Birkhoff subfibrations

Alan Cigoli

UNIVERSITÀ DI TORINO

(joint work with S. Mantovani)

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Mal'tsev algebras

[Gran '01] For an exact Mal'tsev category $\mathcal{E}$ with coequalizers, the subcategory $\mathbf{Mal}(\mathcal{E})$ of Mal'tsev algebras is a Birkhoff subcategory, i.e. a regular epi reflective subcategory of $\mathcal{E}$, closed under quotients.

$$I\colon \mathcal{E} \to \mathbf{Mal}(\mathcal{E}), \quad X \mapsto X/[\nabla_X, \nabla_X].$$

Notice: when $\mathcal{E}$ is pointed, $I(X) = X/[X, X]$ and $\mathbf{Mal}(\mathcal{E}) = \mathbf{Ab}(\mathcal{E})$.

[Janelidze, Kelly '94] Each Birkhoff subcategory of an exact Mal'tsev category yields an admissible Galois structure (with respect to regular epi). In particular this is true for the adjunction

$$\mathbf{Mal}(\mathcal{E}) \underset{H}{\overset{I}{\underset{\perp}{\leftrightarrows}}} \mathcal{E}.$$

This amounts to the fact that in a pullback of the form

$$\begin{array}{ccc} B \times_{HI(B)} H(X) & \xrightarrow{\pi_2} & H(X) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle H(\phi)} \\ B & \xrightarrow{\eta_B} & HI(B), \end{array}$$

with ϕ a regular epi, π_2 is a unit up to iso.

Case study: Mal($\mathcal{C}/B$)

Let us focus on $\mathcal{E} = \mathcal{C}/B$, where $\mathcal{C}$ is a semi-abelian category which is also peri-abelian [Bourn '10] and B an object of $\mathcal{C}$.

Consequences of the hypothesis:

- For each normal subobject $L \trianglelefteq X$, $[L, L] \trianglelefteq X$ [Gray, Van der Linden '15];
- The reflection $I \colon \mathcal{C}/B \to \mathbf{Mal}(\mathcal{C}/B)$ simplifies to

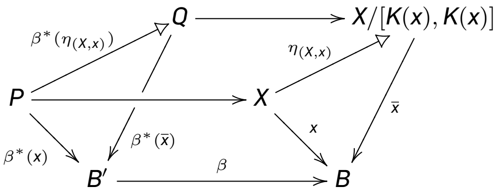
$$\begin{array}{ccc} X & \xrightarrow{\eta_{(X,x)}} & X/[K(x), K(x)] \\ & {}_{x}\searrow \quad \swarrow_{\overline{x}} & \\ & B. & \end{array}$$

(some questions about commutator theory are involved)

Question

Since $\mathcal{C}/B$ is a fibre of the fibration $\mathrm{Cod}\colon \mathbf{Arr}(\mathcal{C}) \to \mathcal{C}$, how does the reflection above interact with change of base?

Units are stable under change of base:



We are going to "saturate" regular epi in order to set up a new Galois structure on each fibre. We consider morphisms f in $\mathcal{C}/B$ such that $\beta^* f$ is a regular epi for some $\beta\colon B' \to B$ in $\mathcal{C}$. We call such morphisms proquotients. In the present context, proquotients are precisely those morphisms f in $\mathcal{C}/B$ whose restriction to kernels is a regular epi, i.e. such that $i_B^*(f)$ is a regular epi, where $i_B\colon 0 \to B$ is the initial arrow.

Fact: Mal'tsev algebras are stable under proquotients. Namely, if (A, a) is a Mal'tsev algebra in $\mathcal{C}/B$ and $f\colon (A, a) \to (X, x)$ is a proquotient, then (X, x) is a Mal'tsev algebra as well.

Proquotients in a regular category

Definition

Let $\mathcal{E}$ be a regular category. A morphism $f\colon X \to Y$ in $\mathcal{E}$ is called a pro-quotient if $1_X \times f\colon X^2 \to X \times Y$ (or equivalently $f \times 1_X$) is a regular epi.

Examples: regular epi, product projections, mono into a terminal object.

If product projections are regular epi (e.g. if $\mathcal{E}$ pointed), then proquotient = regular epi.

Proposition

Let $\mathcal{E}$ be a regular category. Then the following properties hold for the class $\mathcal{P}$ of proquotients:

1. it is closed under composition;
2. it is stable under pullback;
3. if $g \cdot f$ is in $\mathcal{P}$ and f is a regular epi, then g is in $\mathcal{P}$.

$\mathcal{F}$-Birkhoff subcategories

Definition

Let $\mathcal{E}$ be a regular category and let $\mathcal{X}$ be a regular epi reflective subcategory of $\mathcal{E}$:

$$\mathcal{X} \underset{H}{\overset{I}{\rightleftarrows}} \mathcal{E}.$$

Let $\mathcal{F}$ be a class of morphisms in $\mathcal{E}$ containing all regular epi.
$\mathcal{X}$ is an $\mathcal{F}$-Birkhoff subcategory of $\mathcal{E}$ when it is closed under $\mathcal{F}$-images, i.e. if for each $f \colon X \to Y$ in $\mathcal{F}$, Y belongs to $\mathcal{X}$ as soon as X does.

We are interested in the case $\mathcal{F} = \mathcal{P}$, proquotients.

Examples:

- $\mathbf{Mal}(\mathcal{C}/B)$ is a $\mathcal{P}$-Birkhoff subcategory of $\mathcal{C}/B$ (with $\mathcal{C}$ peri-abelian);
- If $\mathcal{E}$ is protomodular, $\mathbf{Sub}_1(\mathcal{E})$ is a $\mathcal{P}$-Birkhoff subcategory of $\mathcal{E}$. In particular, if $\mathcal{E}$ is a regular category and X an object of $\mathcal{E}$, $\mathbf{EqRel}_X(\mathcal{E})$ is a $\mathcal{P}$-Birkhoff subcategory of $\mathbf{Gpd}_X(\mathcal{E})$. Here proquotients are internal functors f such that $\Pi_1(f)$ is a regular epi.

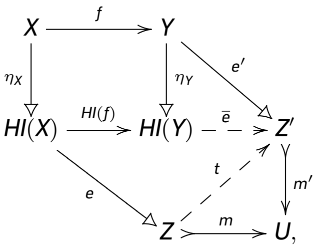
Proposition

Let $\mathcal{E}$ be a regular category and $\mathcal{F}$ a class of morphisms in $\mathcal{E}$ such that:

1. $\mathcal{F}$ contains all regular epimorphims;
2. $\mathcal{F}$ is closed under post-composition with regular epimorphims;
3. If $g \cdot f$ is in $\mathcal{F}$ and f is a regular epi, then g is in $\mathcal{F}$.

A regular epi reflective subcategory $\mathcal{X}$ of $\mathcal{E}$ is $\mathcal{F}$-Birkhoff if and only if for each $f \colon X \to Y$ in $\mathcal{F}$ the naturality square of f is a pushout.

Sketch of the proof



Admissibility with respect to proquotients

A $\mathcal{P}$-Birkhoff subcategory $\mathcal{X}$ of a regular category $\mathcal{E}$ is admissible (with respect to proquotients) in the sense of Janelidze's Galois theory if the reflector $I\colon \mathcal{E} \to \mathcal{X}$ preserves pullbacks of the form

$$\begin{array}{ccc} B \times_{HI(B)} H(X) & \xrightarrow{\pi_2} & H(X) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle H(\phi)} \\ B & \xrightarrow{\eta_B} & HI(B) \end{array}$$

with ϕ proquotient.

Unlike Birkhoff subcategories (with respect to regular epi), $\mathcal{P}$-Birkhoff subcategories need not be admissible: a counterexample is given by **Mal**(**NARng**) in **NARng**.

However:

- If $\mathcal{C}$ is peri-abelian, $\mathbf{Mal}(\mathcal{C}/B)$ is an admissible subcategory of $\mathcal{C}/B$;
- If $\mathcal{E}$ is protomodular, $\mathbf{Sub}_1(\mathcal{E})$ is an admissible subcategory of $\mathcal{E}$. In particular, if $\mathcal{E}$ is a regular category and X an object of $\mathcal{E}$, $\mathbf{EqRel}_X(\mathcal{E})$ is an admissible subcategory of $\mathbf{Gpd}_X(\mathcal{E})$.

In some cases, a characterization of coverings is available.

Proposition

Let B be a group, and $f\colon (X,x) \to (Y,y)$ a proquotient in $\mathbf{Gp}/B$. The following are equivalent:

1. f is a covering with respect to the admissible adjunction

$$\mathbf{Mal}(\mathbf{Gp}/B) \underset{H}{\overset{I}{\underset{\hookrightarrow}{\overset{\leftarrow}{\perp}}}} \mathbf{Gp}/B;$$

2. $i_B^*(f)$ is a central extension of groups;
3. $[K(f), K(x)] = 0$.

For example, if $\partial\colon H \to G$ is a crossed module and $q\colon G \to B$ its cokernel, then

$$\begin{array}{ccc} H & \xrightarrow{\ \partial\ } & G \\ & {}_{0}\searrow \quad \swarrow_{q} & \\ & B & \end{array}$$

is a covering.

Let's have a look at the proof of admissibility of $\mathbf{Mal}(\mathcal{C}/B)$ for $\mathcal{C}$ peri-abelian.

Let (X, x) be an object of $\mathcal{C}/B$ and $f\colon (A, a) \to I(X, x)$ a proquotient in $\mathbf{Mal}(\mathcal{C}/B)$.

$$\begin{array}{ccc} (P,p) & \xrightarrow{\pi_2} & H(A,a) \\ {\scriptstyle \pi_1}\downarrow & p.b. & \downarrow{\scriptstyle H(f)} \\ (X,x) & \xrightarrow{\eta_{(X,x)}} & HI(X,x) \end{array} \quad \xrightarrow{i_B^*} \quad \begin{array}{ccc} K(p) & \xrightarrow{\pi_2} & K(a) \\ {\scriptstyle \pi_1}\downarrow & p.b. & \downarrow{\scriptstyle i_B^* H(f)} \\ K(x) & \xrightarrow{i_B^*(\eta_{(X,x)})} & \mathbf{ab}(K(x)). \end{array}$$

Admissibility of the adjunction

$$\mathbf{Mal}(\mathcal{C}/B) \underset{H}{\overset{I}{\leftrightarrows}} \mathcal{C}/B \quad (\perp)$$

relies on admissibility of

$$\mathbf{Ab}(\mathcal{C}) \overset{\mathbf{ab}}{\leftrightarrows} \mathcal{C} \quad (\perp)$$

and on the fact that, for each object X, $[X, X]$ is a characteristic subobject of X [C.,Montoli '15].

Fibred aspects

Definition

A reflection in **Fib**$(\mathcal{C})$ is just an adjunction $I \dashv H$

$$\begin{array}{ccc} \mathcal{X} & \underset{H}{\overset{I}{\underset{\perp}{\leftrightarrows}}} & \mathcal{Y} \\ & {}_F\searrow \qquad \swarrow_G & \\ & \mathcal{C} & \end{array}$$

in **Fib**$(\mathcal{C})$ where H makes $(\mathcal{X}, F)$ a full subfibration of $(\mathcal{Y}, G)$.
We speak of a regular epi reflection when the unit components are regular epimorphisms.

Proposition

Let $H\colon (\mathcal{X}, F) \to (\mathcal{Y}, G)$ be a full subfibration. TFAE:

1. H has a left adjoint in **Fib**$(\mathcal{C})$;
2. i) for each B in $\mathcal{C}$, $H_B\colon \mathcal{X}_B \to \mathcal{Y}_B$ has a left adjoint I_B;

   ii) for each Y in $\mathcal{Y}_B$ and each cartesian arrow $k\colon X \to I_B(Y)$, the pullback of η_Y along $H(k)$ is a unit up to iso.

As a special case of the previous situation, we consider a full subfibration of the codomain fibration $\mathrm{Cod}\colon \mathbf{Arr}(\mathcal{C}) \to \mathcal{C}$ of a semi-abelian category $\mathcal{C}$.

Theorem

Let $\mathcal{C}$ be a semi-abelian category and $H\colon (\mathcal{X}, F) \to (\mathbf{Arr}(\mathcal{C}), \mathrm{Cod})$ a full subfibration. Then the following are equivalent:

1. H has a left adjoint that gives rise to a regular epi reflection in $\mathbf{Fib}(\mathcal{C})$;
2. i) The restriction $H_0\colon \mathcal{X}_0 \to \mathcal{C}/0 \cong \mathcal{C}$ has a left adjoint I_0, such that each unit component is a regular epi with characteristic kernel.

   ii) For each B in $\mathcal{C}$, the square

$$\begin{array}{ccc} \mathcal{X}_B & \overset{H_B}{\hookrightarrow} & \mathcal{C}/B \\ {\scriptstyle i_B^*}\downarrow & & \downarrow{\scriptstyle i_B^*} \\ \mathcal{X}_0 & \underset{}{\overset{H_0}{\hookrightarrow}} & \mathcal{C} \end{array}$$

is a pullback in $\mathbf{Cat}$.

Actually, 2.i) is sufficient to recover the entire regular epi reflection in $\mathbf{Fib}(\mathcal{C})$.

Birkhoff subfibrations

Definition

Given a regular epi reflection

$$(\mathcal{X}, F) \underset{H}{\overset{I}{\underset{\perp}{\leftrightarrows}}} (\mathbf{Arr}(\mathcal{C}), \mathrm{Cod})$$

in $\mathbf{Fib}(\mathcal{C})$, $(\mathcal{X}, F)$ is called a $\mathcal{P}$-Birkhoff subfibration of $(\mathbf{Arr}(\mathcal{C}), \mathrm{Cod})$ if for each B in $\mathcal{C}$, the restriction

$$\mathcal{X}_B \underset{H_B}{\overset{I_B}{\underset{\perp}{\leftrightarrows}}} \mathcal{C}/B$$

of $I \dashv H$ to the fibre over B makes $\mathcal{X}_B$ a $\mathcal{P}$-Birkhoff subcategory of $\mathcal{C}/B$, where $\mathcal{P}$ is the class of proquotients.

Proposition

Each Birkhoff reflection

$$\mathcal{X}_0 \underset{H_0}{\overset{I_0}{\underset{\perp}{\leftrightarrows}}} \mathcal{C},$$

whose unit components have characteristic kernel, determines (up to iso) a unique $\mathcal{P}$-Birkhoff reflection in $\mathbf{Fib}(\mathcal{C})$, whose restriction to the fibre over 0 is $I_0 \dashv H_0$.

Examples

By means of the above proposition, we can obtain examples of $\mathcal{P}$-Birkhoff subfibrations of $(\mathbf{Arr}(\mathcal{C}), \mathsf{Cod})$ by taking

1. $\mathcal{C}$ an abelian category and $\mathcal{X}_0$ any Birkhoff subcategory of $\mathcal{C}$.
2. $\mathcal{C} = \mathbf{Gp}$ and $\mathcal{X}_0$ any subvariety; characteristic subgroups are precisely the subgroups closed under automorphisms, and so are the kernels of the units of an adjunction.
3. $\mathcal{C}$ a semi-abelian category which is also peri-abelian and $\mathcal{X}_0 = \mathbf{Ab}(\mathcal{C})$. In which case, for each B, $\mathcal{X}_B = \mathbf{Mal}(\mathcal{C}/B)$.
4. $\mathcal{C}$ a semi-abelian category satisfying (NH) and (SH) and $\mathcal{X}_0$ its subcategory of n-nilpotent or n-solvable objects, for any $n > 0$. Here the condition (NH) guarantees that for an object X, the iterated Higgins commutators

$$\begin{cases} [X,X]_0^{\mathrm{Nil}} = X \\ [X,X]_{n+1}^{\mathrm{Nil}} = [X,[X,X]_n] \quad \text{for } n \geq 0 \ \text{(nilpotent case)} \end{cases}$$

$$\begin{cases} [X,X]_0^{\mathrm{Sol}} = X \\ [X,X]_{n+1}^{\mathrm{Sol}} = [[X,X]_n,[X,X]_n] \quad \text{for } n \geq 0 \ \text{(solvable case)} \end{cases}$$

are characteristic subobjects of X.

Admissibility

Proposition

Let $\mathcal{C}$ be a semi-abelian category and let $(\mathcal{X}, F)$ be a $\mathcal{P}$-Birkhoff subfibration of $(\mathbf{Arr}(\mathcal{C}), \mathrm{Cod})$. Then, for each B in $\mathcal{C}$, the restriction

$$\mathcal{X}_B \underset{H_B}{\overset{I_B}{\underset{\hookrightarrow}{\overset{\leftarrow}{\perp}}}} \mathcal{C}/B$$

of $I \dashv H$ to the fibre over B is an admissible Galois structure with respect to proquotients.

The latter might be seen as a prototype of Galois structure in $\mathbf{Fib}(\mathcal{C})$...

Thank you!