul>	has a conservation interview Since	fees the triangle inequality. It is then It 5. So M <sup>4</sup> is metricoble	fore a metric on M*. It is continuous by Lemma		
pointed measures of Z.			$dist_Z(z_l, x_l) \leq dist_Z(z_l, y_l) + dist_Z(y_l, x_l) \leq$	Bocases $M_{46}$ is proper, this implies that $[L_1]_{1=1}^{-1}$ has a convergent subsequence. Since $dist_Z(t_2, t_3) \le dist_Z(t_2, t_3) + dist_Z(t_3, t_3) \le t_{460} + 1/1$ tends to zero as $t \to \infty$ , To show that M <sup>n</sup> is separable, let $\mathbb{F}_2^n$ be the set of all $[M, p] \in M^n$ such that M is				
Proof			this implies that {x <sub>i</sub> } <sup>2</sup> <sub>i=1</sub> has a convergent subsc The proposition now follows from Lemma	this implies that {x <sub>i</sub> } <sup>1</sup> <sub>i=1</sub> has a convergent subsequence as required. The proposition now follows from Lemma A.2.		at are rational-valued. Note that $F_0^{\alpha}$ is countable, at $F^{\alpha}$ be the set of all $(M, s) \in M^{\alpha}$ such that $M$		
For each i, j, there exist a complete	$B_{\mathcal{Z}}(p_1, R) \cap X_1 \subset N^{\varrho}_{\mathcal{Z}}(X_2, \epsilon), \qquad B_{\mathcal{Z}}(p_2, R) \cap X_2 \subset N^{\varrho}_{\mathcal{Z}}(X_1, \epsilon).$		LEMMA B 5	is finite. An exercise shows that the closure of $P_0^{-1}$ contains $P_{-}^{-1}$ So it sufficient to show the first index of the second state of				
$(\phi_U(M_1), \phi_U(p_1)), (\psi_U(M_1)))$	E C C C C C C C C C C C C C C C C C C C		For any $\{M, p\} \in M^n$ and $c, R > 0$ , the set $N_{c,R}$	Lenson $G_{ij} = 0$ sense in $m^{-1}$ to sume in $m^{-1}$ to sume in $m^{-1}$ to sume $m^{-1}$ to sum $m^{-1}$ t				
• $((\phi_{ij}), vel_{M_i}^{(k)}, \phi_{ij}(p_i))$ and	A sequence $\{(X_i, p_i)\}_{i=1}^{\infty}$ of pointed closed subsets of Z converges to $(X_{\infty}, p_{\infty})$ in		1 Proof	in the space of Radon measures on $M$ in the weak* topology. So there exist measures $\mu^{(k)} = \kappa d^{\mu}(M)$ such that $\lim_{t\to\infty} \mu^{(k)} = v d^{(k)}_M$ for every $k$ . Let $X_k$ be a finite subset				
pointed measures of Y <sub>17</sub> for o	the nointed Hausdorff topology if, for every $\epsilon R > 0$ , there is an I such that $i > I$		Let $\{[M_i, p_i]\}_{i=1}^{N_i}$ be a sequence in $M^n \setminus N_{e,n}($ . If $[M_{N_i}, p_{in}] \in N_{e,n}(M, p)$ , then there exist an $e$	Let $[M_{ij}, p_i]_{ij}^{(0)}$ is a sequence in $M' \setminus k_{ij}, (M, j)$ which converges to $[M_{ij}, p_{ij}]$ . of $M$ containing $\{p\} \cup \bigcup_{k=1}^{n} m_k p_k p_k^{(k)}$ such that $\bigcup_{i=1}^{n} X_i$ is dense in $M$ . We may $H(M_{ij}, m_k) \in X_i \to M$ such that $\bigcup_{i=1}^{n} X_i$ is dense in $M$ . We may				
pseudometric dist <sub>21</sub> on Z <sup>*</sup> satisfying	implies that $(X_i, p_i)$ and $(X_{\infty}, p_{\infty})$ are $(\epsilon, R)$ -related.		and (M, p) are (c', R') related. Choose c'', R	and $(M, p)$ are $(c', R')$ related. Choose $c'', R'' > 0$ so that $c'' + c' < c$ and $R < min(R'' - 2c'', R'' - 2c')$ . By Proposition B.4, there is an $c$ such that $(M, n)$ and		$a_i^{(k)}$ . By definition $[X_i, p]$ converges to $[M, p]$ in		
<ul> <li>If x<sub>i</sub> ∈ M<sub>i</sub> ⊂ Z<sup>'</sup>, then dist</li> <li>If x<sub>i</sub> ∈ M<sub>j</sub>, x<sub>j</sub> ∈ M<sub>j</sub>, then dist</li> </ul>	$x_i \in M_i, x_j \in M_j$ , then dis		(M <sub>10</sub> , p <sub>10</sub> ) are (c <sup>*</sup> , R <sup>*</sup> ) related. Lemma II.3 no	$(M_{nc}, p_{cb})$ are $(e^n, R^n)$ -related. Lemma II.3 now implies that $[M_c, p_c] \in N_{c,R}(M, p)$ .		Po are donse in M* as claimed.		
This induces an equivalence relation Z <sup>1</sup> / ~ with the metric dist_y={[x].]	n 170 - 4862-(1, 1): 1.8( (2, 4852)) fe the metric		This contradiction proves that the complement	of N <sub>1,R</sub> (M, p) is closed. □	Acknowledgments: I am grateful to los Abbt for an excellent lecture of	Peter Samak for telling me about [30], to Mik- n J <sup>2</sup> , Betti mumbers, to Gibor Field for sharing		
completion of (Z <sup>#</sup> , datge). For each	h / there is a canonical isometric embedding p; :	<ul> <li>(p<sub>1</sub>(M<sub>1</sub>), φ<sub>1</sub>(p<sub>1</sub>)), (φ<sub>1</sub>(M<sub>1</sub>), φ<sub>2</sub>(p<sub>1</sub>)) are (c<sub>1</sub>, R<sub>1</sub>) related;</li> <li>for every k = 1,, θ<sub>1</sub> ((φ<sub>1</sub>), vo<sup>(M)</sup><sub>(M</sub>, φ<sub>1</sub>(p<sub>1</sub>)), ((φ<sub>1</sub>), vo<sup>(M)</sup><sub>(M</sub>, φ<sub>2</sub>(p<sub>2</sub>)) are (c<sub>1</sub>,</li> </ul>	We can now prove Theorem 3.1, which sta able.	tes that M* is separable and metriz-	a rough draft of a proof of the close	ed manifold case of Theorem 4.1, and to Naser		
separable.	processor concorrege is more in 2, 30 Z is	$R_i$ ) related. By contacting Z with the closure of the images of the $M_i$ 's we may assume without	And different by		tatebuaten Sastan för pointing or was impired by discussions and talk	n errors in previous versions. Part of this work i at the American Institute of Mathematics work-		
For any <i>i</i> , <i>j</i> , there is a map $x_{ij} : q$ $q_1(x_i)$ if $x_i \in M_i$ and $x_{ij}(\psi_{ij}(x_j))$	$\phi_{ij}(M_i) \cup \psi_{ij}(M_j) \rightarrow Z$ such that $w_{ij}(\phi_{ij}(x_i)) = -\phi_j(x_j)$ if $x_j \in M_j$ . This map is distance non-	loss of generality, that the union $\bigcup_{i=1}^{N} \psi_i(M_i)$ is dense in Z. Without loss of gener-	First we show that M <sup>*</sup> is metricable. For [ <i>M</i> , <i>p</i> ]	[,[M <sup>*</sup> , µ <sup>*</sup> ] ∈ M <sup>*</sup> , let	shop "L2 invariants and their relati California, in September 2011. Thi	ves for finitely generated groups" in Palo Alto, a work is supported in part by National Science		
increasing: dist_{F_{12}}(x, y) \ge dist_{Z}(x, y)	$(v_{ij}(x), v_{ij}(y))$ . Since $((\phi_{ij}), vol_{M_j}^{(k)}, \phi_{ij}(p_i))$ and	atty, we may also assume that each M <sub>4</sub> ⊂ Z and φ <sub>4</sub> is the inclusion map. This helps simplify notation.	p((M, p), (M', p')) - in	fe+ 1	Foundation (NSF) grant DMS 0968	352 and NSF CAREER Award DMS 0954606.		
and $((p_1), vol_{M_1}^{(k)}, v_1(p_1))$ are $(c_i, R_i)$ -related. Similarly, $(p_i(M_1), v_1(p_1))$ is		We claim that Z is proper. It suffices to show that every hall centered at $p_{00}$ is sequentially compact. So let $R > 0$ , and let $(x_1)_{x=1}^{N} \subset B_T(p_{00}, R)$ . There is a	where the information over all $c R > 0$ such	R+2C	References			
$(\varphi_j(M_j), \varphi_j(p_j))$ are $(e_{ij}, R_{ij})$ relations	aled as required.	sequence $\{y_i\}_{i=1}^{\infty}$ such that, for each $i$ , $dist_{\mathcal{X}}(\mathbf{x}_i, y_i) < 1/i$ and $y_i \in M_{d(i)}$ for some $a(i)$ . It sufficient to down that a subsequence of $(1, 1)^{\infty}$ is compared. If there is some	related. In order to check the triangle inequalit	$[s, let [M_i, p_i] \in M^*$ (for $i = 1, 2, 3$ ),	[1] M. ANÎRE, N. BERGERON, L. BE	ENGEL T GELANDER, N. NIEGOV, I. RAIMMATET,		
LEMMA B.3	and and the state of the state of the	<i>j</i> such that $\{y_i\}_{i=1}^{n} \cap M_j$ is infinite, then, since $M_j$ is proper, it follows that there is	and suppose that $(M_1, p_1), (M_2, p_2)$ are $(\epsilon_1, R, (\epsilon_3, R_2)$ -related for some $\epsilon_1, \epsilon_2, R_1, R_2 > 0$ . By	1) retained and (M <sub>2</sub> , p <sub>2</sub> ), (M <sub>3</sub> , p <sub>3</sub> ) are Lemma B.3,	and L SAMET, On the group groups, proprint, arXiv:12	eth of L <sup>2</sup> -issuriants for sequences of lattices in Lie 30.2961v2 [math.RT]. (570)		
$R (M_1, p_1), (M_2, p_2)$ are $(e_1, R_1)$ related, then $(M_1, p_1), (M_2, p_3)$ are	$f_i(M_i, p_1), (M_2, p_2)$ are $(e_1, K_1)$ ensure and $(M_2, p_2), (M_3, p_3)$ are $(e_2, K_2)$ . related, then $(M_1, p_1), (M_3, p_3)$ are $(e_1 + e_2, K_3)$ -related where $K_3 = \min(K_1 - C)$ . Cherwise, $\lim_{k \to \infty} n(i) - i \infty$ .		dM bl M bl so to t	AMA ALMA AD SOLOT		f treed equivalence relations, Israel J. Math. 64 (1988), OI 10.1007/04/02882422. (587)		
$2\epsilon_2, R_2 - 2\epsilon_1$ ).		$dist_Z(p_i, y_i) \leq dist_Z(p_i, p_{2k}) + dist_Z(p_{2k}, x_i) + dist_Z(x_i, y_i) \leq \epsilon_{abc} + R + 1/L$	$p_1(w_1, p_1) (w_2, p_2) = t_1 + t_2 + \min(R)$	$(1 - 2\epsilon_2, R_2 - 2\epsilon_1) + 2\epsilon_1 + 2\epsilon_2$	[3] D. ALDOUS and R. LYONS, Pro Probab. 12 (2007), 1454-1	rener on animodular random actuority, Electron, J. 1508, MR 2354165, DOI 10.1214/EJPv12-463, 1598.		
Proof This follows from Lange 10	41	In other words, $y_i \in B_Z(p_i, R + 1/i + \epsilon_{n(i)})$ . If i is large enough then $R_{n(i)} >$	$-\epsilon_1 + \epsilon_2 + \min(R)$	$(1 + 2x_1, R_2 + 2x_2)$	591) [4] L. RENAMEN and G. R. THLAM	Accurrence of doinbuilonal limits of Snite planar		
and renews from Lemmas B.2 and		$R + 1/i + \epsilon_{ab}$ . Because $(M_{ab}), p_{ab}$ .). $(M_{m}, p_{m})$ are $(\epsilon_{ab}), R_{ab}$ ) related.	$\leq \left(\epsilon_1 + \frac{1}{R_1 + 2\epsilon_1}\right)$	$+(c_2 + \frac{1}{R_2 + 2c_2}).$	graphs, Electron. J. Proba DOI 10.1214/01P-04-94	A 6 (2001), 13 pp. MR 1873300. (270		
PROPOSITION B.4 A sequence {[M <sub>1</sub> , p <sub>2</sub> ]] <sup>∞</sup> , ⊂ M <sup>n</sup> con	nerges to $[M_{20}, p_{20}] \in M^{\bullet}$ if and only if for every	$B_Z(p_1, R + 1/i + \epsilon_{n(2)}) \cap M_{n(2)} \subset N_Z^2(M_{\infty}, \epsilon_{n(2)}).$	By minimizing the right hand side over all c1, c2	R <sub>1</sub> , R <sub>2</sub> such that (M <sub>1</sub> , p <sub>1</sub> ), (M <sub>2</sub> , p <sub>2</sub> )	[5] C. I. BEREP and P. W. KONES, & 179 (1977) 1–30 MR 40	landorf discussion and Rivisian groups, Arts Math.		
		So there exists $\tau_i \in M_\infty$ with ${\rm dist}_Z(y_i,\tau_i) \le \epsilon_{\alpha(j)}.$ Note that	are (e1, R2)-related and (M2, P2), (M3, P3) are	$(e_2, R_2)$ -related, we see that $\rho$ satis-	[6] K. BORSUK, On the imbedding Math. 35 (2010).	of systems of compacts in simplicial complexes, Fund.		
					man, or (1968), 217-234	and the second second		

### **DEFINITIONS** I

A **seminorm** on a category  $\underline{C}$  is a function  $\| \cdot \| \colon \underline{C}_1 \to [0, \infty]$  such that (N1)  $\| \text{id}_X \| = 0$  for every object  $X \in \underline{C}_0$ ,

(N2)  $||f;g|| \le ||f|| + ||g||.$ 

An isomorphism  $f: X \to Y$  with inverse  $g: Y \to X$  is called a **norm isomorphism** if ||f|| = ||g|| = 0.

A seminorm is called a **norm** if for all objects X, Y the following holds

(N3) if there are modulators  $f: X \to Y$  and  $g: Y \to X$ , then X and Y are norm isomorphic; and

(N4) if for all 
$$\varepsilon > 0$$
 there is  $f \colon X \to Y$  with  $||f|| \le \varepsilon$ , then there is  $f \colon X \to Y$  with  $||f|| = 0$ .

Induced metric:

$$d^+_{\parallel \cdot \parallel}(X,Y) \coloneqq rac{1}{2} \left( d_{\parallel \cdot \parallel}(X,Y) + d_{\parallel \cdot \parallel}(Y,X) \right)$$

Left dual:

$$\|f\|^{*\mathrm{L}} \coloneqq \sup_{f'}^0 \left(\|f'\| - \|f'\,;\,f\|\right)$$

(where  $\sup_{x\in X}^a f(x) = \sup\{a\} \cup \{f(x) \mid x \in X\}$ )

Alternative approaches:

- Kubiś (2017)
- Perrone (2023)

# DEFINITIONS II NORMS FROM CAPACITIES

### By a concrete category with generalized subobjects

 $(\underline{C}; GS)$  we understand a concrete category  $(\underline{C}, F)$  additionally endowed with an extension



and a selection function GS assigning to each  $X \in \underline{C}_0$  a subset of Sub(*SX*), called (generalized) subobjects , such that

1. for each  $X \in \underline{C}_0$  the order preserving induced functor

$$|\operatorname{\mathsf{GS}}|(X)\colon\operatorname{\mathsf{GS}}(X) o\operatorname{\mathsf{Sub}}(F(X)),\quad C\mapsto(F_{\mathcal{S}})_1(C)$$

is well-defined.,

2. if  $f: X \to Y$  and  $C \in GS(Y)$ , then there is a  $B \in GS(X)$ (written  $B = f^*(C)$ ) with  $|B| = (Ff)^*(|C|)$ , that is maximal in GS(Y) with this property.

Note:  $|C| := (F_S C)$ (source  $C) \subseteq F(X)$ .

A **precapacity** *w* on a concrete category with subobjects (<u>*C*</u>; GS) is a function

$$c: \bigsqcup_{X \in \underline{SC}_0} \mathsf{GS}(X) \to [-\infty, \infty]$$

and it is called a **capacity** if it is monotone or antimonotone, i.e. for  $B, C \in GS(X)$  with  $B \subseteq C$  we have  $c(\text{source } B) \leq c(\text{source } C)$ .

Each precapacity gives rise to a norm by the assignment

$$\|f\|_{c} \coloneqq \sup_{\substack{C \in \mathsf{GS}(Y), \\ c(C) < \infty}}^{0} cf^{*}C - c(C).$$

### METRIC SPACES

 $\underline{\mathsf{MET}} = \text{compact metric spaces} \\ \mathcal{M} = (\mathcal{M}, \mathcal{d}_{\mathcal{M}}) \text{ and multivalued maps.}$ 

$$\begin{array}{c|c} \underline{SMET} \\ s & & F_s \\ \underline{MET} & \xrightarrow{F} & \underline{SET} \end{array}$$

$$F(\mathcal{M}) = \mathcal{P}(M \times M)$$
  

$$F(f) = \lambda P.\{(y, y') \mid y \in f[x], y' \in f[x'], (x, x') \in P\},$$
  

$$\underline{SMET} = \underline{SET}$$
  

$$GS(\mathcal{M}) = \mathcal{P}(F(\mathcal{M}))$$

$$c_{\mathsf{diam}} \coloneqq \lambda A. \sup_{P \in A} \inf_{p \in P} d_{\mathcal{M}}(p).$$

$$\|f\|_{\text{diam}} = \sup_{\substack{x, x' \in M, \\ y \in f[x], y' \in f[x']}} |x \, x'| - |y \, y'|.$$

Generalizing a classical theorem of Feudenthal and Hurewicz we show that  $\|f\|_{diam}$  is a norm on <u>MET</u>. Moreover the metric  $d^+_{diam}$  induced by this norm is almost the Gromov-Hausdorff distance  $d_{GH}$ ; to be precise the identity map

 $(\{ \substack{\text{isometry classes of} \\ \text{compact metric spaces} \}, d_{\mathsf{GH}}) \rightarrow (\{ \substack{\text{isometry classes of} \\ \text{compact metric spaces} \}, d_{\mathsf{diam}}^+)$ 

is 2-Lipschitz with Cauchy continuous inverse.

#### Lemma behind

Let  $\mathcal{M}, \mathcal{M}'$  be compact metric spaces. For all L, I with  $I > L \ge 0$  it holds for sufficiently small  $\delta > 0$  that for every  $h: \mathcal{M}' \to \mathcal{M}'$  with  $\|h\|_{\text{diam}} < \delta$  and  $d_{\text{diam}}(\mathcal{M}', \mathcal{M}) \le L$  we have that

- ▶  $h_*(M')$  is *I*-dense, and
- $||h||_{\text{diam}}^{*L} \leq 4I + C\delta \text{ where } C = C(I L, \mathcal{M}).$

#### **Core of the Proof**

 $\mathbb{N} \xrightarrow{\mathsf{pack}_{\mathcal{M}}} \mathsf{GS}(\mathcal{M}) \xrightarrow{c_{\mathsf{diam}}} [0,\infty]$ 

### FUTURE WORK

Current state: Insall and Luckhardt (2021). New version under way.

- Generalize lemma from last slide. Apply to other categories (approximation theorems in its own rights).

 $\forall \varepsilon > \mathbf{0} \colon \exists i_{\varepsilon} \colon \forall i \to i' \text{ with } i \geq i_{\varepsilon} \colon \|\mathbf{X}_{i \to i'}\|, \varepsilon.$ 

The set of morphisms between  $\vec{X}$  and  $\vec{Y}$  is given by all  $f_j^i \in \underline{C}[\vec{X}, \vec{Y}] = \lim_i \operatorname{colim}_j \underline{C}[X_i, Y_j]$  such that  $\limsup_{ij} \|f_j^i\|^{*L} \le \limsup \|f_j^i\| < \infty$ .

▶ Define a norm on this category by means of a Choquet style integral: For a directed set  $I = (I, \leq)$  and an order preserving function  $F: I \rightarrow [0, 1]$ , thought of as the distribution of a probability measure, set for  $f \in \underline{C}[\vec{X}, \vec{Y}]$ 

$$\int f(i) \,\mathrm{d}\dot{F} \coloneqq \int 1 - F(\sup\{i \mid f(i) \le t\}) \,\mathrm{d}t$$

where  $f(i) := \inf\{ ||g|| | g \in \underline{C}[X_i, Y_i]$  with  $\iota_{ij}(g) = \operatorname{pr}_i f \}$ , where  $\iota_{ij}$  is the universal map  $C[X_i, Y_i] \to \operatorname{colim}_{i \in I} C[X_i, Y_i]$ .

 Generalize the notion of a normed category to 2-categories. This generalization should for instance capture coarse structure.

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