Birkhoff's variety theorem for relative algebraic theories

Yuto Kawase

Kyoto University

CT 2023

Abstract

Main theorem I (informal)

(Single-sorted) algebraic theories = finitary monads on $\mathbf{Set}$
$\downarrow$ generalize
$\mathscr{A}$-relative algebraic theories = finitary monads on $\mathscr{A}$ ($\mathscr{A}$:LFP category)

Here, we define "$\mathscr{A}$-relative algebraic theory" via partial Horn theory.

Main theorem II (informal)

Birkhoff's variety theorem relative to $\mathbf{Set}$
$\downarrow$ generalize
Birkhoff's variety theorem relative to $\mathscr{A}$ ($\mathscr{A}$:LFP category)

Contents

Finitary monads and algebraic theories

Definition

A (single-sorted) algebraic theory, which is also called an equational theory, consists of:

- a set Ω of *operations*,
- for each $\omega \in \Omega$, a natural number $\mathrm{ar}(\omega) \in \mathbb{N}$,
- a set E of *equations*.

Definition

Let (Ω, E) be a single-sorted algebraic theory. A model of (Ω, E) consists of:

- a set A,
- for each $\omega \in \Omega$, a mapping $[\![\omega]\!]_A : A^{\mathrm{ar}(\omega)} \to A$.

satisfying all equations in E.

Finitary monads and algebraic theories

There is a classical result about the correspondence between algebraic theories and finitary monads.

Fact

The following two classes of categories coincide.

- *Categories of models of single-sorted algebraic theories*
- *Eilenberg-Moore categories of finitary monads on* $\mathbf{Set}$

$$\text{single-sorted algebraic theories} = \text{finitary monads on } \mathbf{Set}!$$

Finitary monads and algebraic theories

More is true:

- single-sorted algebraic theories = finitary monads on $\mathbf{Set}$
- S-sorted algebraic theories = finitary monads on $\mathbf{Set}^S$
- "ordered" algebraic theories = finitary monads on $\mathbf{Pos}$ [Adámek, Ford, Milius, Schröder, 2021]

In this talk,

$$\mathbf{Set},\ \mathbf{Set}^S,\ \mathbf{Pos} \quad \overset{\text{generalize}}{\rightsquigarrow} \quad \text{locally finitely presentable (LFP) categories}$$

Locally finitely presentable categories and partial Horn logic

Definition

A category $\mathscr{A}$ is locally finitely presentable (LFP)

$\overset{\text{def}}{\Leftrightarrow}$ It is cocomplete and has a set $\mathcal{G}$ of f.p.objects such that every object is a filtered colimit of objects from $\mathcal{G}$.

A purely categorical sentence

LFP categories are characterized as categories of models of various kinds of logical theory.

Fact

The following classes of categories coincide:

- *LFP categories,*
- *Categories of models of cartesian theories,*
- *Categories of models of essentially algebraic theories,*
- *Categories of models of partial Horn theories.*

Locally finitely presentable categories and partial Horn logic

A small category $\mathscr{C}$ consists of...

- a set $\mathrm{ob}\mathscr{C}$ ("objects"),
- a set $\mathrm{mor}\mathscr{C}$ ("morphisms"),
- a function $\mathrm{id}\colon \mathrm{ob}\mathscr{C} \to \mathrm{mor}\mathscr{C}$ ("identities"),
- a function $\mathrm{d}\colon \mathrm{mor}\mathscr{C} \to \mathrm{ob}\mathscr{C}$ ("domain"),
- a function $\mathrm{c}\colon \mathrm{mor}\mathscr{C} \to \mathrm{ob}\mathscr{C}$ ("codomain"), and
- a partial function $\circ\colon \mathrm{mor}\mathscr{C} \times \mathrm{mor}\mathscr{C} \rightharpoonup \mathrm{mor}\mathscr{C}$ ("composition").

- We can define "the theory of small categories" as a partial Horn theory.
- Partial Horn theory = a logical theory which can deal with partial functions (and relations).

Locally finitely presentable categories and partial Horn logic

We introduce *partial Horn theory*.

Definition

A multi-sorted first-order signature (or S-sorted signature) Σ consists of:

- a set S of sorts,
- a set Σ_{f} of function symbols,
- a set Σ_{r} of relation symbols

such that

- for each $f \in \Sigma_{\mathrm{f}}$ an arity $f : s_1 \times \cdots \times s_n \to s \, (s_i, s \in S)$ is given,
- for each $R \in \Sigma_{\mathrm{r}}$ an arity $R : s_1 \times \cdots \times s_n \, (s_i \in S)$ is given.

Locally finitely presentable categories and partial Horn logic

Let Σ be an S-sorted signature.

- A term $\tau ::= x \mid f(\tau_1, \ldots, \tau_n)$, where $f \in \Sigma_{\mathrm{f}}$;
- A Horn formula $\varphi ::= \top \mid \varphi \wedge \varphi' \mid \tau = \tau' \mid R(\tau_1, \ldots, \tau_n)$, where $R \in \Sigma_{\mathrm{r}}$;
- A context $\cdots$ $\vec{x} = (x_1, \ldots, x_n)$ (a finite tuple of distinct variables).

The notation $\vec{x}.\varphi$ [resp. $\vec{x}.\tau$] means that all variables of φ [τ] are in the context $\vec{x}$. (*Horn formula [term]-in-context*)

Definition

1. A Horn sequent over Σ is an expression of the form

$$\varphi \vdash^{\vec{x}} \psi \qquad (\text{“}\varphi \text{ implies } \psi\text{”})$$

   (φ, ψ are Horn formulas over Σ in the same context $\vec{x}$.)
2. A partial Horn theory $\mathbb{T}$ over Σ is a set of Horn sequents over Σ.

What is the difference between ordinary Horn theory and partial Horn theory?

$\rightsquigarrow$ It lies in the concept of models.

	(ordinary) Horn theory	partial Horn theory
Axiom	Horn sequent $\varphi \vdash^{\vec{x}} \psi$	Horn sequent $\varphi \vdash^{\vec{x}} \psi$
Interpretation of function symbols	total map $M_{\vec{s}} \xrightarrow{[\![f]\!]_M} M_s$	partial map $M_{\vec{s}} \xrightharpoonup{[\![f]\!]_M} M_s$
Interpretation of relation symbols	subset $[\![R]\!]_M \subseteq M_{\vec{s}}$	subset $[\![R]\!]_M \subseteq M_{\vec{s}}$
Validity of φ	"φ holds."	"All terms in φ are defined and φ holds."
Validity of $\varphi \vdash^{\vec{x}} \psi$	"If φ holds then ψ holds."	"If all terms in φ are defined and φ holds, then all terms in ψ are defined and ψ holds."

Especially,

An equation $\tau = \tau$ holds iff the value of the partial map $[\![\tau]\!]_M$ is defined.

So, we will use the abbreviation $\tau\downarrow$ for $\tau = \tau$.

Locally finitely presentable categories and partial Horn logic

Notation

Let $\mathbb{T}$ be a partial Horn theory over an S-sorted signature Σ.

$\mathbb{T}\text{-PMod}$: the category of (partial) models of $\mathbb{T}$

Fact (well-known)

A category $\mathscr{A}$ is LFP iff there exists a partial Horn theory $\mathbb{T}$ satisfying $\mathscr{A} \simeq \mathbb{T}\text{-PMod}$.

Locally finitely presentable categories and partial Horn logic

Example (small categories)

We can define the partial Horn theory $\mathbb{T}_{\mathrm{cat}}$ of small categories as follows:
The $S := \{\mathrm{ob}, \mathrm{mor}\}$-sorted signature Σ_{cat} consists of:

$$\mathrm{id} : \mathrm{ob} \to \mathrm{mor}, \quad \mathrm{d} : \mathrm{mor} \to \mathrm{ob}, \quad \mathrm{c} : \mathrm{mor} \to \mathrm{ob}, \quad \circ : \mathrm{mor} \times \mathrm{mor} \to \mathrm{mor}.$$

The partial Horn theory $\mathbb{T}_{\mathrm{cat}}$ over Σ_{cat} consists of:

$$\top \xmapsto{x : \mathrm{ob}} \mathrm{id}(x){\downarrow}, \qquad (\text{ id is total. })$$

$$\top \xmapsto{f : \mathrm{mor}} \mathrm{d}(f){\downarrow} \wedge \mathrm{c}(f){\downarrow}, \qquad (\text{ d and c are total. })$$

$$\mathrm{d}(g) = \mathrm{c}(f) \xmapsto{g, f : \mathrm{mor}} (g \circ f){\downarrow},$$
$$(g \circ f){\downarrow} \xmapsto{g, f : \mathrm{mor}} \mathrm{d}(g) = \mathrm{c}(f), \qquad (\ g \circ f \text{ is defined iff } \mathrm{d}(g) = \mathrm{c}(f).\ )$$

and so on.
$\rightsquigarrow$ We have $\mathbb{T}_{\mathrm{cat}}\text{-PMod} \cong \mathbf{Cat}$.

Relative algebraic theories

In this part, we fix a partial Horn theory $\mathbb{S}$ over an S-sorted signature Σ. Now, we present a new "algebraic concept" relative to $\mathbb{S}$.

Definition ([K])

A $\mathbb{S}$-relative signature consists of:

- a set Ω of operators;
- for each $\omega \in \Omega$, a Horn formula-in-context $\vec{x}.\varphi$, written $\mathrm{ar}(\omega)$ and called *arity of* ω;
- for each $\omega \in \Omega$, a sort $s \in S$, written $\mathrm{type}(\omega)$ and called *type of* ω.

Given $\mathbb{S}$-relative signature Ω, we can extend Σ to an S-sorted signature $\Sigma + \Omega$ by adding $\omega \in \Omega$ as a function symbol $\omega\colon s_1 \times \cdots \times s_n \to s$, where $\mathrm{ar}(\omega) = (x_1{:}s_1, \ldots, x_n{:}s_n).\varphi$ and $\mathrm{type}(\omega) = s$.

Definition ([K])

1. A Horn sequent $\varphi \vdash_{\vec{x}} \psi$ over $\Sigma + \Omega$ is called an $\mathbb{S}$-relative judgement if φ is over Σ.
2. An $\mathbb{S}$-relative algebraic theory consists of:
   - an $\mathbb{S}$-relative signature Ω,
   - a set E of $\mathbb{S}$-relative judgements.

Relative algebraic theories

$\mathbb{S}$: a partial Horn theory over an S-sorted signature Σ.

Definition ([K])

Let Ω be an $\mathbb{S}$-relative signature. An Ω-algebra $\mathbb{A}$ consists of:

- a partial $\mathbb{S}$-model A,
- for each $\omega \in \Omega$, a total map $[\![\omega]\!]_{\mathbb{A}} : [\![\mathrm{ar}(\omega)]\!]_A \to A_{\mathrm{type}(\omega)}$.

Definition ([K])

An Ω-algebra $\mathbb{A}$ is called a model of an $\mathbb{S}$-relative algebraic theory (Ω, E) if all $\mathbb{S}$-relative judgements belonging to E are valid in $\mathbb{A}$.

Notation

Let (Ω, E) be an $\mathbb{S}$-relative algebraic theory.

$Alg(\Omega, E)$: the category of models of (Ω, E) and $(\Sigma + \Omega)$-homomorphisms

$\mathbb{S}$: a partial Horn theory

	(Multi-sorted) algebraic theory (Ω, E)	$\mathbb{S}$-relative algebraic theory (Ω, E)
Base category	$\mathbf{Set}^S$	$\mathbb{S}\text{-PMod}$
Operator	$s_1 \times \cdots \times s_n \xrightarrow{\omega} s$	$(x_1{:}s_1, \ldots, x_n{:}s_n).\varphi \xrightarrow{\omega} s$
Axiom	equation $\tau = \tau'$	$\mathbb{S}$-relative judgement $\varphi \overset{\vec{x}}{\longmapsto} \psi$

$$Alg(\Omega, E) \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Set}^S \quad (F \dashv U)$$

$$Alg(\Omega, E) \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{S}\text{-PMod} \quad (F \dashv U)$$

In $\mathbb{S}$-relative algebraic theory ...

- Each operator $\omega \in \Omega$ needs not be total, but its domain must be defined by "$\mathbb{S}$'s language."
- We can use (Horn) implications as axioms, but its precondition must not contain any operator $\omega \in \Omega$. Preconditions must be written in "$\mathbb{S}$'s language."

Relative algebraic theories

$\mathbb{S}$: a partial Horn theory

Main theorem I [K]

The following are equivalent for a category $\mathscr{C}$.

1. $\mathscr{C}$ is finitary monadic over $\mathbb{S}\text{-PMod}$, i.e., there exists a finitary monad T over $\mathbb{S}\text{-PMod}$ satisfying $\mathscr{C} \simeq \mathbb{S}\text{-PMod}^T$.
2. $\mathscr{C}$ is a category of models of an $\mathbb{S}$-relative algebraic theory, i.e., there exists an $\mathbb{S}$-relative algebraic theory (Ω, E) satisfying $\mathscr{C} \simeq Alg(\Omega, E)$.

$\mathbb{S}$-relative algebraic theories = finitary monads on $\underline{\mathbb{S}\text{-PMod}}$ (an arbitrary LFP category)

↑ generalize

(single-sorted) algebraic theories = finitary monads on $\mathbf{Set}$

Corollary (well-known)

Cat *is finitary monadic over* **Quiv** *(the category of quivers, or directed graphs).*

Proof.

Define the partial Horn theory $\mathbb{S}_{\mathrm{quiv}}$ of quivers as follows:

$$S_{\mathrm{quiv}} := \{\mathrm{e}, \mathrm{v}\}, \quad \Sigma_{\mathrm{quiv}} := \{\mathrm{s}, \mathrm{t} : \mathrm{e} \to \mathrm{v}\}, \quad \mathbb{S}_{\mathrm{quiv}} := \{\top \overset{f:\mathrm{e}}{\vdash} \mathrm{s}(f)\downarrow \wedge\, \mathrm{t}(f)\downarrow\}.$$

Then, we can define an $\mathbb{S}_{\mathrm{quiv}}$-relative algebraic theory (Ω, E) such that $Alg(\Omega, E) \simeq \mathbf{Cat}$.

$$\Omega := \{\circ, \mathrm{id}\};$$

$$\mathrm{ar}(\circ) := (g, f : \mathrm{e}).\mathrm{s}(g) = \mathrm{t}(f), \quad \mathrm{type}(\circ) := \mathrm{e};$$

$$\mathrm{ar}(\mathrm{id}) := (x : \mathrm{v}).\top, \quad \mathrm{type}(\mathrm{id}) := \mathrm{e};$$

$$E := \left\{ \begin{array}{c} \top \overset{x:\mathrm{v}}{\vdash} \mathrm{s}(\mathrm{id}(x)) = x \wedge \mathrm{t}(\mathrm{id}(x)) = x, \\ \mathrm{s}(g) = \mathrm{t}(f) \overset{g,f:\mathrm{e}}{\vdash} \mathrm{s}(g \circ f) = \mathrm{s}(f) \wedge \mathrm{t}(g \circ f) = \mathrm{t}(g), \\ \top \overset{f:\mathrm{e}}{\vdash} f \circ \mathrm{id}(\mathrm{s}(f)) = f \wedge \mathrm{id}(\mathrm{t}(f)) \circ f = f, \\ \mathrm{s}(h) = \mathrm{t}(g) \wedge \mathrm{s}(g) = \mathrm{t}(f) \overset{h,g,f:\mathrm{e}}{\vdash} (h \circ g) \circ f = h \circ (g \circ f) \end{array} \right\}$$

Our main theorem finishes the proof. □

Fact (Birkhoff's variety theorem)

(Ω, E): a single-sorted algebraic theory. $\mathscr{E} \subseteq Alg(\Omega, E)$: fullsub. Then, the following are equivalent.

1. *$\mathscr{E}$ is definable by equations, i.e., $\mathscr{E} = Alg(\Omega, E + {}^{\exists}E')$.*
2. *$\mathscr{E} \subseteq Alg(\Omega, E)$ is closed under:*
   - *products,*
   - *subobjects,*
   - *surjective images.*

Fact (multi-sorted version, existing work)

(Ω, E): an S-sorted algebraic theory. $\mathscr{E} \subseteq Alg(\Omega, E)$: fullsub. Then, the following are equivalent.

1. *$\mathscr{E}$ is definable by equations, i.e., $\mathscr{E} = Alg(\Omega, E + {}^{\exists}E')$.*
2. *$\mathscr{E} \subseteq Alg(\Omega, E)$ is closed under:*
   - *products,*
   - *subobjects,*
   - *surjective images,*
   - *filtered colimits.*

Birkhoff's variety theorem

Question

Is it possible to generalize Birkhoff's theorem to our relative algebraic theories?

Birkhoff's variety theorem

$\mathbb{S}$: a partial Horn theory over an S-sorted signature Σ.

Proposition

Let (Ω, E) be an $\mathbb{S}$-relative algebraic theory.
Then, $Alg(\Omega, E + E') \subseteq Alg(\Omega, E)$ is closed under:

- products,
- filtered colimits.

In general, $Alg(\Omega, E + E') \subseteq Alg(\Omega, E)$ is not closed under:

- surjections (surjective images),
- subobjects.

We will modify them as follows:

surjections $\overset{\text{modify}}{\rightsquigarrow}$ U-retractions
subobjects $\overset{\text{modify}}{\rightsquigarrow}$ Σ-closed subobjects

Birkhoff's variety theorem

$\mathbb{S}$: a partial Horn theory over an S-sorted signature Σ.

Main theorem II [K]

Let (Ω, E) be an $\mathbb{S}$-relative algebraic theory. Consider the forgetful functor $U\colon\ Alg(\Omega, E) \to \mathbb{S}\text{-PMod}$. Then, the following are equivalent for a full subcategory $\mathscr{E} \subseteq Alg(\Omega, E)$.

1. There exists a set of $\mathbb{S}$-relative judgements E' satisfying $\mathscr{E} = Alg(\Omega, E + E')$.
2. $\mathscr{E} \subseteq Alg(\Omega, E)$ is closed under:
   - products,
   - Σ-closed subobjects,
   - U-retracts, (A morphism p is called a *U-retraction* if $U(p)$ is a retraction.)
   - filtered colimits.

This generalizes existing Birkhoff's theorem in the following sense:

$\mathbb{S}$:=(the theory of sets) $\rightsquigarrow$ the original version of Birkhoff's theorem

$\mathbb{S}$:=(the theory of S-sorted sets) $\rightsquigarrow$ the S-sorted version of Birkhoff's theorem

Birkhoff's variety theorem: closed monomorphisms

$\mathbb{T}$: a partial Horn theory over an S-sorted signature Σ.

Definition ([K])

A subobject $A \subseteq B$ in $\mathbb{T}$-PMod is called $\mathbb{T}$-closed (or Σ-closed) if the following diagrams form pullback squares for any $f, R \in \Sigma$.

$$\begin{array}{ccc} A_{s_1} \times \cdots \times A_{s_n} & \hookleftarrow & \mathrm{Dom}(\llbracket f \rrbracket_A) \\ \downarrow & & \downarrow \\ B_{s_1} \times \cdots \times B_{s_n} & \hookleftarrow & \mathrm{Dom}(\llbracket f \rrbracket_B) \end{array} \qquad \begin{array}{ccc} A_{s_1} \times \cdots \times A_{s_n} & \hookleftarrow & \llbracket R \rrbracket_A \\ \downarrow & & \downarrow \\ B_{s_1} \times \cdots \times B_{s_n} & \hookleftarrow & \llbracket R \rrbracket_B \end{array}$$

Definition (informal)

$A \subseteq B$ is Σ-closed $\overset{\text{def}}{\Leftrightarrow}$ all structures of A are induced from those of B.

$$S := \{*\}, \quad \Sigma_{\mathrm{mon}} := \{e : 1 \to *, \quad \cdot : * \times * \to *\},$$

$$\mathbb{T}_{\mathrm{mon}} := \left\{ \begin{array}{c} \top \vdash e\downarrow, \quad \top \vdash^{x,y} x \cdot y\downarrow, \\ \top \vdash^{x,y,z} (x \cdot y) \cdot z = x \cdot (y \cdot z), \\ \top \vdash^{x} x \cdot e = x = e \cdot x \end{array} \right\}.$$

Then, we have $\mathbb{T}_{\mathrm{mon}}\text{-}\mathrm{PMod} \cong \mathbf{Mon}$.

An inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ in $\mathbf{Mon}$ is $\mathbb{T}_{\mathrm{mon}}$-closed.

$$\Sigma'_{\mathrm{mon}} := \Sigma_{\mathrm{mon}} + \{\bullet^{-1} : * \to *\},$$

$$\mathbb{T}'_{\mathrm{mon}} := \mathbb{T}_{\mathrm{mon}} + \left\{ \begin{array}{c} x^{-1}\downarrow \vdash^{x} x^{-1} \cdot x = e = x \cdot x^{-1}, \\ x \cdot y = e = y \cdot x \vdash^{x,y} x^{-1} = y \end{array} \right\}.$$

Then, we have $\mathbb{T}'_{\mathrm{mon}}\text{-}\mathrm{PMod} \cong \mathbf{Mon}$.

The inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ in $\mathbf{Mon}$ is not $\mathbb{T}'_{\mathrm{mon}}$-closed.

$\mathbb{T}$-closedness depends on $\mathbb{T}$!

$\mathbb{S}$: a partial Horn theory over an S-sorted signature Σ.

Main theorem II (recall)

Let (Ω, E) be an $\mathbb{S}$-relative algebraic theory. Consider the forgetful functor $U : Alg(\Omega, E) \to \mathbb{S}\text{-PMod}$. Then, the following are equivalent for a full subcategory $\mathscr{E} \subseteq Alg(\Omega, E)$.

1. There exists a set of $\mathbb{S}$-relative judgements E' satisfying $\mathscr{E} = Alg(\Omega, E + E')$.
2. $\mathscr{E} \subseteq Alg(\Omega, E)$ is closed under:
   - products,
   - Σ-closed subobjects, $\leftarrow$ depending on syntax
   - U-retracts,
   - filtered colimits.

References

[ARV11] J. Adámek, J. Rosický, and E. M. Vitale. *Algebraic theories*. Vol. 184. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2011.

[AR94] J. Adámek and J. Rosický. *Locally presentable and accessible categories*. Vol. 189. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1994.

[Adá+21] J. Adámek et al. "Finitary monads on the category of posets". In: *Math. Structures Comput. Sci.* 31.7 (2021), pp. 799–821.

[Joh02] P. T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vol. 2*. Vol. 44. Oxford Logic Guides. The Clarendon Press, Oxford University Press, Oxford, 2002.

[Kaw23] Y. Kawase. *Birkhoff's variety theorem for relative algebraic theories*. 2023. arXiv: 2304.04382 [math.CT].

[KP93] G. M. Kelly and A. J. Power. "Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads". In: *J. Pure Appl. Algebra* 89.1-2 (1993), pp. 163–179.

[PV07] E. Palmgren and S. J. Vickers. "Partial horn logic and Cartesian categories". In: *Ann. Pure Appl. Logic* 145.3 (2007), pp. 314–353.

The definition of homomorphisms

Definition ([PV07])

A Σ-homomorphism $h : M \to N$ between partial Σ-structures consists of:

- a total map $h_s : M_s \to N_s$ for each sort $s \in S$

such that for each function symbol $f : s_1 \times \cdots \times s_n \to s$ in Σ and relation symbol $R : s_1 \times \cdots \times s_n$ in Σ, there exist total maps (dashed arrows) making the following diagrams commute.

$$
\begin{array}{ccccc}
M_{s_1} \times \cdots \times M_{s_n} & \hookleftarrow & \mathrm{Dom}([\![f]\!]_M) & \xrightarrow{[\![f]\!]_M} & M_s \\
\downarrow{\scriptstyle h_{s_1} \times \cdots \times h_{s_n}} & & \downarrow{\scriptstyle \exists} & & \downarrow{\scriptstyle h_s} \\
N_{s_1} \times \cdots \times N_{s_n} & \hookleftarrow & \mathrm{Dom}([\![f]\!]_N) & \xrightarrow[{[\![f]\!]_N}]{} & N_s
\end{array}
$$

$$
\begin{array}{ccc}
M_{s_1} \times \cdots \times M_{s_n} & \hookleftarrow & [\![R]\!]_M \\
\downarrow{\scriptstyle h_{s_1} \times \cdots \times h_{s_n}} & & \downarrow{\scriptstyle \exists} \\
N_{s_1} \times \cdots \times N_{s_n} & \hookleftarrow & [\![R]\!]_N
\end{array}
$$

An example of ordered algebra

Example (posets)

We present the partial Horn theory $\mathbb{T}_{\mathrm{pos}}$ of posets. Let $S := \{*\}$, $\Sigma_{\mathrm{pos}} := \{\leq : * \times *\}$. The partial Horn theory $\mathbb{T}_{\mathrm{pos}}$ over Σ_{pos} consists of:

$$\top \vdash_{x} x \leq x, \quad x \leq y \wedge y \leq x \vdash_{x,y} x = y, \quad x \leq y \wedge y \leq z \vdash_{x,y,z} x \leq z.$$

Then, we have $\mathbb{T}_{\mathrm{pos}}\text{-}\mathrm{PMod} \simeq \mathbf{Pos}$.

Example ($\mathbf{Pos}$-relative algebras)

We present a $\mathbf{Pos}$-relative algebraic theory (Ω, E).

$$\Omega := \{-\}, \quad \mathrm{ar}(-) := (x,y).y \leq x, \quad \mathrm{type}(-) := *$$

$$E := \{\, x \leq y \vdash_{x,y,z} (x - z) \leq (y - z), \quad y \leq z \vdash_{x,y,z} (x - z) \leq (x - y) \,\}.$$

Then, a model of (Ω, E) is just a "poset with subtraction". For example, $\mathbb{N}$ with usual subtraction is a model of (Ω, E).

Example (partial Boolean algebras)

The (single sorted) partial Horn theory $\mathbb{S}_{\mathrm{rsrel}}$ of reflexive and symmetric relations is defined as follows:

$$\Sigma_{\mathrm{rsrel}} := \{\odot : * \times *\}, \quad \mathbb{S}_{\mathrm{rsrel}} := \{\top \xrightarrow{x} x \odot x, \quad x \odot y \xrightarrow{x,y} y \odot x\}.$$

We define a $\mathbb{S}_{\mathrm{rsrel}}$-relative algebraic theory (Ω, E) as follows:

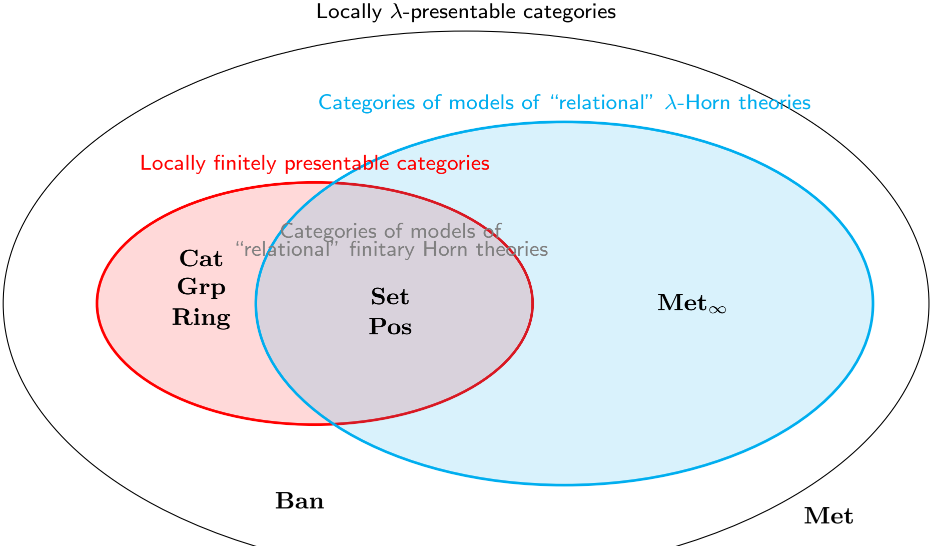
$$\Omega := \{0, 1, \neg, \vee, \wedge\};$$

$$\mathrm{ar}(0) = \mathrm{ar}(1) := ().\top, \quad \mathrm{ar}(\neg) := x.\top, \quad \mathrm{ar}(\vee) = \mathrm{ar}(\wedge) := (x,y).x \odot y;$$

$$E := \left\{ \begin{array}{c} \top \xrightarrow{x} x \odot 0,\ x \odot 1; \qquad x \odot y \xrightarrow{x,y} x \odot \neg y; \\ x \odot y,\ y \odot z,\ z \odot x \xrightarrow{x,y,z} x \odot (y \vee z),\ x \odot (y \wedge z); \\ x \odot y,\ y \odot z,\ z \odot x \xrightarrow{x,y,z} (x \vee y) \vee z = x \vee (y \vee z),\ (x \wedge y) \wedge z = x \wedge (y \wedge z); \\ x \odot y \xrightarrow{x,y} x \vee y = y \vee x,\ x \wedge y = y \wedge x; \\ x \odot y \xrightarrow{x,y} (x \wedge y) \vee x = x,\ x \wedge (y \vee x) = x; \\ \top \xrightarrow{x} x \vee 0 = x,\ x \wedge 1 = x,\ x \vee \neg x = 1,\ x \wedge \neg x = 0; \\ x \odot y,\ y \odot z,\ z \odot x \xrightarrow{x,y,z} (x \wedge y) \vee z = (x \vee z) \wedge (x \vee z); \\ x \odot y,\ y \odot z,\ z \odot x \xrightarrow{x,y,z} (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z) \end{array} \right\}$$

An algebra of (Ω, E) is called *partial Boolean algebra* in [**berg2012**].

Our main theorem I is a direct generalization of the finitary and $\mathbf{Set}$-enriched case of C.Ford, S.Milius, and L.Schröder's result in [C. Ford et al., 2021]. They described (enriched) λ-accessible monads on a category belonging to a special class of locally λ-presentable categories. That class is categories of models of "relational" λ-Horn theories.



A characterization of f.p.objects

$\mathbb{T}$: a partial Horn theory over an S-sorted signature Σ.

Theorem ([K])

For any Horn formula $\vec{x}.\varphi$, the functor $\mathbb{T}\text{-PMod} \ni A \mapsto [\![\vec{x}.\varphi]\!]_A \in \mathbf{Set}$ is representable, i.e., there exists a partial model $\langle \vec{x}.\varphi \rangle_{\mathbb{T}}$ satisfying

$$\mathbb{T}\text{-PMod}(\langle \vec{x}.\varphi \rangle_{\mathbb{T}}, A) \cong [\![\vec{x}.\varphi]\!]_A \quad (\forall A \in \mathbb{T}\text{-PMod}).$$

Theorem ([K])

The following are equivalent for each object $A \in \mathbb{T}\text{-PMod}$.

1. *A is finitely presentable in $\mathbb{T}\text{-PMod}$.*
2. *There exists a Horn formula $\vec{x}.\varphi$ over Σ satisfying $A \cong \langle \vec{x}.\varphi \rangle_{\mathbb{T}}$.*

Preservation theorems for partial Horn theories

Definition ([K])

Let $\rho : (S, \Sigma, \mathbb{S}) \to (S', \Sigma', \mathbb{T})$ be a theory morphism between partial Horn theories. A ρ-relative judgment is a Horn sequent $\varphi^\rho \vdash_{\vec{x}^\rho} \psi$, where $\vec{x}.\varphi$ is a Horn formula-in-context over Σ and $\vec{x}^\rho.\psi$ is a Horn formula-in-context over Σ'.

Theorem ([K])

Let $\rho \colon \mathbb{S} \to \mathbb{T}$ be a theory morphism between partial Horn theories. Then, for every replete full subcategory $\mathscr{E} \subseteq \mathbb{T}\text{-PMod}$, the following are equivalent:

1. *$\mathscr{E}$ is definable by ρ-relative judgments, i.e., there exists a set $\mathbb{T}'$ of ρ-relative judgments satisfying $\mathscr{E} = (\mathbb{T} + \mathbb{T}')\text{-PMod}$.*
2. *$\mathscr{E} \subseteq \mathbb{T}\text{-PMod}$ is closed under products, $\mathbb{T}$-closed subobjects, U^ρ-retracts, and filtered colimits.*

Preservation theorems for partial Horn theories

Taking ρ as the trivial one $\rho : (S, \varnothing, \varnothing) \to (S, \Sigma, \mathbb{T})$, we obtain the first corollary:

Corollary ([K])

Let $\mathbb{T}$ be a partial Horn theory over Σ. Then, for every replete full subcategory $\mathscr{E} \subseteq \mathbb{T}\text{-PMod}$, the following are equivalent:

1. *$\mathscr{E}$ is definable by Horn formulas, i.e., there exists a set E of Horn formulas satisfying $\mathscr{E} = (\mathbb{T} + \mathbb{T}')\text{-PMod}$, where $\mathbb{T}' := \{\top \overset{\vec{x}}{\longmapsto} \varphi\}_{\vec{x}.\varphi \in E}$.*
2. *$\mathscr{E} \subseteq \mathbb{T}\text{-PMod}$ is closed under products, Σ-closed subobjects, surjections, and filtered colimits.*

Taking ρ as the identity $\mathbb{T} \to \mathbb{T}$, we obtain the second corollary:

Corollary ([K])

Let $\mathbb{T}$ be a partial Horn theory over Σ. Then, for every replete full subcategory $\mathscr{E} \subseteq \mathbb{T}\text{-PMod}$, the following are equivalent:

1. *$\mathscr{E}$ is definable by Horn sequents, i.e., there exists a set $\mathbb{T}'$ of Horn sequents satisfying $\mathscr{E} = (\mathbb{T} + \mathbb{T}')\text{-PMod}$.*
2. *$\mathscr{E} \subseteq \mathbb{T}\text{-PMod}$ is closed under products, Σ-closed subobjects, and filtered colimits.*

Filtered colimit elimination

Theorem ([K])

Let $(S, \Sigma, \mathbb{S})$ *be a partial Horn theory. Assume that:*

- S *is finite,*
- *For every model* M *of* $\mathbb{S}$*, "the largest quotient"* $M \to QM$ *is a retraction in* $\mathbb{S}\text{-PMod}$.

Let (Ω, E) *be an* $\mathbb{S}$*-relative algebraic theory. Then, for every replete full subcategory* $\mathscr{E} \subseteq Alg(\Omega, E)$*, the following are equivalent:*

1. $\mathscr{E} \subseteq Alg(\Omega, E)$ *is closed under products,* Σ*-closed subobjects, and* U*-local retracts.*
2. $\mathscr{E} \subseteq Alg(\Omega, E)$ *is closed under products,* Σ*-closed subobjects,* U*-retracts, and filtered colimits.*
3. $\mathscr{E} \subseteq Alg(\Omega, E)$ *is definable by* $\mathbb{S}$*-relative judgments.*

Example

The following partial Horn theories satisfy the assumptions in the above theorem:

- The theory of sets.
- The theory of finite sorted sets.
- The theory of posets.