Edgewise Subdivision, CULF maps, and Right Fibrations

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Simplicial spaces and ∞-categories

Δ: simplicial category with objects $[n] = \{0 \to 1 \to \cdots \to n\}$ for $n \geq 0$

$\mathrm{sSpace} = \mathrm{PSh}(\Delta) = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Space})$: ∞-category of simplicial spaces

$\Delta^n = \mathrm{hom}(-, [n]) \in \mathrm{sSpace}$: standard n-simplex

$X \in \mathrm{sSpace}$ is Segal or a Segal space if the following are pullbacks:

$$\begin{tikzcd} X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d, "d_0"] \\ X_n \arrow[r, "d_n"'] & X_{n-1} \end{tikzcd}$$

Equivalently: $X_n \xrightarrow{\sim} X_1 \times_{X_0} X_1 \cdots X_1 \times_{X_0} X_1$

Definition

$f \in X_1$ is an equivalence if there are $\sigma, \tau \in X_2$ as follows:

$$\begin{tikzcd} & 1 \arrow[dr] \arrow[d, phantom, "\sigma"] & \\ 0 \arrow[ur, "f"] \arrow[rr, "\mathrm{id}"'] & {} & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[rr, "\mathrm{id}"] \arrow[dr] & {} \arrow[d, phantom, "\tau"] & 2 \\ & 1 \arrow[ur, "f"'] & \end{tikzcd}$$

$X_1^{\mathrm{eq}} \subset X_1$ is the subspace spanned by the equivalences

Definition (Rezk)

$X \in \mathbf{sSpace}$ is a complete Segal space if it is Segal and $s_0 : X_0 \xrightarrow{\sim} X_1^{\mathrm{eq}}$

$\mathbf{CSS} \subset \mathbf{sSpace}$: subcategory of complete Segal spaces – $\mathbf{CSS} \simeq \mathbf{Cat}_\infty$

Active-inert factorization system on Δ

- $f\colon [n] \longrightarrow\mathrel{\mkern-12mu}| \; [m]$ is active if it is endpoint-preserving: $f(0) = 0$ and $f(n) = m$
- $f\colon [n] \rightarrowtail [m]$ is inert if it is distance-preserving: $f(i+1) = f(i) + 1$

An arbitrary map in Δ factors uniquely: $f = f_{\mathrm{int}} \circ f_{\mathrm{act}}$

$$
\begin{tikzcd}
{[n]} \arrow[rr, "f"] \arrow[dr, "f(\ ) - f(0)"'] & & {[m]} \\
& {[f(n) - f(0)]} \arrow[ur, tail, "+f(0)"'] &
\end{tikzcd}
$$

$(\Delta_{\mathrm{act}}, \Delta_{\mathrm{int}})$ factorization system on Δ

Decomposition spaces / 2-Segal spaces

Pushouts of active along inert exist in Δ:

$$\begin{tikzcd}
{[n]} \arrow[r] \arrow[d, tail, "+a"'] & {[m]} \arrow[d, tail, "+a"] \\
{[k]} \arrow[r] & {[m+k-n]} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Definition (Dyckerhoff & Kapranov; Gálvez-Carillo, Kock, Tonks)

A simplicial space X is a decomposition space or 2-Segal space if it sends active-inert pushout squares to pullback squares.

for $0 < i < n$:

$$\begin{tikzcd}
X_{n+1} \arrow[r, "d_{i+1}"] \arrow[d, tail, "d_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d, tail, "d_0"] \\
X_n \arrow[r, "d_i"'] & X_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{n+1} \arrow[r, "d_i"] \arrow[d, tail, "d_{n+1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d, tail, "d_n"] \\
X_n \arrow[r, "d_i"'] & X_{n-1}
\end{tikzcd}$$

Right fibrations of simplicial spaces

Last-endpoint-preserving maps generate the left class in a factorization system on **sSpace**, with right class the right fibrations and left class the final maps

$$
\begin{tikzcd}
\Delta^k \arrow[r] \arrow[d, "\ell"'] & Y \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dashed, "\exists!"] & X
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
Y_n \arrow[r, "\ell^*"] \arrow[d, "p_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_k \arrow[d, "p_k"] \\
X_n \arrow[r, "\ell^*"'] & X_k
\end{tikzcd}
$$

Factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{E}$:

- $\mathcal{L}$ and $\mathcal{R}$ span replete subcategories of $\mathsf{Arr}\,\mathcal{E}$
- $\mathcal{L} \perp \mathcal{R}$
- Maps admit factorizations: $A \xrightarrow{\ell} C \xrightarrow{r} B$

ex: when Σ a set, $\mathcal{E}$ presentable: $({}^{\perp}(\Sigma^{\perp}), \Sigma^{\perp})$

Comprehensive factorization

Comprehensive factorization system on Cat_∞ (Joyal 2008; Ayala–Francis 2020)

- left class = final functors
- right class = right fibrations

Proposition

The (final map, right fibration) factorization system on sSpace *restricts to the (final functor, right fibration) factorization system on* $\mathsf{CSS} \simeq \mathsf{Cat}_\infty$

CULF maps

Active maps generate the left class in a factorization system on **sSpace**, with right class the CULF maps[1] and left class the ambifinal maps

$$
\begin{tikzcd}
\Delta^k \arrow[r] \arrow[d, "a"'] & Y \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dashed, "\exists!"] & X
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
Y_n \arrow[r, "a^*"] \arrow[d, "p_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_k \arrow[d, "p_k"] \\
X_n \arrow[r, "a^*"'] & X_k
\end{tikzcd}
$$

[1] Conservative and Unique Lifting of Factorizations

CULF maps and decomposition spaces

CULF maps are the most important class of maps for decomposition spaces

CULF maps induce coalgebra homomorphisms on incidence coalgebras associated to decomposition spaces (when defined)

Proposition

If X is a decomposition space and $Y \to X$ is CULF, then Y is decomposition space.

Example

If X is a decomposition space, then $\mathrm{Dec}\, X \to X$ is CULF

Edgewise subdivision

$Q\colon \Delta \to \Delta$ with $Q[n] = [n]^{\mathrm{op}} \star [n] \cong [2n+1]$

$$\begin{tikzcd} \mathsf{sSpace} \arrow[rr, "Q^*" description] & & \mathsf{sSpace} \arrow[ll, bend right=40, "Q_!"'] \arrow[ll, bend left=40, "Q_*"] \end{tikzcd}$$

$(\mathrm{sd}\, X)_n = (Q^* X)_n = X_{2n+1}$

$\mathrm{sd}(NC) = N(\mathrm{tw}\, C)$

$$\begin{tikzcd} 0 \arrow[r] & 1 \\ 0' \arrow[u] & 1' \arrow[l] \arrow[u] \end{tikzcd}$$

Theorem (Bergner, Osorno, Ozornova, Rovelli, Scheimbauer 2020)

$X \in \mathsf{sSpace}$ is a decomposition space if and only if $\mathrm{sd}\, X \in \mathsf{sSpace}$ is a Segal space

Edgewise subdivision of CULF maps

Lemma

$p\colon Y \to X$ is CULF if and only if $\operatorname{sd}(p)\colon \operatorname{sd} Y \to \operatorname{sd} X$ is a right fibration.

Proof.

$(\Rightarrow)$

$$
\begin{tikzcd}
\Delta^0 \arrow[r] \arrow[d, "\ell"'] & \operatorname{sd} Y \arrow[d, "\operatorname{sd}(p) = Q^*(p)"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \operatorname{sd} X
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
\Delta^1 \arrow[r] \arrow[d, "Q_!(\ell)"'] & Y \arrow[d, "p"] \\
\Delta^{2n+1} \arrow[r] \arrow[ur, dashed] & X,
\end{tikzcd}
$$

$(\Leftarrow)$ For $n \geq 1$, the map $\Delta^1 \rightarrow\!\!\shortmid \Delta^{2n}$ is a retract of $\Delta^1 \rightarrow\!\!\shortmid \Delta^{2n+1}$. For $\Delta^1 \rightarrow\!\!\shortmid \Delta^0$:

$$
\begin{tikzcd}
Y_0 \arrow[r, "s_0"] \arrow[d] & Y_1 \arrow[d] \\
X_0 \arrow[r, "s_0"] & X_1
\end{tikzcd}
\quad \text{is a retract of the left square:} \quad
\begin{tikzcd}
Y_1 \arrow[r, "s_1"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Y_2 \arrow[r, "d_1"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Y_1 \arrow[d] \\
X_1 \arrow[r, "s_1"] & X_2 \arrow[r, "d_1"] & X_1
\end{tikzcd}
$$

Theorem (Hackney–Kock)

If $X \in \mathsf{sSpace}$, then there is an equivalence of ∞-categories $\mathsf{CULF}_{/X} \simeq \mathsf{RFib}_{/\,\mathsf{sd}\,X}$

Proof outline.
The $Q^* \dashv Q_*$ adjunction on sSpace induces:

$$\begin{tikzcd} \mathsf{sSpace}_{/X} \arrow[r, bend left, "\mathsf{sd}_X"] \arrow[r, phantom, "\perp"] & \mathsf{sSpace}_{/\,\mathsf{sd}\,X} \arrow[l, bend left, "U"] \end{tikzcd}$$

$$\begin{tikzcd} \widetilde{Z} \arrow[r] \arrow[d, "U(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q_* Z \arrow[d, "Q_*(p)"] & & Z \arrow[d, "p"] \\ X \arrow[r, "\eta_X"'] & Q_* Q^* X & \arrow[l, phantom, "\leftsquigarrow"] & Q^* X \end{tikzcd}$$

$\mathrm{CULF}_{/X} \simeq \mathrm{RFib}_{/\,\mathrm{sd}\,X}$

$$\begin{tikzcd} \mathrm{sSpace}_{/X} \arrow[r, bend left, "\mathrm{sd}_X"] \arrow[r, phantom, "\perp"] & \mathrm{sSpace}_{/\,\mathrm{sd}\,X} \arrow[l, bend left, "U"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathrm{CULF}_{/X} \arrow[r, bend left, "\mathrm{sd}_X"] \arrow[r, phantom, "\simeq"] & \mathrm{RFib}_{/\,\mathrm{sd}\,X} \arrow[l, bend left, "U"] \end{tikzcd}$$

- Q_* takes right fibrations to CULF maps
- Q^* preserves pullbacks
- $\eta\colon \mathrm{id} \Rightarrow Q_*Q^*$ is cartesian on CULF maps
- $\varepsilon\colon Q^*Q_* \Rightarrow \mathrm{id}$ is cartesian on right fibrations

$$\begin{tikzcd} Q^*\widetilde{Z} \arrow[r] \arrow[d, "Q^*U(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q^*Q_*Z \arrow[r, "\varepsilon_Z"] \arrow[d, "Q^*Q_*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "p"] \\ Q^*X \arrow[r, "Q^*\eta_X"] \arrow[rr, bend right, equal] & Q^*Q_*Q^*X \arrow[r, "\varepsilon_{Q^*X}"] & Q^*X \end{tikzcd}$$

$\varepsilon\colon Q^*Q_* \Rightarrow \mathrm{id}$ cartesian on right fibrations? consider other adjunction $Q_! \dashv Q^*$

Lemma

The unit η' of the $Q_! \dashv Q^*$ adjunction is a final functor at Δ^n

Proof.
η'_{Δ^n} preserves terminal objects, so is final:

$$(0 \to 1) = \Delta^1 \to Q^*Q_!\Delta^1 = Q^*\Delta^3 = \left(\begin{tikzcd} & & & 00 \arrow[d] \\ & & 11 \arrow[r] \arrow[d] & 01 \arrow[d] \\ & 22 \arrow[r] \arrow[d] & 12 \arrow[r] \arrow[d] & 02 \arrow[d] \\ 33 \arrow[r] & 23 \arrow[r] & 13 \arrow[r] & 03 \end{tikzcd}\right)$$

Decomp is the ∞-category of decomposition spaces and CULF maps

Theorem (Hackney–Kock)

If $X \in \mathsf{Decomp}$, then

$$\mathsf{Decomp}_{/X} = \mathsf{CULF}_{/X} \simeq \mathsf{RFib}_{/\,\mathsf{sd}\,X} \simeq \mathsf{RFib}_{/\widehat{\mathsf{sd}\,X}} \simeq \mathsf{PSh}(\widehat{\mathsf{sd}\,X})$$

where $Y \to \widehat{Y}$ is the completion of Rezk

Proof.

- previous theorem
- Bergner *et al.* 2020; Boavida 2018
- Grothendieck construction

Free decomposition spaces

Lemma

If $B\mathbb{N}$ is the monoid of natural numbers, then $\mathsf{tw}\, B\mathbb{N} \simeq \Delta_{\mathrm{int}}$

Example

There is an equivalence $\mathsf{PSh}(\Delta_{\mathrm{int}}) \simeq \mathsf{PSh}(\mathsf{tw}\, B\mathbb{N}) \simeq \mathsf{Decomp}_{/B\mathbb{N}}$

Theorem

Left Kan extension along $\Delta_{\mathrm{int}} \to \Delta$ *sends*

- *objects of* $\mathsf{PSh}(\Delta_{\mathrm{int}})$ *to decomposition spaces*
- *morphisms to CULF maps*
- *terminal object to* $B\mathbb{N}$

The induced functor $\mathsf{PSh}(\Delta_{\mathrm{int}}) = \mathsf{PSh}(\Delta_{\mathrm{int}})_{/*} \to \mathsf{Decomp}_{/B\mathbb{N}}$ *is equivalence above*