Towards a varietal encyclopedia of internal categories

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Outline

Congruence modular varieties

Brief recalls about internal categories

Internal categories in Gumm categories

Internal categories in congruence modular varieties

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Internal categories in congruence modular varieties

A congruence modular variety is a variety in which the modular formula holds for congruences:

$$(T \vee S) \wedge R = T \vee (S \wedge R), \text{ for any triple } (T, S, R) \text{ such that} : T \subset R$$

- There is a characterization of congruence modudar varieties by terms and equations.
- One simple non-Mal'tsev example is given with the generalized right complemented semi-group: two binary operations: $\circ$ and $\star$, and two axioms:

$$x \circ (x \star y) = y \circ (y \star x)$$

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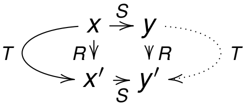
In 1983, Gumm characterized them in ”geometric terms” by the validity of the Shifting Lemma: given any triple of equivalence relations (T, S, R) such that $R \cap S \subset T$ on an algebra A, the following left hand side situation implies the dotted right hand side one:



- The main interest of the Shifting lemma is that it is freed of any condition involving finite colimits.
- Thanks to the Yoneda embedding, it keeps a meaning in any finitely complete category $\mathbb{E}$. This led, in 2004, to the notion of Gumm category introduced by M. Gran and myself.

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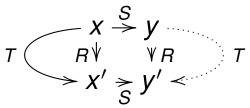

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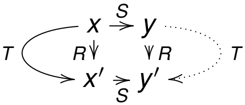

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- Thanks to the Yoneda embedding, it keeps a meaning in any finitely complete category $\mathbb{E}$. This led, in 2004, to the notion of Gumm category introduced by M. Gran and myself.

Any Mal'tsev variety in a congruence modular one.
Any regular Mal'tev category is a Gumm one.

- In any congruence modular variery or Gumm category, the Cube Lemma holds: for any triple of equivalence relations (T, S, R) on an object X such that $R \cap S \subset T$, the plain arrows imply the dotted one:

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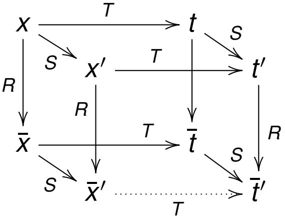

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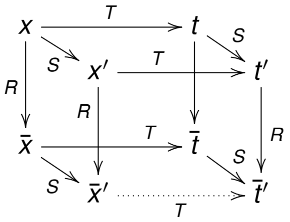

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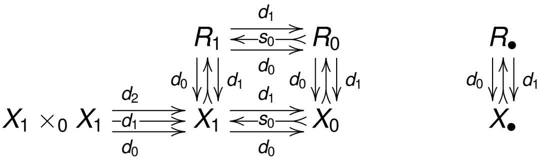
Main Result: Given any internal category $X_\bullet$ in a Gumm category:



together with a vertical internal equivalence relation $R_\bullet$ on the underlying reflexive graph of $X_\bullet$.

- Then the upper horizontal reflexive graph $R_\bullet$ is underlying an internal category.

Main Result: Given any internal category $X_\bullet$ in a Gumm category:

$$\begin{array}{ccccccc}
 & & R_1 & \underset{d_0}{\overset{d_1}{\underset{\xleftarrow{\;s_0\;}}{\rightrightarrows}}} & R_0 & \qquad & R_\bullet \\
 & & d_0 \downarrow\uparrow\downarrow d_1 & & d_0 \downarrow\uparrow\downarrow d_1 & & d_0 \downarrow\uparrow\downarrow d_1 \\
X_1 \times_0 X_1 & \underset{d_0}{\overset{d_2}{\xrightarrow[\;d_1\;]{}}} & X_1 & \underset{d_0}{\overset{d_1}{\underset{\xleftarrow{\;s_0\;}}{\rightrightarrows}}} & X_0 & & X_\bullet
\end{array}$$

together with a vertical internal equivalence relation $R_\bullet$ on the underlying reflexive graph of $X_\bullet$.

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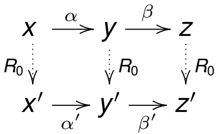
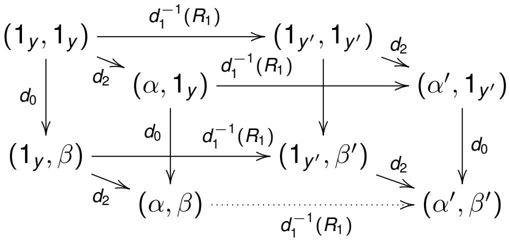
Proof.

Consider the following diagram in $X_1 \times_0 X_1$:

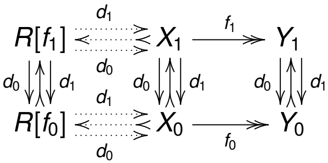


it gives rise to the following cube situation:



which means that $\alpha R_1 \alpha'$ and $\beta R_1 \beta'$ implies $\beta.\alpha R_1 \beta' \alpha'$. □

This give rises to many simple applications in the Gumm regular context we shall need later on. Given any regular epimorphisms between reflexive graphs as in the right hand side:



When $X_\bullet$ is an internal category, so is the vertical $R[f_\bullet]$.

- Accordingly, provided that the factorization $f_1 \times_0 f_1 : X_1 \times_0 X_1 \to Y_1 \times_0 Y_1$ is a regular epimorphism, the reflexive graph $Y_\bullet$ is an internal category as well.
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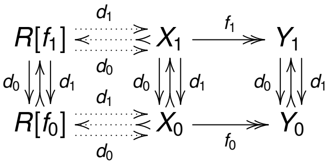

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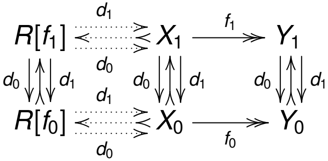

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An internal category $X_\bullet$ is reflexive graph in $\mathbb{E}$ as on the right hand side:

$$X_1 \times_0 X_1 \underset{d_0}{\overset{d_2}{\underset{\cdots d_1 \cdots}{\rightrightarrows}}} X_1 \underset{d_0}{\overset{d_1}{\underset{\longleftarrow s_0}{\rightrightarrows}}} X_0$$

together with a multiplication d_1 satisfying the well known axioms:
1) unit axioms:
1') domain unit axiom: $d_1((1_{d_0(\alpha)}, \alpha) = \alpha$;
1") codomain unit axiom: $d_1(\alpha, 1_{d_1(\alpha)}) = \alpha$;
2) incidence axioms:
2') domain incidence axiom: $d_0(d_1(\alpha, \beta)) = d_0(\alpha)$
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3) associativity axiom: $d_1(\alpha, d_1(\beta, \gamma)) = d_1(d_1(\alpha, \beta), \gamma)$.

- These are simplicial axioms for specific 3-truncated simplicial objects.

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From the simplicial notations, the Illusie "shifting" comonad $\mathbb{C}$ on simplicial objects $Simpl\mathbb{E}$, gives rise to:

$$\mathbb{C}(X_\bullet): \qquad X_1 \times_0 X_1 \times_0 X_1 \underset{d_0}{\overset{d_2}{\overrightarrow{\underrightarrow{\;\;d_1\;\;}}}} X_1 \times_0 X_1 \underset{d_0}{\overset{d_1}{\overrightarrow{\underleftarrow{\;\;s_0\;\;}}}} X_1$$

which show that this monad is stable on the subcategory $Cat\mathbb{E}$

This internal category $\mathbb{C}(X_\bullet)$ is nothing but the collection of the coslice categories of $X_\bullet$.

- With any $X_\bullet$ we can associate two meaningful parameters: 1) the endosome given by the following pullback in the fiber $Cat_{X_0}\mathbb{E}$:

$$\begin{array}{ccc} EndX_\bullet & \rightarrowtail & X_\bullet \\ \downarrow\uparrow & & \downarrow \\ \Delta_{X_0} & \rightarrowtail & \nabla_{X_0} \end{array}$$

- It is the collection of the endomorphisms of $X_\bullet$ and determines a monoid in the fiber $Pt_{X_0}\mathbb{E}$.

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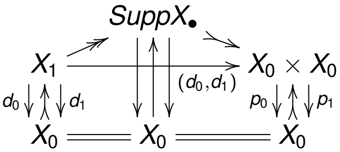

- **Proposition**

  When $\mathbb{E}$ is regular, a category $X_\bullet$ is a groupoid if and only if its endosome is group and its support an equivalence relation.

  - It is clear that a groupoid is such that its endosome is group and its support an equivalence relation.

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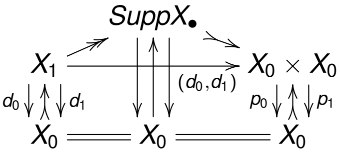

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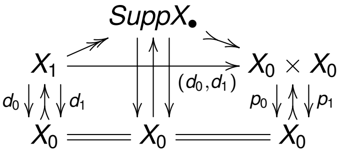

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The converse is obtained in two steps:

Lemma

In any category $\mathbb{E}$, when the endosome of an internal category $X_\bullet$ is a group, then the internal category $X_\bullet^{op} \times_0 X_\bullet$ is groupoid:

$$\begin{array}{ccc} X_\bullet^{op} \times_0 X_\bullet & \xrightarrow{p} & X_\bullet \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow \\ X_\bullet^{op} & \longrightarrow & \nabla_{X_0} \end{array}$$

Proof.

Given any map (g, f) in $X_\bullet^{op} \times_0 X_\bullet$: $\quad x \underset{g}{\overset{f}{\rightleftarrows}} y$

If the endosome of $X_\bullet$ is a group,
- 1) then $g.f$ is an isomorphism; so g is a split epimorphism.
- 2) $f.g$ is an isomorphism as well; so g is a split monomorphism.
Accordingly g is an isomorphism, and f as well by duality.
So, $X_\bullet^{op} \times_0 X_\bullet$ is groupoid. □

Lemma

When $SuppX_\bullet$ is an equivalence relation, the horizontal projection p:

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X_\bullet^{op} & \twoheadrightarrow & (SuppX_\bullet)^{op} & \rightarrowtail & \nabla_{X_0}
\end{array}$$

is a regular epimorphism.

Proof.

If the support $SuppX_\bullet$ is an equivalence relation, we get the isomorphism $\simeq$. So, the projection p is necessarily a regular epimorphism since so is $X_\bullet^{op} \twoheadrightarrow (SuppX_\bullet)^{op}$. □

- Then, when, moreover, $X_\bullet^{op} \times_0 X_\bullet$ is a groupoid, so is $X_\bullet$.

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It is well known that in a Mal'tsev variety or category $\mathbb{E}$,
any internal category is is necessarily an internal groupoid.

- These are not the only contexts. Martins-Ferreira, Rodelo, and van der Linden (2014) showed:

Proposition

In any regular category $\mathbb{E}$ the two following conditions are equivalent:
- any preorder is an equivalence relation;
- any internal category is a groupoid.
This is the case in particular for any n-permutable category.

- Then, from a work Chadja and Rachunek (1983), we get:

Proposition

A variety $\mathbb{V}$ is n-permutable for some integer n if and only if any internal category is a groupoid.

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In a Gumm category $\mathbb{E}$, starting from results of G. Janelidze and M.C. Pedicchio on pseudogroupoids (2001), M. Gran and myself (2004) showed that on a reflexive graph $X_\bullet$:

1) there is at most one multiplication satisfying the unit axioms and the domain incidence axiom;

2) any multiplication satisfying the domain unit axiom is left cancellable;

3) any multiplication satisfying axioms 1) and 2) is associative;

4) the inclusion functor $Cat\mathbb{E} \rightarrowtail Gph\mathbb{E}$ is a full inclusion.

Clearly, left cancellability is equivalent to requiring that any morphism in the category $X_\bullet$ is a a monomorphism.

- Accordingly, in a Gumm category, on a reflexive graph $X_\bullet$, there is at most one structure of internal category which is necessarily left and right cancellable,
- or equivalently in which any morphism is both monomorphic and epimorphic.
- It is worth to give a name to this specific class of categories. I propose nearly groupoid.
- Nearly groupoids are stable under subobjects in $Cat\mathbb{E}$. Any subcategory of a groupoid is a nearly groupoid.
- Seen as a special kind of category, a preorder is a nearly groupoid.

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Let $X_\bullet$ be an internal category.

The following conditions are equivalent:
1) it is a nearly groupoid;
2) the shifted categories of $X_\bullet$ and of $X_\bullet^{op}$ are preorders:

$$X_1 \times_0 X_1 \times_0 X_1 \underset{d_0}{\overset{d_2}{\underset{-d_1\rightarrow}{\rightrightarrows}}} X_1 \times_0 X_1 \underset{d_0}{\overset{d_1}{\underset{\leftarrow s_0 -}{\rightrightarrows}}} X_1$$

$$X_1 \times_0 X_1 \times_0 X_1 \underset{d_1}{\overset{d_3}{\underset{-d_2\rightarrow}{\rightrightarrows}}} X_1 \times_0 X_1 \underset{d_1}{\overset{d_2}{\underset{\leftarrow s_1 -}{\rightrightarrows}}} X_1$$

3) in set theoretical terms: any slice and coslice of the category in question is a preorder.

There is an observation that M. Gran and myself did not draw:

the unicity of the internal category structure implies that:
any unitary magma structure on a split epimorphism $(f, s) : X \rightleftarrows Y$ in the fiber $Pt_Y\mathbb{E}$ is a commutative monoid,

- because any split epimorphism can be considered as a specific kind of reflexive graph which, moreover, coincides with its dual.
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In any Gumm category, a unitary magma structure on a split epimorphism (f, s) is necessarily underlying a left and right cancellable commutative monoid.

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Corollary

In any Gumm category, a unitary magma structure on a split epimorphism (f, s) is necessarily underlying a left and right cancellable commutative monoid.

Accordingly any internal category in a Gumm category is a nearly groupoid such that its endosome:

$$\begin{array}{ccc} EndX_\bullet & \rightarrowtail & X_\bullet \\ \downarrow\uparrow & & \downarrow \\ \Delta_{X_0} & \rightarrowtail & \nabla_{X_0} \end{array}$$

is commutative. Or, in other words, any internal category $X_\bullet$ is a commutative nearly groupoid.

Outline

Congruence modular varieties

Brief recalls about internal categories

Internal categories in Gumm categories

Internal categories in congruence modular varieties

In this context, the characterization of the internal categories is more constrained. For that we shall need some more results.
In the same way as in the Mal'tsev varieties, we get:

- **Proposition**

  In a congruence modular variety $\mathbb{V}$, any split epimorphism $(f, s) : X \rightleftarrows Y$ has a universal associated abelian group object in the fiber $Pt_Y\mathbb{V}$.

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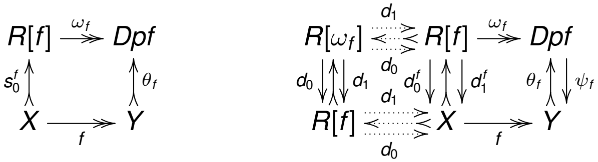
In the same way as in the Mal'tsev varieties, we get:

- **Proposition**

  In a congruence modular variety $\mathbb{V}$, any split epimorphism $(f, s) : X \rightleftarrows Y$ has a universal associated abelian group object in the fiber $Pt_Y\mathbb{V}$.

Start with the following left hand side pushout:



and complete the diagram with the kernel relations.

- In any $\mathbb{E}$, on the vertical left hand side, we get a reflexive relation. It is symmetric since the twisting isomorphism on $R[f]$ produces an involutive isomorphism on Dpf.
- Since $\mathbb{V}$ is a Gumm category, according to our main result the vertical left hand side reflexive relation is transitive. So, we get a let hand side vertical equivalence relation.
- Moreover, in any congruence modular variety $\mathbb{V}$, the equivalence relations $R[d_0^f]$ and $R[\omega_f]$ do permute, which is equivalent to saying that the downward square indexed by 0 is a regular pushout.

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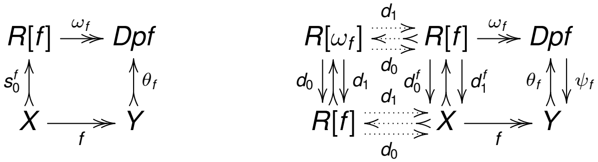

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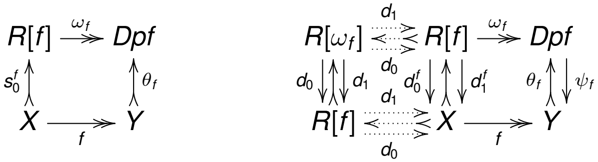

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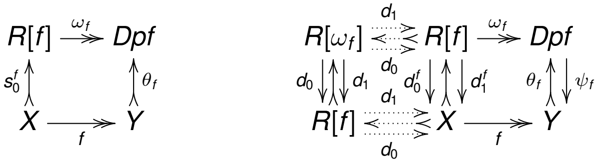

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Accordingly the vertical right hand side split epimorphism $Dpf \rightleftarrows Y$ is endowed with a structure of internal groupoid (=abelian group).

- We can then show that it is the universal abelian group object associated with (f, s) by adding the left hand side part:

- and by checking that the comparison morphism $\epsilon(f, s)$ is a regular epimorphism.
- So, the inclusion $Ab(Pt_Y\mathbb{V}) \rightarrowtail Pt_Y\mathbb{V}$ is stable under monomorphism.

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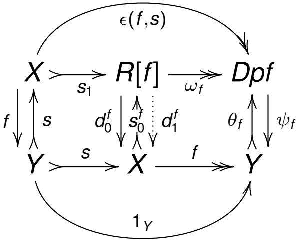

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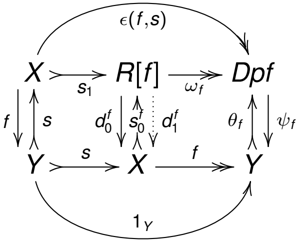

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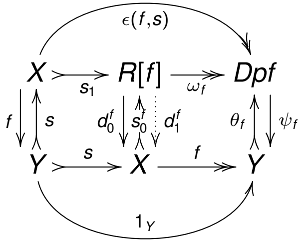

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Theorem

In a congruence variety $\mathbb{V}$, any unitary magma structure on a split epimorphism (f, s) is necessarily an abelian group.

Proof.

In any Gumm category, any unitary magma structure on a split epimorphism (f, s) is necessarily a commutative left and right cancellable monoid.

- So, since any variety is exact, we can then universally embed any unitary magma structure on (f, s) into an abelian group:

$$\begin{array}{ccc} X & \rightarrowtail & Ab(f,s) \\ {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} & & {\scriptstyle \bar{f}}\downarrow\uparrow{\scriptstyle \bar{s}} \\ Y & = & Y \end{array}$$

  mimicking the construction of $\mathbb{Z}$ from $\mathbb{N}$.
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Corollary

Any internal category in a congruence modular variety $\mathbb{V}$ is a nearly groupoid whose endosome is an abelian group.

- we have then many applications, for instance:

Proposition

An internal category $X_\bullet$ in a congruence modular variety $\mathbb{V}$ is a groupoid if and only if its support in an equivalence relation.

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In a congruence distributive variety,
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