Dalhousie University
Mathematics & Statistics

The tangent categories of algebras over an Operad

Joint work with Sacha Ikonicoff & Jean-Simon Pacaud Lemay

Marcello Lanfranchi

Supervisors: Dorette Pronk ∗ Geoffrey Cruttwell

Many geometries, one language

There are many different kinds of geometries

Many geometries, one language

Algebraic geometry

There are many different kinds of geometries

Many geometries, one language

Algebraic geometry

There are many different kinds of geometries

Differential geometry

Many geometries, one language

Algebraic geometry

There are many different kinds of geometries

Synthetic differential geometry

Differential geometry

Many geometries, one language

Algebraic geometry

Non-commutative geometry

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All of them share some notion of differentiability...

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Is there a common language to describe them all?

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All of them share some notion of differentiability...

Is there a common language to describe them all?

Could this language be tangent category theory?

Many geometries, one language

This is a tough question... and we don’t have an answer

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This is a tough question... and we don’t have an answer

BUT

Many geometries, one language

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BUT

Operads generate tangent categories

The main result of this talk

Many geometries, one language

This is a tough question... and we don't have an answer

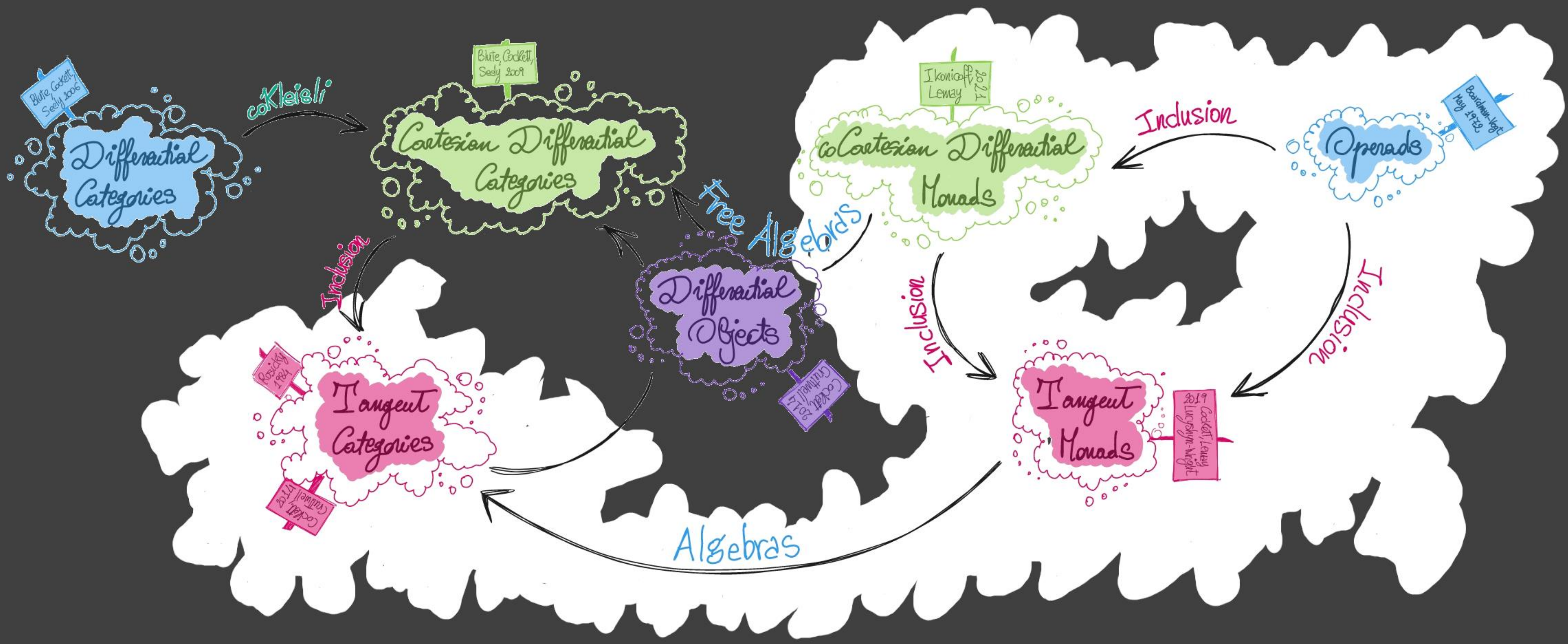
The main result of this talk

BUT

Operads generate geometrical theories

new models of geometries described with tangent categories

The world of differential categories



The program

Language for geometry

Tangent Categories

The program

Language for geometry

Tangent Categories

Monads that generate tangent cats

Tangent monads

The program

Language for geometry

Monads that generate tangent cats

Tangent Categories

Tangent monads

Operads and algebras

The program

Language for geometry

Tangent Categories

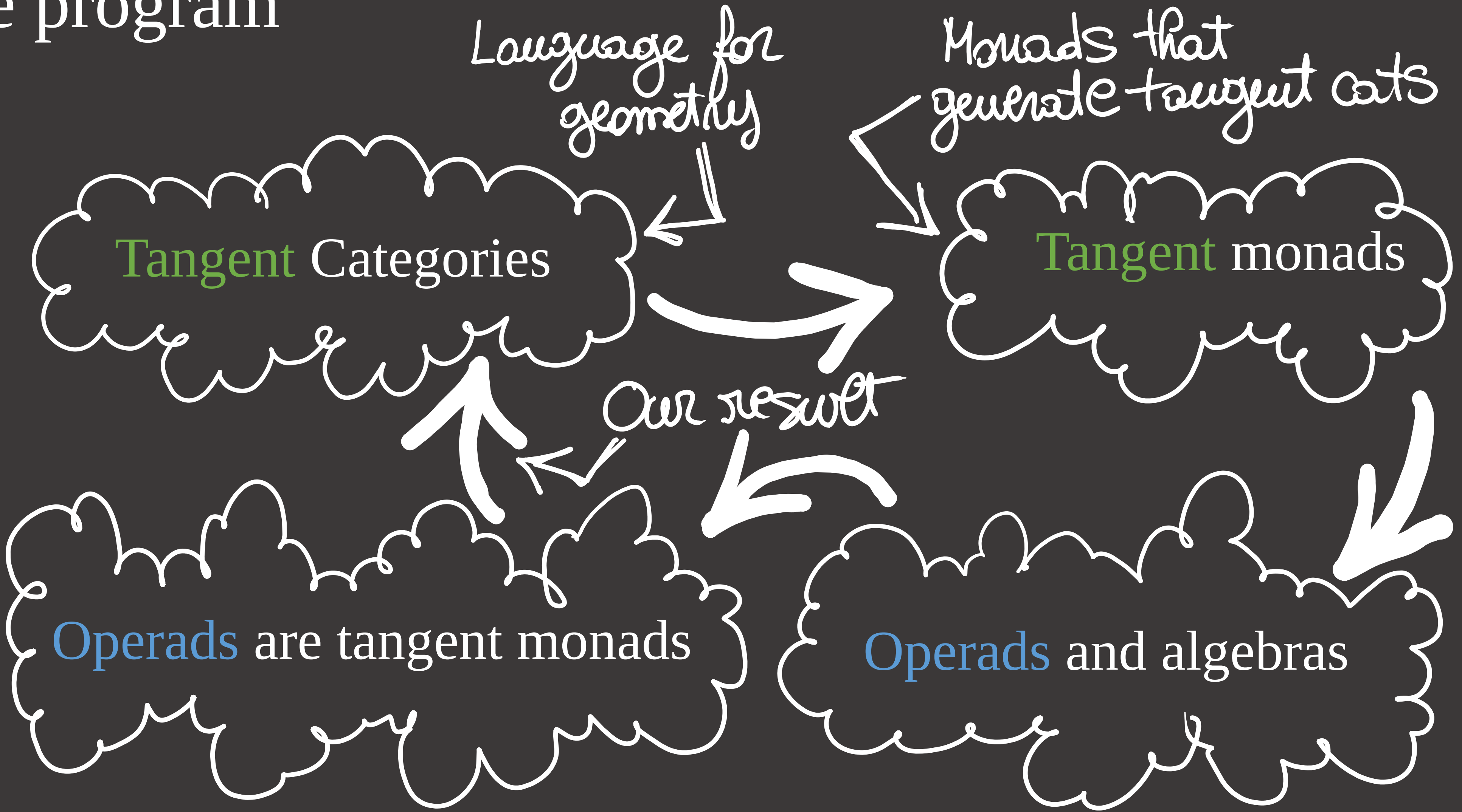
Monads that generate tangent cats

Tangent monads

Operads and algebras

Operads are tangent monads

The program


Language for geometry
Monads that generate tangent cats
Tangent Categories
Tangent monads
Our result
Operads are tangent monads
Operads and algebras

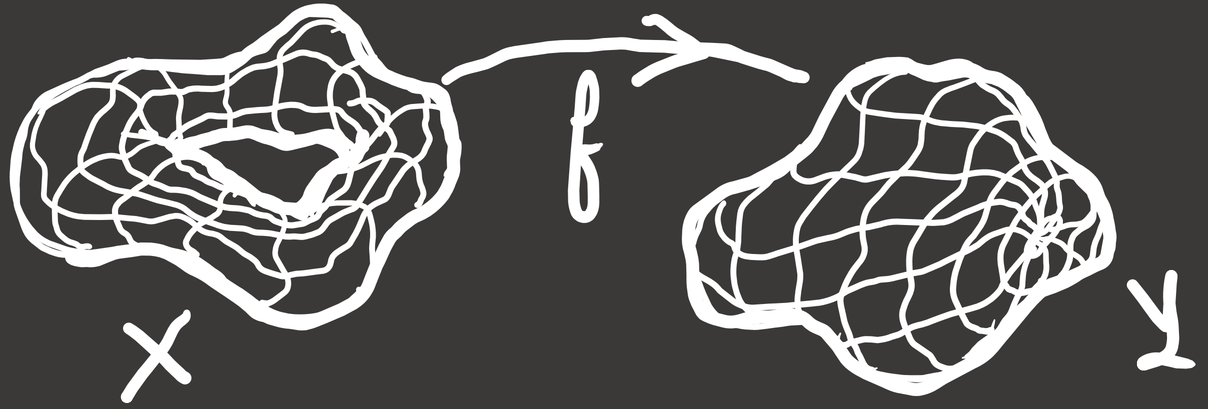
Tangent categories

Rosicky 1984
Cockett, Cruttwell 2014

"Tangent category theory
is a categorical language
for differential geometry."

Tangent categories

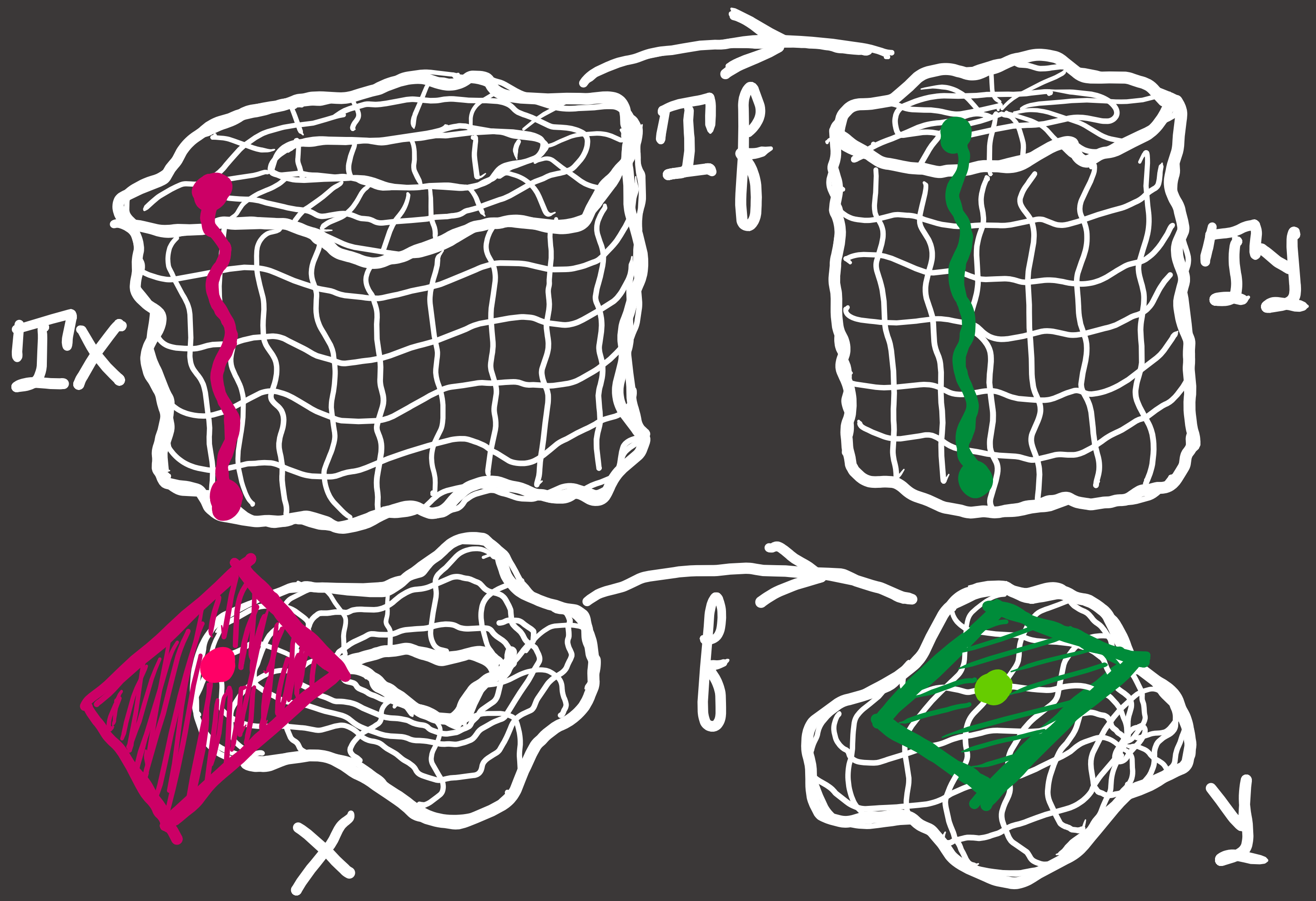
A category



Tangent categories

A category

An endofunctor T



Tangent categories

A category

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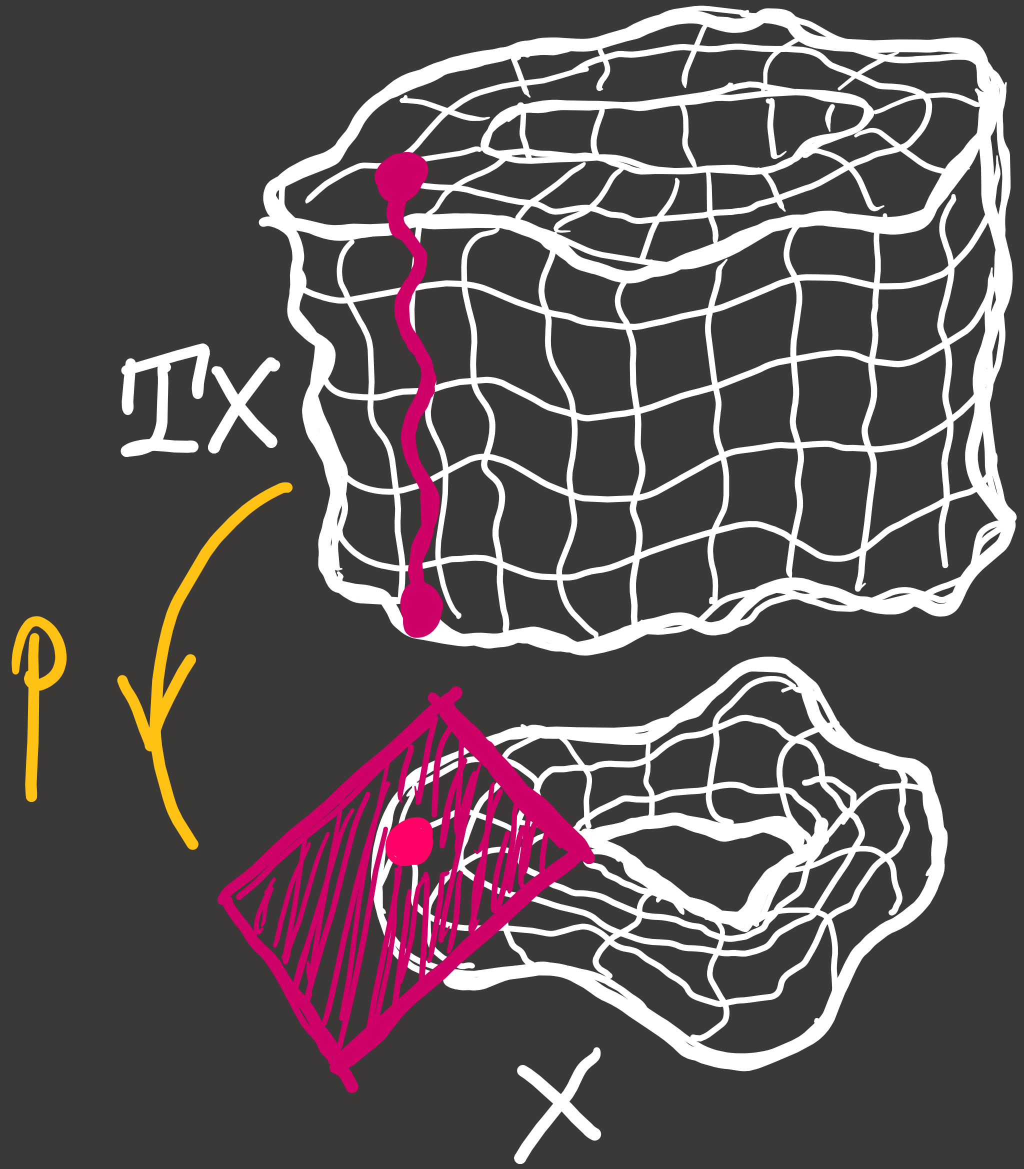
A projection p



Tangent categories

A category

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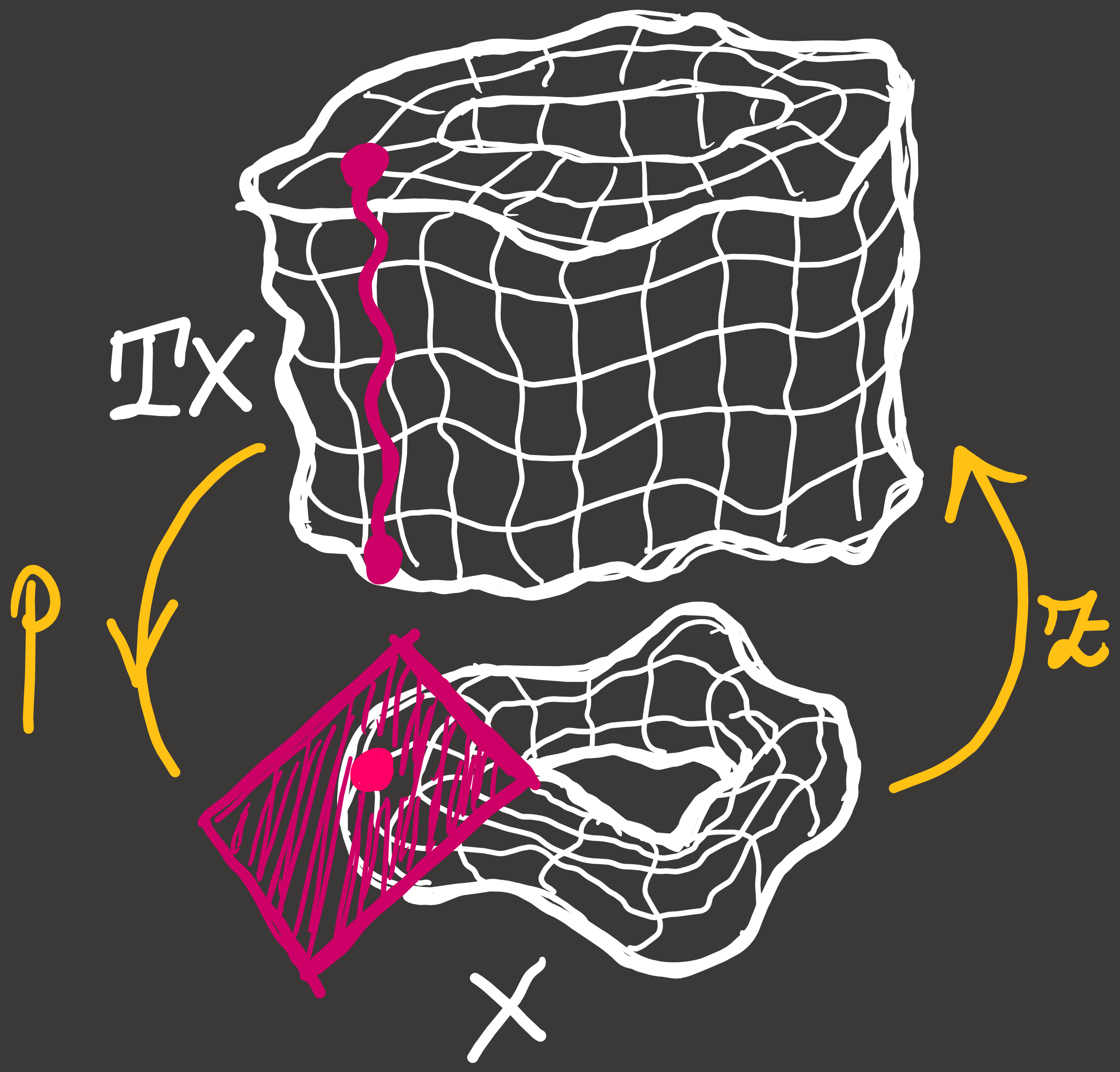
A zero morphism z



Tangent categories

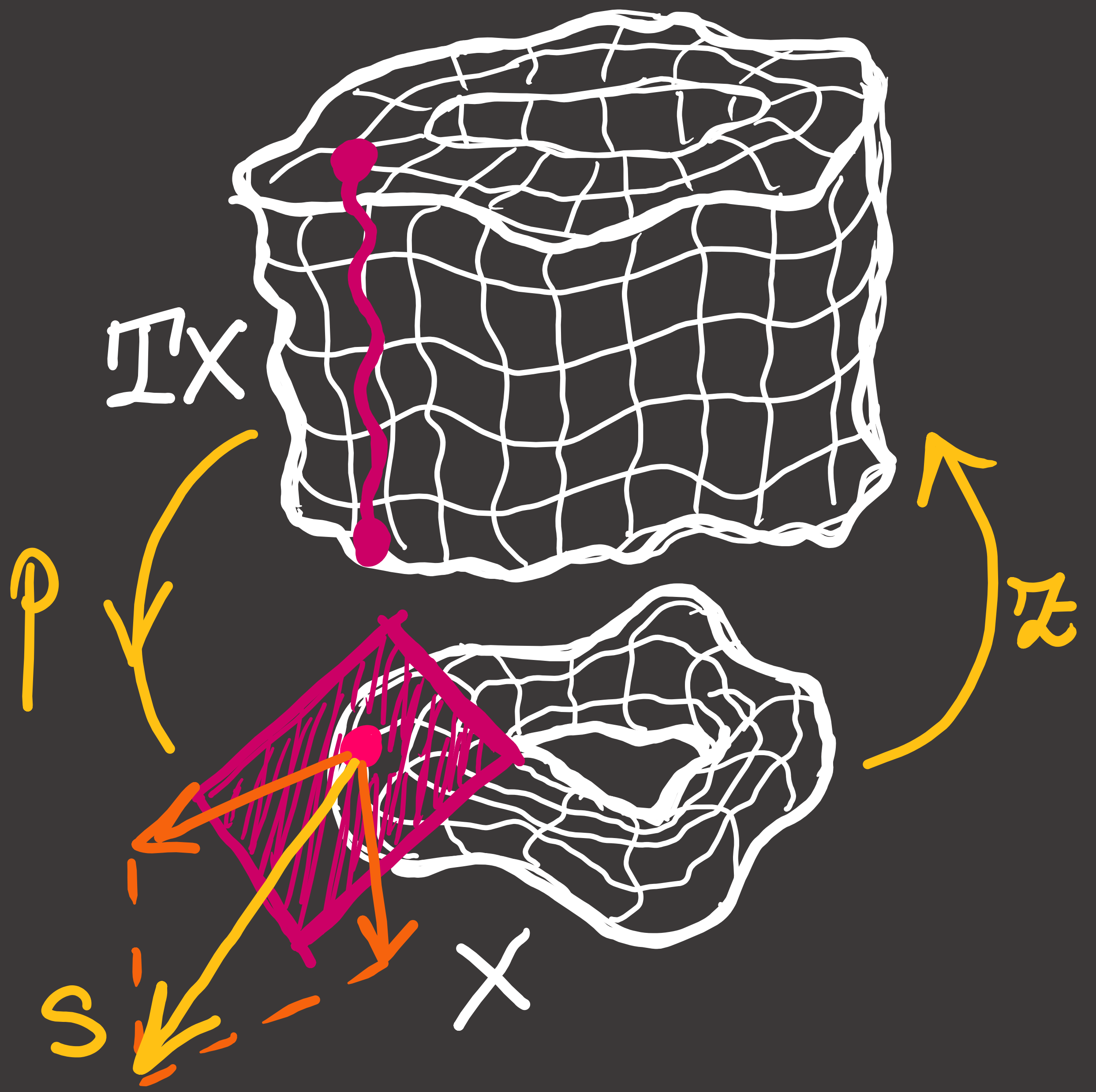
A category

An endofunctor T

A projection p

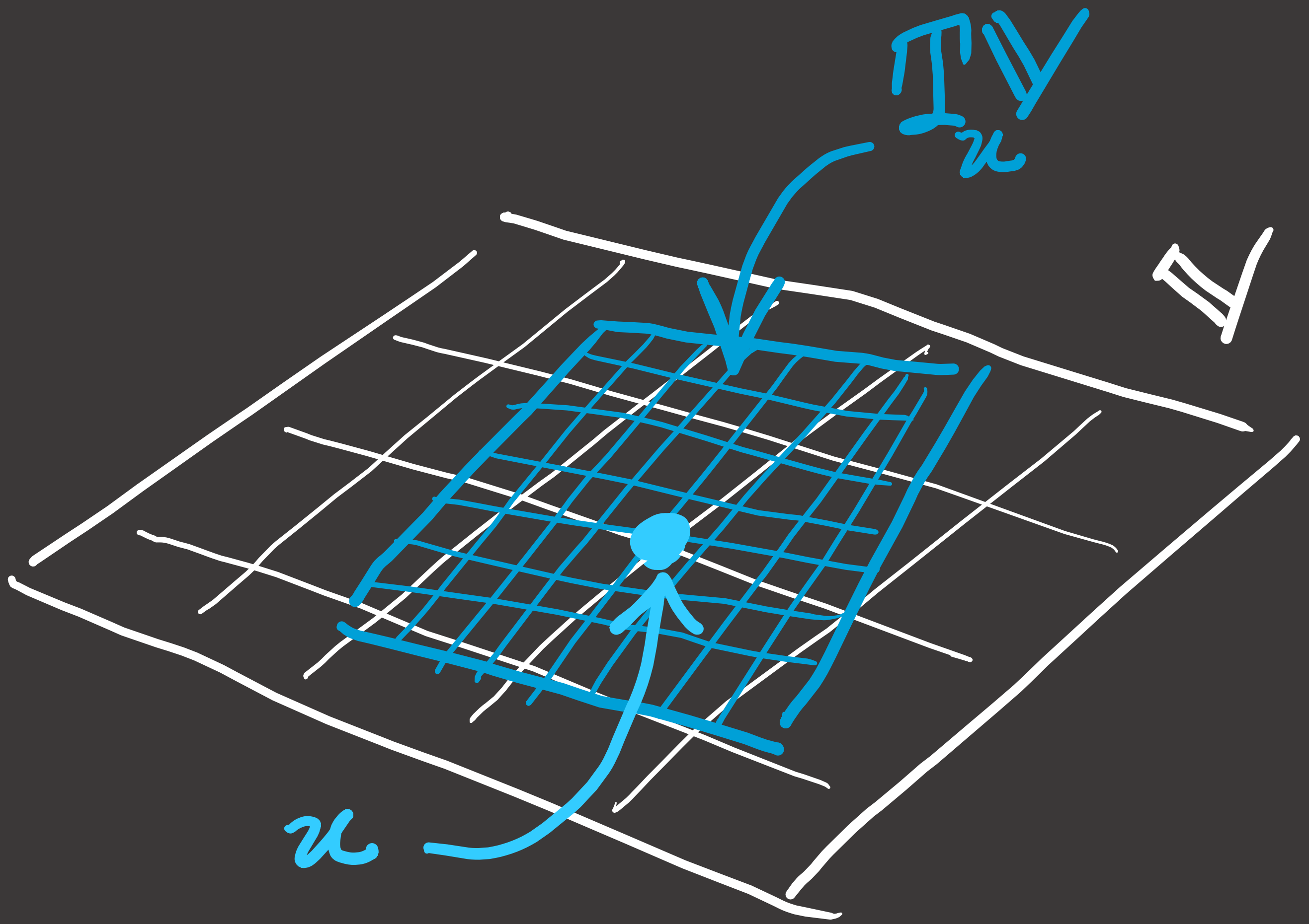
A zero morphism z

A sum morphism s



Tangent categories

A vertical lift l



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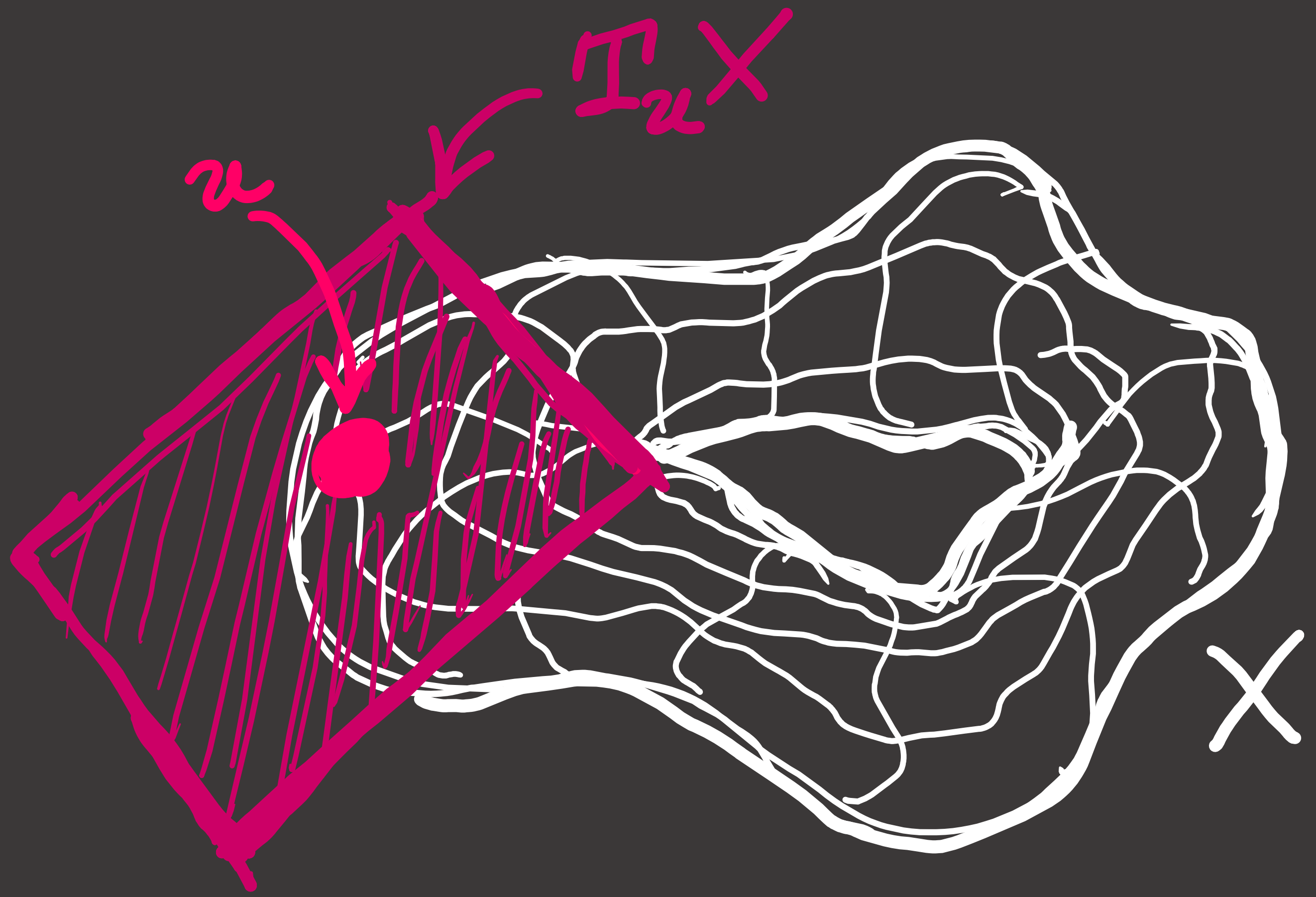

Tangent categories

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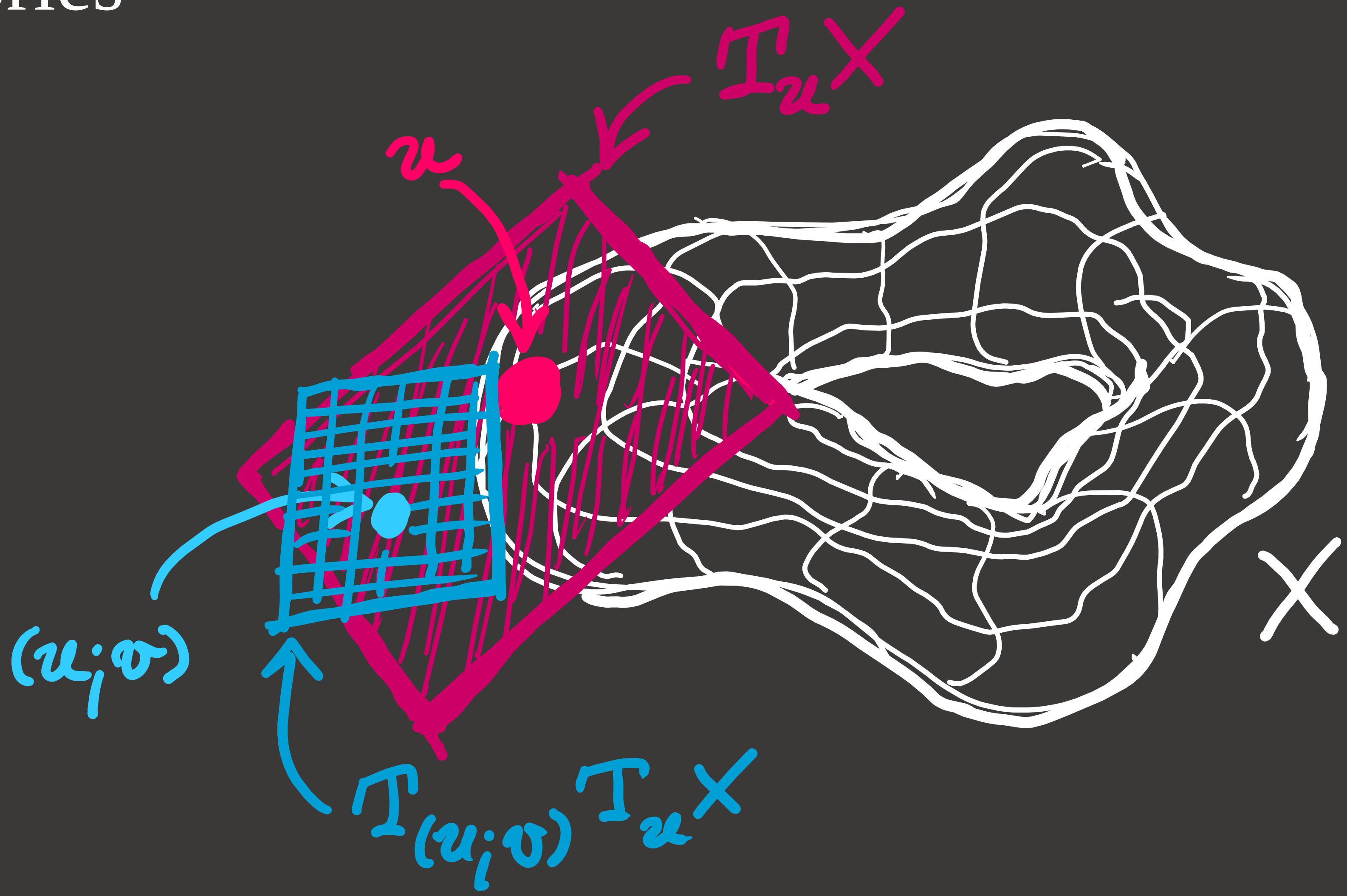
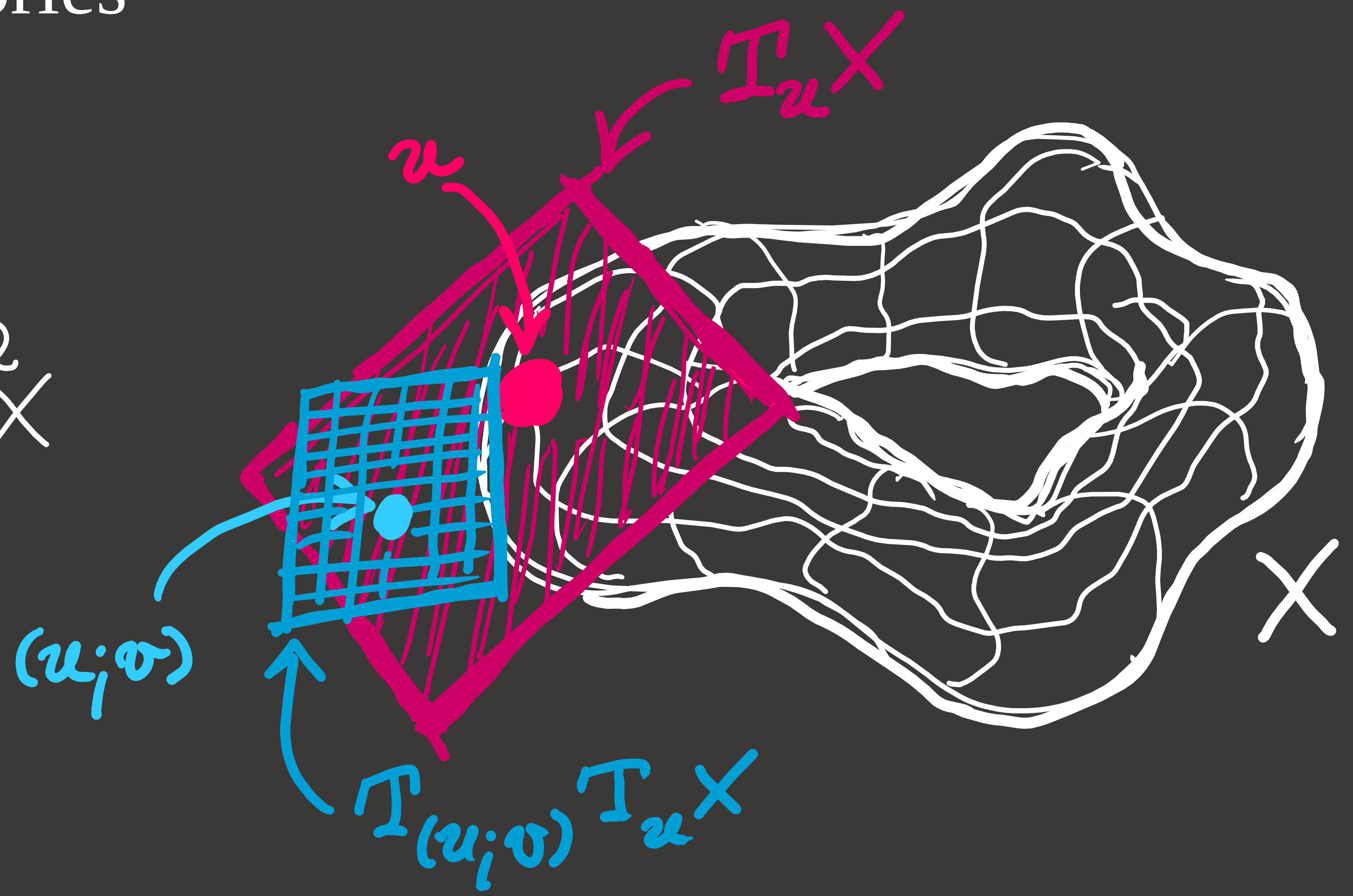


Tangent categories

A vertical lift l

$$\ell : TX \to T^2X$$



Tangent categories

A canonical flip

Idea! Symmetry of the Hessian matrix:

$\partial_u \partial_v = \partial_v \partial_u$

Tangent categories

A canonical flip:

Idea!
Symmetry of the Hessian matrix:
$\partial_u \partial_v = \partial_v \partial_u$

$$
\begin{tikzcd}
& T_2^2 X \arrow[d, "s_T"'] & \\
& T^2 X \arrow[d, bend left, "p_T"] & \\
T_2 X \arrow[r, "s"] & T X \arrow[u, bend left, "z_T"] \arrow[r, shift left, "p"] & X \arrow[l, shift left, "z"]
\end{tikzcd}
$$

Tangent categories

A canonical flip

Idea!
Symmetry of the Hessian matrix:
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$$
\begin{tikzcd}
& T_2^2 X \arrow[d, "s_T"'] & & T_2^2 X \arrow[d, "Ts"] \\
& T^2 X \arrow[dr, bend right, "p_T"] & & T^2 X \arrow[dl, bend left, "Tp"'] \\
T_2 X \arrow[rr, "s"] & & TX \arrow[ul, bend left, "z_T"] \arrow[ur, bend right, "Tz"'] \arrow[r, shift left, "p"] & X \arrow[l, shift left, "z"]
\end{tikzcd}
$$

Tangent categories

A canonical flip :

$c : T^2X \to T^2X$

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T^2_2X \arrow[d, "s_T"'] & & T^2_2X \arrow[d, "Ts"] \\
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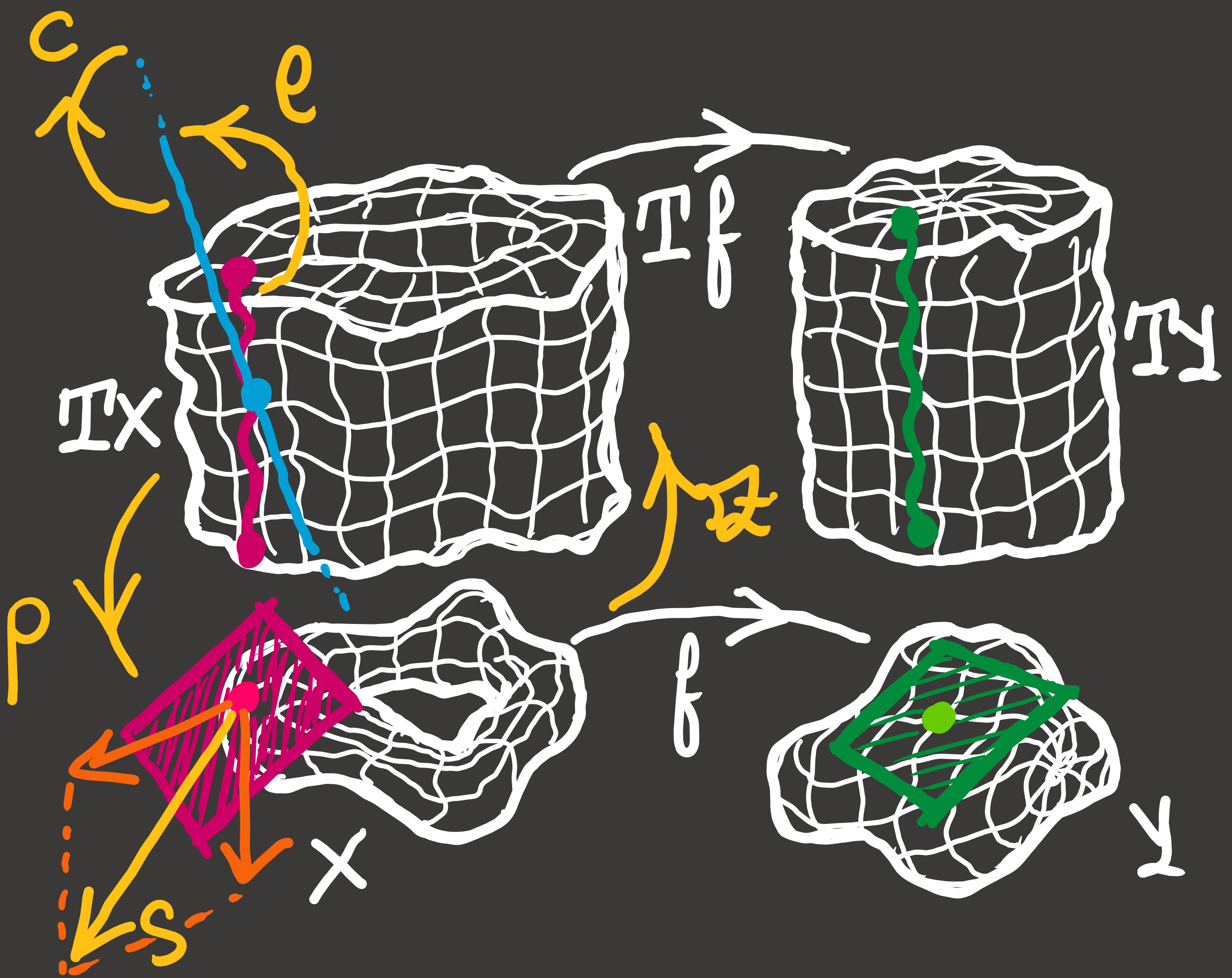
$$
\begin{tikzcd}
& T_2^2X \arrow[r, "c_2"'] \arrow[d, "s_T"'] & T_2^2X \arrow[d, "Ts"] & \\
& T^2X \arrow[r, "c"] \arrow[d, bend left, "p_T"] & T^2X \arrow[dl, bend left, "Tp"'] & \\
T_2X \arrow[r, "s"] & TX \arrow[u, bend left, "z_T"] \arrow[ur, bend right, "Tz"'] \arrow[r, shift left, "p"] & X \arrow[l, shift left, "z"] &
\end{tikzcd}
$$

Tangent categories

Differential geometry



Tangent categories

Products ≅ coProducts

Categories with biproducts

Every category with biproducts has a canonical tangent structure

$$TX := X \oplus X$$

Tangent monads

Cockett, Lemay, Lucyshyn-Wright 2020

“Tangent monads are machines that produce tangent categories.”

Tangent monads

A tangent monad is a monad is the 2-category of tangent categories

Tangent monads

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A tangent category

$(\mathbb{X}, \mathbb{T})$

category

tangent structure

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A tangent category

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A monad

$S : \mathbb{X} \to \mathbb{X}$

endofunctor

unit

$\eta : X \to SX$

multiplication

$\mu : S^2X \to SX$

Tangent monads

A tangent monad is a monad is the 2-category of tangent categories

A tangent category

A monad

A distributive law

$(\mathbb{X}, \mathbb{T})$

category

tangent structure

monad

$\lambda : \mathrm{S}\mathrm{T}X \to \mathrm{T}\mathrm{S}X$

tangent bundle functor

$\mathrm{S} : \mathbb{X} \to \mathbb{X}$

endofunctor

unit

$\eta : X \to \mathrm{S}X$

$\mu : \mathrm{S}^2X \to \mathrm{S}X$

multiplication

Tangent monads

Cockett, Lemay, Lucyshyn-Wright 2020

Theorem.
The Eilenberg-Moore category of a tangent monad is a tangent category.

Tangent monads

Cockett, Lemay, Lucyshyn-Wright 2020

Theorem.
The Eilenberg-Moore category of a tangent monad is a tangent category.

Algebras of the monad

$$\begin{array}{c}(\mathsf{Alg}_S, \mathbb{T}_S)\\ \Big\downarrow U \\ (\mathbb{X}, \mathbb{T})\end{array}$$

Forgetful functor

Lifted tangent structure

Base tangent category

Tangent monads

Cockett, Lemay, Lucyshyn-Wright 2020

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The Eilenberg-Moore category of a tangent monad is a tangent category.

Algebras of the monad

$$\begin{tikzcd} (\mathrm{Alg}_S, \mathbb{T}_S) \arrow[d, "U"] \\ (\mathbb{X}, \mathbb{T}) \end{tikzcd}$$

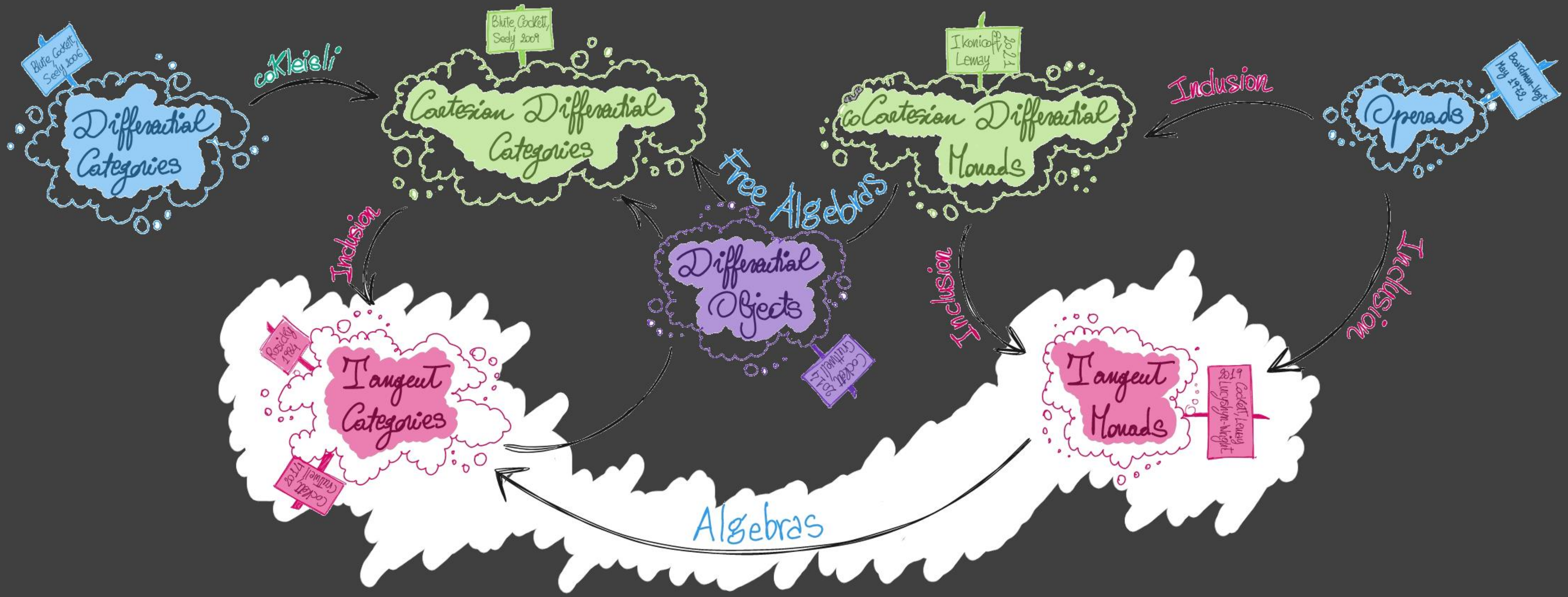
Forgetful functor

Lifted tangent structure

Base tangent category

$$\mathbb{T}_S(\overbrace{SA \xrightarrow{a} A}^{S\text{-algebra}}) := \underbrace{STA \xrightarrow{\lambda} TSA}_{\text{Distributive law}} \xrightarrow{Ta} TA$$

Tangent monads



Tangent monads

Ikonicoff, Lemay 2021
Ikonicoff, Lanfranchi, Lemay 2023

Products
||2
coProducts

Theorem.
Let (X, B) be the tangent category of biproducts. A tangent monad (S, λ) over (X, B) is equivalent to a coCartesian differential monad, that is a monad S equipped with a differential combinator:

$$\partial: SX \to S(X \oplus X)$$

Tangent monads

Tangent monads

Cockett, Lemay, Lucyshyn-Wright 2020

Theorem.
Let X be the category with biproducts. If (S, ∂) is a cCDM over X, then Alg_S is a tangent category.

Tangent monads

Cockett, Lemay, Lucyshyn-Wright 2020

Theorem.
Let X be the category with biproducts. If (S, ∂) is a cCDM over X, then Alg_S is a tangent category.

Moreover, if Alg_S has reflexive coequalizers then also Alg_S^{op} is a tangent category.

Operads and algebras

"Operads are
machines that produce
algebraic objects."

Operads and algebras

We just need Biproducts and limits & colimits

Take the symmetric monoidal category of vector spaces (this can be generalized)

Operads and algebras

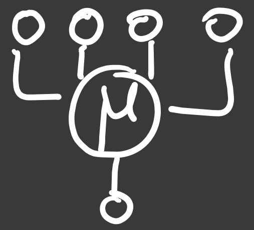
We just need Biproducts and limits & colimits

Take the symmetric monoidal category of vector spaces (this can be generalized)

An operad is:

- $\{\mathcal{P}(n)\}$

Vector spaces



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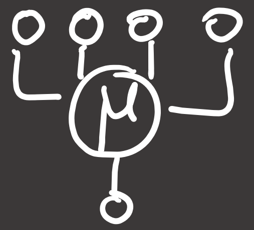

- $\{\mathcal{P}(n)\}$
- $1_{\mathcal{P}} \in \mathcal{P}(1)$

A unit

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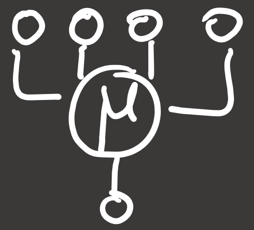

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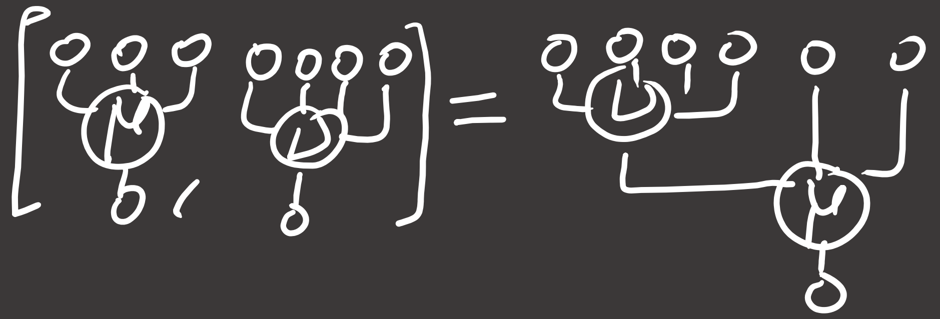
Partial compositions

- γ_k



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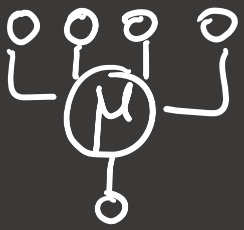
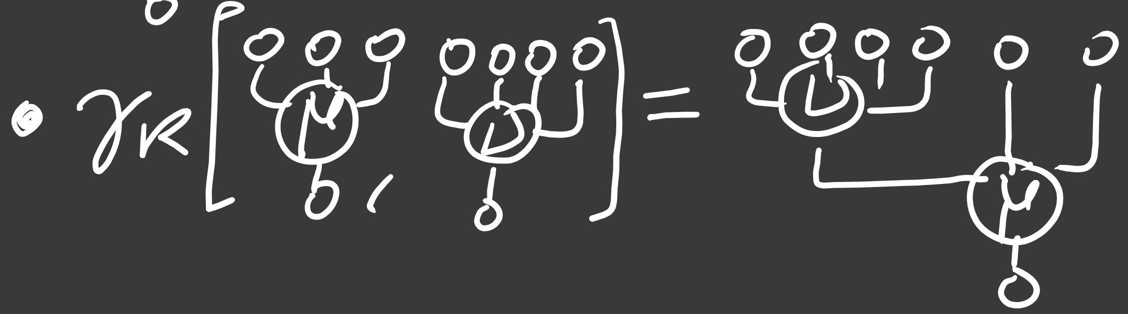
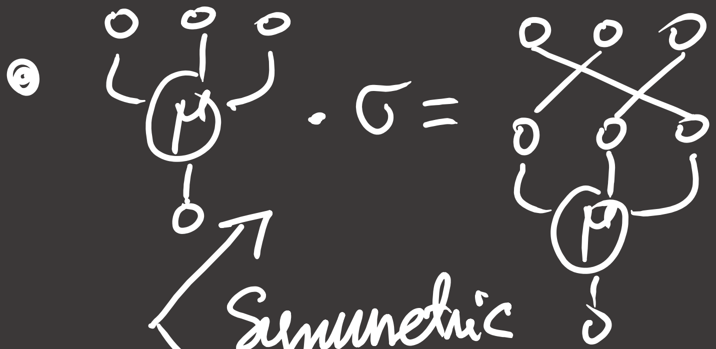
- $\{\mathcal{P}(n)\}$ Vector spaces 
-  $\in \mathcal{P}(1)$ A unit
-  Partial compositions
-  Symmetric actions

Operads and algebras

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Vector spaces

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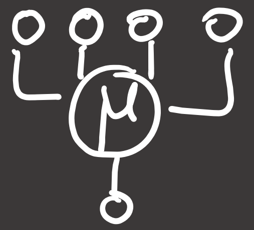
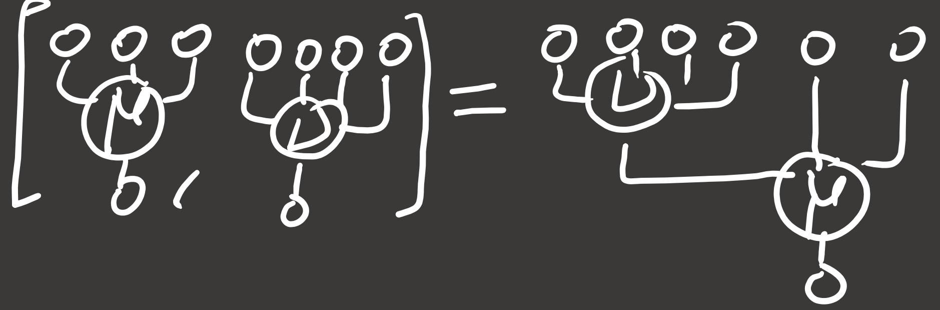
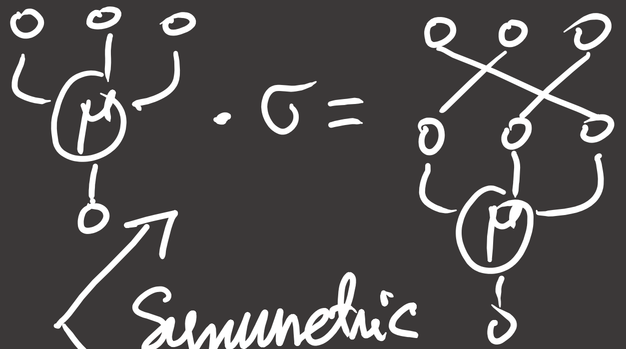


A unit

-  $\in \mathcal{P}(1)$

Partial compositions

- γ_k 

- 

Symmetric actions

this gives a monad!

Operads and algebras

We just need Biproducts and limits & colimits

Take the symmetric monoidal category of vector spaces (this can be generalized)

An algebra of an operad is:

- a vector space A

Operads and algebras

We just need Biproducts and limits & colimits

Take the symmetric monoidal category of vector spaces (this can be generalized)

An algebra of an operad is:

- a vector space A
- $\vartheta_n : \mathcal{P}(n) \otimes A^{\otimes n} \to A$

  $\vartheta_n(\mu; a_1, \dots, a_n) = \mu(a_1, \dots, a_n)$

Operads and algebras

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Take the symmetric monoidal category of vector spaces (this can be generalized)

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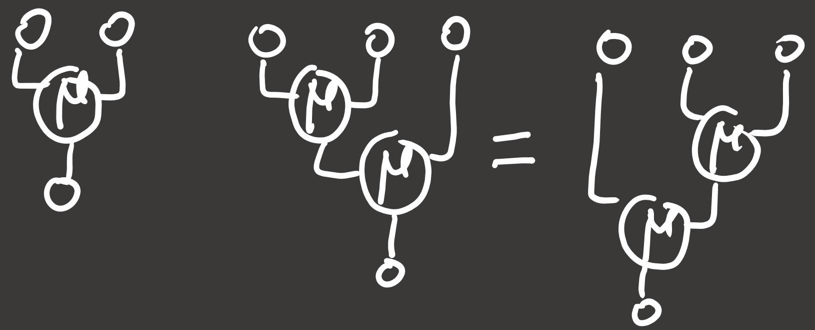
An algebra is a representation of the operad!

Operads and algebras: examples

Ass



Associative algebras

Operads and algebras: examples

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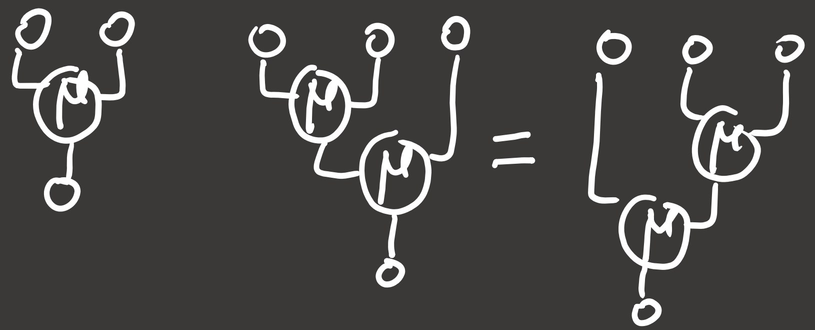
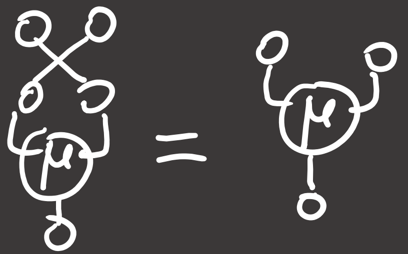


Associative algebras

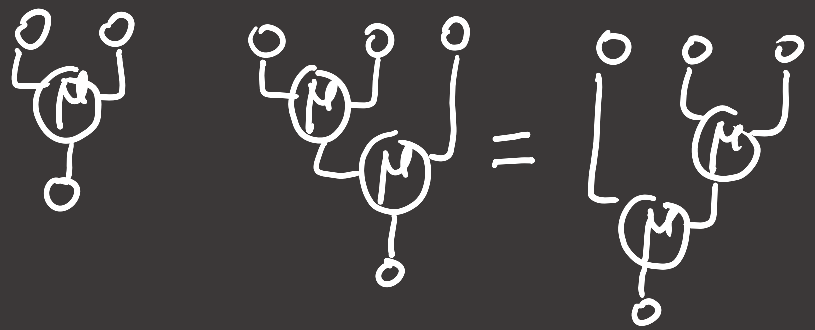
Com



Commutative algebras

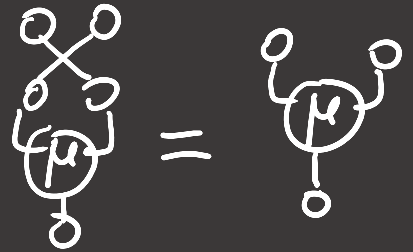
Operads and algebras: examples

Ass



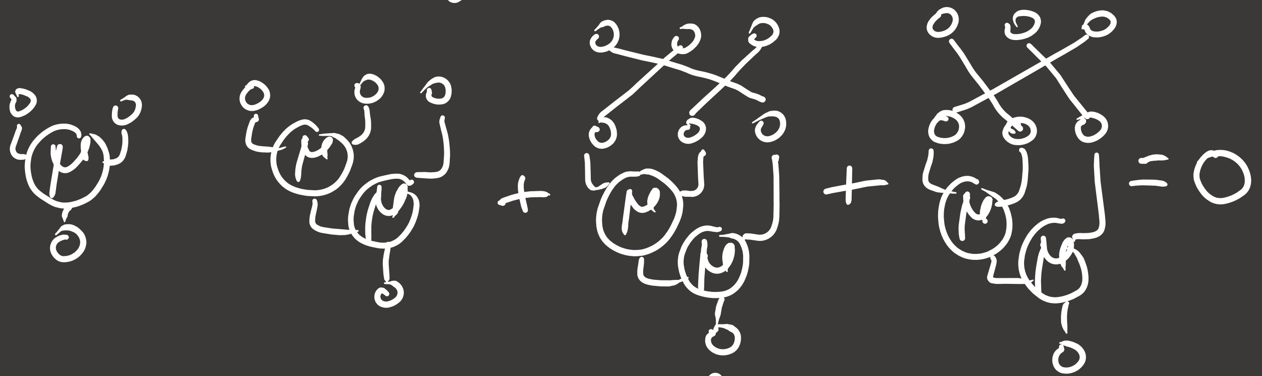
Associative algebras

Com



Commutative algebras

Lie



Lie algebras

Operads are tangent monads

"Operads are
machines that produce
tangent categories."

Operads are tangent monads

This is the new stuff

"Operads are machines that produce geometrical theories."

Operads are tangent monads

Ikonicoff, Lanfranchi, Lemay 2023

Theorem.
The monad associated to an operad is a coCartesian differential monad

Operads are tangent monads

Ikonicoff, Lanfranchi, Lemay 2023

$\mathcal{P} \leftarrow$ Operad

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Operads are tangent monads

Ikonicoff, Lanfranchi, Lemay 2023

Theorem.
The monad associated to an operad is a coCartesian differential monad

$\mathcal{P}$ Operad

$$S_{\mathcal{P}} X = \bigoplus_{n} \mathcal{P}(n) \otimes_{S_n} X^{\otimes n}$$

Monad associated to $\mathcal{P}$

Operads are tangent monads

Ikonicoff, Lanfranchi, Lemay 2023

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The monad associated to an operad is a coCartesian differential monad

$\mathcal{P}$ ← Operad

$$S_{\mathcal{P}} X = \bigoplus_n \mathcal{P}(n) \otimes_{\mathbb{S}_n} X^{\otimes n}$$

Monad associated to $\mathcal{P}$

Differential combinator

$$\partial : S_{\mathcal{P}} X \longrightarrow S_{\mathcal{P}}(X \oplus X)$$

$$\partial(\mu; x_1, \ldots, x_n) = \sum_{k=1}^{n} (\mu; (x_1, 0), \ldots, (0, x_k), \ldots, (x_n, 0))$$

Operads are tangent monads

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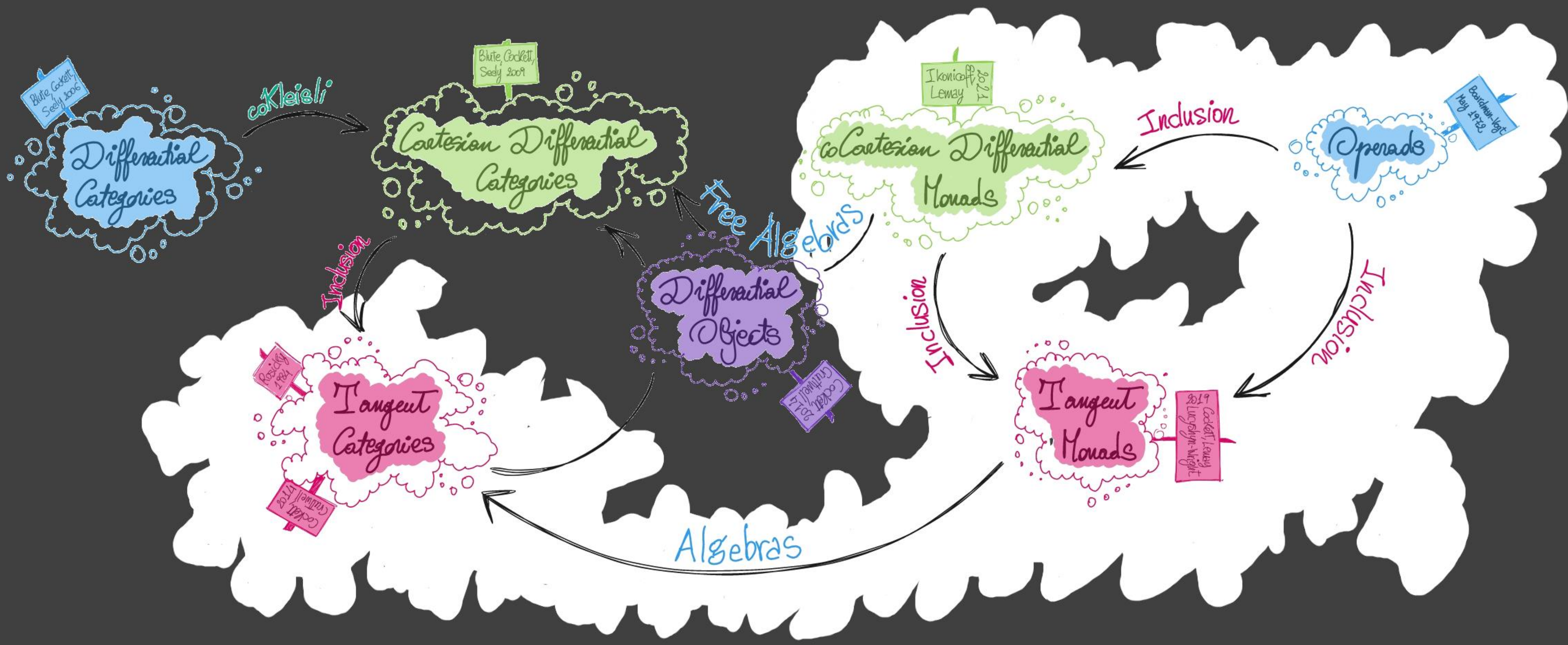
Operads are tangent monads



Operads generate tangent categories

Ikonicoff, Lanfranchi, Lemay 2023

Theorem.
Let P be an operad.
The categories Alg_P and Alg_P^{op} are tangent categories.

The main result of this talk!

Operads generate tangent categories

Ikonicoff, Lanfranchi, Lemay 2023

Theorem.
Let P be an operad. The tangent bundle functor $T\colon Alg_P \to Alg_P$ is the functor so defined:

$$TA \coloneqq A \ltimes A$$

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$A \ltimes A$

$A \oplus A$ as a vector space

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$A \ltimes A$

$A \oplus A$ as a vector space

$$\mu((a_1,b_1),\dots,(a_m,b_m)) = \sum_{k=1}^{m} \big(\mu(a_1,\dots,a_m),\mu(a_1,\dots,b_k,\dots,a_m)\big)$$

Operads generate tangent categories

Ikonicoff, Lanfranchi, Lemay 2023

Theorem.
Let P be an operad. The tangent bundle functor $T^\circ\colon Alg_P^{op} \to Alg_P^{op}$ is the functor so defined:

$$T^\circ A \coloneqq Free_A \Omega_A$$

Operads generate tangent categories

Ikonicoff, Lanfranchi, Lemay 2023

Theorem.
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$$T^\circ A := Free_A \Omega_A$$

$$T^\circ A = Free_A\, \Omega_A$$

Operads generate tangent categories

Ikonicoff, Lanfranchi, Lemay 2023

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A-module of Kähler differentials of A

$$T^\circ A = Free_A \Omega_A$$

Operads generate tangent categories

Ikonicoff, Lanfranchi, Lemay 2023

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Let P be an operad. The tangent bundle functor $T^\circ\colon Alg_P^{op} \to Alg_P^{op}$ is the functor so defined:

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$$T^\circ A = \underbrace{Free_A}_{\text{Free } A\text{-algebra of an } A\text{-module}} \overbrace{\Omega_A}^{A\text{-module of Kähler differentials of } A}$$

Operads generate tangent categories

Ikonicoff, Lanfranchi, Lemay 2023

Theorem.
Let P be an operad. The tangent bundle functor $T°: Alg_P^{op} \to Alg_P^{op}$ is the functor so defined:

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$\mu(a_1, \ldots, a_n) = \mu_A(a_1, \ldots, a_n)$

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$T^\circ A = Free_A\, \Omega_A$

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$da,\ a \in A$

$\mu(a_1, \dots, a_n) = \mu_A(a_1, \dots, a_n)$

$d(\mu(a_1, \dots, a_n)) =$

$\sum_{k=1}^{n} \mu(a_1, \dots, da_k, \dots, a_n)$

New models of tangent categories

Cruttwell, Lemay 2023

Commutative & Unital Algebras

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Commutative & Unital Algebras

Cruttwell and Lemay showed that
$(cAlg^{op}, T^{\circ})$ describes
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$$T^{\circ}A = \overbrace{Sym_A}^{\text{symmetric algebra}}\underbrace{\Omega_A}_{\text{Kähler differentials}}$$

New models of tangent categories

Ginzburg 2005

Associative & Unital Algebras

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We believe that (Alg^{op}, T°) describes such geometry.

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tensor algebra

$$T^{\circ}A = \underbrace{\overbrace{\mathrm{Tens}_A}}\ \underbrace{\Omega_A^{nc}}$$

non-commutative Kähler differentials

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non-commutative Kähler differentials

this is the 1st model of non-commutative geometry described with tangent categories

What next?

What about Lie algebras?
or Poisson algebras...

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or Poisson algebras...

Differential Bundles?
Connections? Cohomology?
Differential equations?

What next?

What about Lie algebras?
or Poisson algebras...

Differential objects = $\mathrm{Mod}^{op}_{\mathcal{O}(1)}$

Differential Bundles?
Connections?
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Differential equations?

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Differential op

Objects = $Mod_{D(1)}$

Differential Bundles?

Connections?

Cohomology?

Differential equations?

∞-Operads $\to$ ∞-Geometry?

Functoriality?

Thanks.

The Rosický Tangent Categories of Algebras over an Operad
https://arxiv.org/abs/2303.05434