Louvain-la-Neuve

2023 CT

International CATEGORY THEORY Conference

Programme and Abstracts

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Schedule of the Conference

	Sunday 02/07	Monday 03/07	Tuesday 04/07	Wednesday 05/07	Thursday 06/07	Friday 07/07	Saturday 08/07
09.00 9.30 10.00		Opening 9.00 - 9.30 George Janelidze 9.30 - 10.30	Vanessa Miemietz	Christina Vasilakopoulou	Luca Reggio	Paolo Perrone	Steve Lack
10.00 10.30			Steve Awodey	Maru Sarazola	Marcelo Fiore	Joachim Kock	Isar Stubbe
10.30 11.00		Coffee break	Coffee break	Coffee break	Coffee break	Coffee break	Coffee break
11.00 11.30		Dominique Bourn	Graham Manuell	Ieke Moerdijk	Paul Taylor	Jiří Adámek	Susan Niefield
11.30 12.00		Diana Rodelo	Nathanael Arkor	Eugenia Cheng	Benedikt Ahrens	Dirk Hofmann	Rory Lucyshyn-Wright
12.00 12.30		Andrea Montoli	John Bourke	Clemens Berger	Matías Menni	Simona Paoli	Tom Leinster
12.30 14.30		Lunch	Lunch	Lunch	Lunch	Lunch	
14.30 15.00		David Jaz Myers / Alan Cigoli	Lurdes Sousa / Jonathan Weinberger	Social activities	Dorette Pronk / -	Giacomo Tendas / Nima Rasekh	
15.00 15.30		Lyne Moser / Yuto Kawase	Luigi Santocanale / Joshua Wrigley	Social activities	Miloslav Štěpán / Ivan Di Liberti	Cipriano Cioffo / Yuki Maehara	
15.30 16.00		Philip Hackney / F. Lucatelli Nunes	Matheus Duzi / Davide Trotta	Social activities	Coffee break	Coffee break	
16.00 16.30		Coffee break	Coffee break	Social activities	Bryce Clarke / Ryuya Hora	Jacopo Emmenegger / Martina Rovelli	
16.30 17.00		Marcello Lanfranchi / Aline Michel	Lawvere Session	Social activities	Poster Session	Rui Prezado / Rhiannon Griffiths	
17.00 17.30	Welcome Drink	Ruben Van Belle / N. Martins-Ferreira	Lawvere Session	Social activities	Poster Session	Poster Session	
17.30 18.00	Welcome Drink	Daniel Luckhardt / Michael Hoefnagel	Lawvere Session	Social activities		Poster Session	
Evening	Drink ends at 19.00				Conference Dinner 19.30 - 0.00		

The plenary talks will take place in room A.10, the parallel talks in room A.02 (if the name of the speaker is on the left) and in room A.03 (if the name of the speaker is on the right).

Poster Sessions

Thursday - Room A.04 A/B

Hisashi Aratake

Nicola Carissimi

Boris Chorny

Alex Corner

Florian De Leger

Matthew Di Meglio

Nicola Di Vittorio

Bojana Femić

Tom Hirschowitz

Jan Jurka

Guillermo Lopez Cafaggi

Arne Mertens

Alex Osmond

Philip Saville

Friday - Room A.04 A/B

Dylan Braithwaite

Elena Caviglia

Cédric de Lacroix

Bo Shan Deval

Arnaud Duvieusart

Jens Hemelaer

Jérémie Marquès

Luca Mesiti

Jason Parker

Sanjiv Ranchod

Alessio Santamaria

Andrew Slattery

Marek Zawadowski

Tony Zorman

Session in honour of Bill Lawvere

Speakers at this commemorative session will highlight some of Bill Lawvere's many contributions to Category Theory and beyond, reflect on their importance, and share personal memories. Anders Kock will give the main presentation, followed by short contributions by Peter Johnstone, Ieke Moerdijk, Matias Menni, Giuseppe Rosolini and George Janelidze.

The session will be chaired by Walter Tholen.

Abstracts

Invited speakers

A new direction in non-pointed categorical algebra

George Janelidze

University of Cape Town

In contrast to semi-abelian categories [8], a larger class of categories, tentatively called *nearly semi-abelian* was introduced in [7]. There are several definitions whose equivalence to each other requires a rather long proof, which, at the same time, motivates them. The (seemingly) simplest definition requires Barr exactness, Bourn protomodularity, the existence of finite coproducts, and the morphism $0 \to 1$ to be a regular epimorphism; it follows that such a category is semi-abelian if and only if that regular epimorphism is an isomorphism (or, equivalently, if and only if the category is quasi-pointed in the sense of [5]). In addition to semi-abelian categories, examples of such categories include all protomodular varieties of universal algebras and cotoposes (=categories dual to elementary toposes). The results presented in this talk use a new notion of *essentially nullary monad*, and a kind of new approach to *theory of ideals* and to *action representability.* Relevant references will also include [6], [4], [1], [2], [3], [10], and [9].

References

[1] Borceux, F. Non-pointed strongly protomodular categories. *Appl. Categ. Structures* 12 (2004), no. 4, 319-338.

[2] Borceux, F.; Janelidze, G; Kelly, G. M. Internal object actions. *Comment. Math. Univ. Carolin.* 46 (2005), no. 2, 235-255.

[3] Borceux, F.; Janelidze, G; Kelly, G. M. On the representability of actions in a semi-abelian category. *Theory Appl. Categ.* 23 (2005), no. 11, 244-286.

[4] Bourn, D. Normal subobjects and abelian objects in protomodular categories. *J. Algebra* 228 (2000), no. 1, 143-164.

[5] Bourn, D. 3×3 Lemma and protomodularity. *J. Algebra* 236 (2001), no. 2, 778-795.

[6] Carboni, A.; Janelidze, G. Modularity and descent. *J. Pure Appl. Algebra* 99 (1995), no. 3, 255-265.

[7] Janelidze, G. Nearly semi-abelian categories. *Invited talk, Conf. "Categories, Rings and Modules" in honour of Alberto Facchini*, January 27-28, 2023.

[8] Janelidze, G.; Márki, L.; Tholen, W. Semi-abelian categories. *J. Pure Appl. Algebra* 168 (2002), no. 2-3, 367-386.

[9] Lapenta, S.; Metere, G.; Spada, L. Relative ideals in homological categories, with an application to MV-algebras. *ArXiv* 2208.12597 [math.CT].

[10] Ursini, A. Normal subalgebras, I. *Appl. Categ. Structures* 21 (2013), no. 3, 209-236.

Enriched accessible categories*

Steve Lack

Macquarie University

Many basic categorical notions, either enriched or unenriched, can be defined in terms of representability of some functor: examples include the notions of limit, colimit, and adjunction. One therefore obtains weakened notions of limit, colimit, or adjunction, upon weakening the requirement of representability of the presheaf in question. An important example arises when one replaces representable functors by small ones. This gives rise to a whole series of notions which we call "virtual". We develop a theory of accessibility based on the resulting notions of virtual colimit, virtual adjoint, virtual reflectivity, and virtual orthogonality class. The resulting theory is less Set-based than existing theories of accessibility, and thus well adapted to the enriched setting.

There are also other possibilities than the small functors, which give rise to various other known variants, such as the "multi"-notions of Diers, the "poly"-notions of Lamarche and others, as well as further possibilities: some or known, others new; some are general, others involve specific enrichment contexts.

*Joint work with Giacomo Tendas.

Categorification in Representation Theory

Vanessa Miemietz

University of East Anglia

In the last 25 years, tremendous progress has been made in representation theory, using higher categorical methods. This area is now called categorification. I will explain and motivate the ideas behind this, before focusing on specific examples.

Universal properties in probability theory *

Paolo Perrone

University of Oxford

In its early days, category theory was developed primarily for algebra, geometry, and topology. Several constructions and results in algebra and geometry have since been expressed in terms of universal properties. This point of view has provided us with a much deeper understanding of these structures, and of algebra and geometry in general. Today it seems almost impossible to imagine, for example, how the field of algebraic topology would look without category theory, as well as the other way around.

In the last few years we have experienced a rise in applications of category theory to domains such as probability, statistics, and information theory. What universal properties do we see in these contexts? The universal constructions of relevance here are less known in the community than their algebraic and geometric counterparts, but they can give us just as well a deep understanding of some aspects of probability. In this talk we will focus on three key examples (or more, depending on time and interest):

- We will show how finite products are fundamentally incompatible with the idea of uncertainty and correlations;
- We will show how the incompatibility above vanishes "at infinity", by expressing the Kolmogorov extension theorem in terms of a cofiltered limit;
- We will formulate De Finetti's theorem as a particular equalizer construction, which only works in categories where the arrows have randomness.

We will use the language of *probability monads* and *Markov categories*, which we will briefly introduce, and for which we will assume no previous knowledge.

*This talk is based on my work so far with Tobias Fritz, Tomáš Gonda, Sean Moss, Eigil F. Rischel, and Dario Stein, as well as other researchers in the categorical probability community.

Resource-sensitive model theory: a categorical view*

Luca Reggio

University College London

Since the pioneering work of Lawvere in the 1960s, category theory has been applied to yield a syntax-independent view of the fundamental structures of logic, encompassing for example first-order logic and extensions to infinitary and higher-order languages.

In this talk, motivated by the needs of finite model theory and descriptive complexity, I will focus on logic fragments defined by restricting access to the *logical resources*. Thus, we may consider e.g. finite-variable logics or logics with bounded quantifier rank. These resource-sensitive logics, along with the corresponding combinatorial parameters of (relational) structures, play a pivotal role in finite model theory.

A key insight, due to Abramsky, Dawar and their collaborators [1, 5], is that—in many cases of interest—these resource-sensitive logic fragments can be described by means of comonads on the category of structures, and the associated combinatorial parameters by means of the Eilenberg-Moore coalgebras for the comonads. This is at the origin of the framework of *game comonads*, which I will outline.

In order to capture the essential properties of game comonads and their categories of coalgebras in a purely axiomatic fashion, I shall introduce the framework of *arboreal categories* [2, 3]. Roughly speaking, these are categories having a dense subcategory of *paths*. Arboreal categories have an intrinsic process structure that allows to define notions such as bisimulation and back-and-forth systems/games, which are then transferred to "extensional" structures, e.g. the category of relational structures, via (resource-indexed) *arboreal adjunctions*.

This can be regarded as a first step towards a *resource-sensitive axiomatic model theory*, opening up a landscape in which the degree of tractability of a logic is related to properties of the corresponding arboreal adjunction. An example of this perspective "in action" is the recent study of homomorphism preservation theorems in logic through the lens of arboreal categories [4].

It is interesting to compare this with the use of accessible categories as an axiomatic framework for abstract elementary classes [6]. Whereas the latter is aimed at extending first-order model theory into the infinite, replacing compactness by λ-accessibility, we are interested in capturing fine structure "down below", typically in fragments of first-order logic. Accordingly, I will explain how Gabriel-Ulmer duality can be applied to show that *finitely accessible* arboreal adjunctions cannot distinguish between structures that are equivalent in the logic $\mathcal{L}_{\infty,\omega}$, thus determining the expressive power of these adjunctions [7].

References

[1] Abramsky, S.; Dawar, A.; Wang, P. The pebbling comonad in finite model theory. *32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, 2017, 1–12.

[2] Abramsky, S.; Reggio, L. Arboreal categories and resources. *48th International Colloquium on Automata, Languages, and Programming (ICALP)*, vol. 198 (2021), 115:1–115:20.

*Talk based on joint works with Samson Abramsky and Colin Riba.

[3] Abramsky, S.; Reggio, L. Arboreal categories: An axiomatic theory of resources. *Preprint available at* `https://arxiv.org/abs/2102.08109`, 2022.

[4] Abramsky, S.; Reggio, L. Arboreal categories and homomorphism preservation theorems. *Preprint available at* `https://arxiv.org/pdf/2211.15808`, 2023.

[5] Abramsky, S.; Shah, N. Relating structure and power: Comonadic semantics for computational resources. *27th EACSL Annual Conference on Computer Science Logic (CSL)*, 2018, 2:1–2:17.

[6] Beke, T.; Rosický, J. Abstract elementary classes and accessible categories. *Annals of Pure and Applied Logic*, vol. 163 (2012), no. 12, 2008–2017.

[7] Reggio, L.; Riba, C. Finitely accessible arboreal adjunctions and Hintikka formulae. *Preprint available at* `https://arxiv.org/abs/2304.12709`, 2023.

Sweedler theory for double categories

Christina Vasilakopoulou

National Technical University of Athens

Back in the late '60s, Moss Sweedler introduced the concept of a "measuring k-coalgebra" as a space of generalized k-algebra maps. A particular case is that of the finite dual of a k-algebra, namely the coalgebra with the property that coalgebra maps into it naturally correspond to algebra maps into the classical linear dual of a coalgebra. Gavin Wraith was the first to observe that measuring coalgebras induce an enrichment of the category of k-algebras in k-coalgebras. Interestingly, in modern terms, this renders the category of k-algebras an example of a semi-Hopf linear category [2]. Anel and Joyal first referred to the (tensored and cotensored) enrichment of dg-algebras in dg-coalgebras along with involved structures related to the bar-cobar construction as "Sweedler theory", whose terminology we follow.

In this talk, we will investigate how this fact of an enrichment of monoids in comonoids, established in a broader context of locally presentable and braided monoidal closed categories [4], can lead to a many-object generalization in the setting of monoidal double categories [7, 1]. In the process of capturing such results in other double categories of interest, it turns out that the structure of an *oplax* monoidal double category [3] is required, which in its trivial one-object case returns the context of a duoidal category and the respective enrichment results therein [5]. Additionally, what comes naturally together with such dual algebraic structures are the fibration and opfibration of modules and comodules over them: pleasantly, these fit into an "enriched fibration" picture [6] in the worked out cases, and are envisioned to provide insight to further cases of interest, for example that of (co)operads and their (co)modules.

References

[1] Aravantinos-Sotiropoulos, V.; Vasilakopoulou, C. Enriched duality in double categories II: V-modules and V-comodules. *In preparation.*

[2] Buckley, M.; Fieremans, T.; Vasilakopoulou, C.; Vercruysse, J. Oplax Hopf algebras. *Higher structures* 5 (2021), no. 1, 71–120.

[3] Gambino, N.; Garner, C.; Vasilakopoulou, C. Monoidal Kleisli bicategories and the arithmetic product of symmetric sequences, *Preprint* arXiv:2206.06858.

[4] Hyland, J.M.E.; Lopez Franco, I.; Vasilakopoulou, C. Hopf measuring comonoids and enrichment. *Proceedings of the London Mathematical Society* 115 (2017), no. 5, 1118–1148.

[5] Lopez Franco, I.; Vasilakopoulou, C. Duoidal categories, measuring comonoids and enrichment. *Preprint* arXiv:2005.01340.

[6] Vasilakopoulou, C. On enriched fibrations. *Cahiers de topologie et géométrie différentielle catégoriques* 50 (2018), no. 4, 354–387.

[7] Vasilakopoulou, C. Enriched duality in double categories: V-categories and V-cocategories. *Journal of Pure and Applied Algebra* 223 (2019), no. 7, 2889–2947.

Abstracts

Plenary, parallel and poster sessions

Strongly Finitary Monads and Varieties of Algebras*

Jiří Adámek

Czech Technical University Prague and Technical University Braunschweig

Classical varieties of algebras, presented by equations between terms, are well known to bijectively correspond to finitary monads on Set. Kurz and Velebil proved a corresponding result for varieties of ordered algebras (presented by inequations between terms): an enriched monad on the cartesian closed category Pos is *strongly finitary* if it is the left Kan extension of its restriction to finite discrete posets. Strongly finitary monads on Pos bijectively correspond to varieties of ordered algebras, see [4].

For a general monoidal closed category $\mathcal{V}$ strong finitarity was introduced by Kelly and Lack [3]: let $J : \mathcal{V}_f \hookrightarrow \mathcal{V}$ denote the full subcategory of finite copowers $I + I + ...I$ of the monoidal unit. An enriched monad T on $\mathcal{V}$ is *strongly finitary* if it is the left Kan extension of its restriction to $\mathcal{V}_f$:

$$T = \mathrm{Lan}_J T \cdot J.$$

An analogous result to that for Pos is, surprisingly, also true in some categories that fail to be locally finitely presentable: metric spaces, complete metric spaces, and complete posets.

Complete Posets

We work in the cartesian closed category CPO of cpos (posets with joins of ω-chains) and continuous maps (preserving joins of ω-chains). Here $J : \mathsf{CPO}_f \hookrightarrow \mathsf{CPO}$ is the full subcategory of finite sets (considered as discrete posets).

Using Kelly's concept of density presentation of the functor J, we prove that an enriched endofunctor on POS is strongly finitary iff it is finitary and preserves coinserters of reflexive pairs. The proof is based on the following

Proposition. In every cartesian closed category directed colimits commute with finite products.

Given a finitary signature, a *continuous algebra* is a cpo with continuous operations. For presentations of classes of continuous algebras one uses equations between *extended terms*. They allow (besides the usual composite terms $t = \sigma(t_1, ..., t_n)$ for n-ary operations σ) also formal joins $t = \bigvee_{n<\omega} t_n$. Let f be an interpretation of variables in an algebra A. Then the interpretation of extended terms in a A is a partial map $t \mapsto f'(t)$: it is defined in $t = \bigvee_{n<\omega} t_n$ iff all $f'(t_n)$ are defined and form a chain; then $f'(t) = \bigvee f'(t_n)$.

A *variety of continuous algebras* is a class presented by a set of equations between extended terms. All varieties form a category with concrete functors (those commuting with the forgetful functors to **CPO**) as morphisms.

Theorem. Every variety of continuous algebras has free algebras, and the resulting monad on CPO is strongly finitary. Conversely, the Eilenberg-Moore category of a strongly finitary monad on CPO is concretely isomorphic to a variety.

Corollary. The category of strongly finitary monads on CPO is dually equivalent to the category of varieties of continuous algebras.

*Joint work with Matěj Dostál and Jiří Velebil. Abstract submitted to CT2023.

Metric Spaces and Complete Metric Spaces

The category Met of metric spaces (*extended*: the distance ∞ is allowed) has as morphisms the nonexpanding maps. It is enriched w.r.t. the sum-metric on the cartesian product: the distances in $X \otimes Y$ are $d((x, y), (x', y')) = d(x, y) + d(x', y')$. Here $\mathsf{M}et_f$ is the category of finite sets (considered as discrete metric spaces).

Mardare et al. [5] introduced the concept of a *(complete) quantitative algebra* which is a (complete) metric space A with n-ary operations that are nonexpanding from A^n (endowed with the maximum metric) to A. A *quantitative equation* is an expression $t =_\varepsilon t'$ where t and t' are terms and $\varepsilon \geq 0$ is a rational number. A quantitative algebra satisfies this equation if the computation of the terms always results in elements of distance at most ε.

A *variety of quantitative algebras* is a class presented by a set of quantitative equations.

Theorem. Every variety of quantitative algebras has free algebras, and the resulting monad on Met is strongly finitary. Conversely, the Eilenberg-Moore category of a strongly finitary monad on Met is concretely isomorphic to a variety.

The same result is true for the full subcategory CMet of complete metric spaces: strongly finitary monads on it are dually equivalent to varieties of complete quantitative algebras.

Corollary. The category of strongly finitary monads on Met or CMet is dually equivalent to the category of varieties of (complete) quantitative algebras.

References

[1] Adámek, J.; Dostál, M.; Velebil, J. Quantitative algebras and a classification of metric monads. arXiv:221001565

[2] Adámek, J.; Dostál, M; Velebil, J. Strongly finitary monads for continuous algebras. arXiv:2301.01034

[3] Kelly, G.M.; Lack, S. Finite-products-preserving functors, Kan extensions and strongly finitary 2-monads. *Appl. Categ. Structures*1(1993), 85-94.

[4] Kurz, A.; Velebil J. Quasivarieties and varieties of ordered algebras: regularity and exactness. *Math. Structures Comput. Sci.* 27 (2016), 1-42.

[5] Mardare, R.; Panangaden, P.; Plotkin, G.D. Quantitative algebraic reasoning. *IEEE Proceedings LICS 2016*, 700-709.

The Univalence Principle*

Benedikt Ahrens

Delft University of Technology

We argue that univalent foundations (UF) are exactly the right foundations to do (higher) category theory in. In particular, we show how, in UF, proofs and constructions on mathematical objects, in particular, on (higher) categories, transfer across equivalence of mathematical structures. The talk is based on [2].

Background

The *equivalence principle* is an informal principle stating that reasoning in mathematics should be invariant under a suitable notion of "sameness" of mathematical structures. The equivalence principle is not usually enforced; indeed, the statement $1 \in \mathbb{N}$ is not invariant under isomorphism of sets.

Blanc [3] and Freyd [5] devised a syntax for logical statements about categories that is invariant under equivalence of categories. The obstacle to being invariant under equivalence of categories is *equality of objects*. To avoid mentioning equality of objects even in the definition of composition, Blanc and Freyd work in a *dependently typed language*; there, morphisms of a category are given, for any pair (a, b) of objects, by a type $\hom(a, b)$, and composition is specified by a family of functions $(\circ)_{a,b,c}$: $\hom(b, c) \times \hom(a, b) \to \hom(a, c)$ for any triple (a, b, c) of objects. Composition of morphisms can thus be specified without an equality hypothesis on the objects of the category.

Makkai [7] extended this idea to a wide class of categorical structures, using a first-order logic with dependent sorts (FOLDS). To this end, he devises a notion of *signature* for (higher-)categorical structures, where a signature specifies the data (sorts and operations on those sorts) of a structure. Such a signature is given by an inverse category specifying the dependency between the sorts. A *model* of such a signature is then given by a suitable functor from that signature into the category of sets. Makkai shows that his FOLDS is invariant under equivalence of structures. By equipping the sort of arrows A explicitly with an equality predicate E, and omitting such an equality predicate on objects O, Makkai disallows the source of non-invariance in his FOLDS. Makkai's work does not provide an invariant foundation of mathematics, but rather an invariant "interface" to set theory. Invariance is restricted to statements; FOLDS does not allow one to express *constructions* on mathematical structures.

The Univalence Principle

Inspired by Makkai's work, Voevodsky designed UF as a foundation of mathematics in which both proofs and constructions on structures can be transported across equivalence of structures. The underlying formalism of UF is Martin-Löf type theory (MLTT), a higher-order logic with dependent types. Intuitively, types represent spaces; this is made precise, for instance, in Voevodsky's interpretation of UF in Kan complexes [6]. Thus, in UF, mathematical objects are built from spaces rather than sets. Any construction on a type/space A in UF is automatically invariant under "paths" $a \rightsquigarrow a'$ in A. Discrete types/spaces play the rôle of sets in UF. To MLTT, Voevodsky added his "univalence axiom" (see, e.g., [6]), which states that constructions on types are invariant under equivalence of types. Formally, the univalence

*Joint work with Paige R. North and Michael Shulman and Dimitris Tsementzis. Abstract submitted to CT2023.

axiom (for a given universe, that is, a given type of types, $\mathcal{U}$) forces paths in $\mathcal{U}$ between two types A and B to coincide with equivalences between A and B.

Working in UF thus gives one the option to build mathematical objects out of discrete spaces ("sets")—whenever equality of elements is desired—or, instead, of more general spaces where such an equality is not desired or feasible. Groups, rings, and other algebraic structures are suitably formalized to have an underlying discrete space. Categories are used differently in different contexts. On the one hand, categories are sometimes used in the spirit of an (essentially) algebraic theory, with equality on objects. On the other hand, categories might be used "categorically", where the use of equality is avoided. UF offer the option of defining two different notions of category, each purpose-built for the aforementioned uses. The first kind, of "strict" category is defined using a *discrete* type of objects. The second kind of category is defined using a not-necessarily-discrete type of objects. However, without imposing any further restriction, such a category has superfluous data in the form of the spatial data on the type of objects. This issue can be solved by imposing a "completeness" condition as in [1]: one asks that the spatial structure coincides with the categorical structure by imposing that for any objects $a, b : \mathcal{C}_0$, the paths $a \rightsquigarrow b$ correspond to isomorphisms $a \cong b$. (Under Voevodsky's interpretation of UF in Kan complexes [6], categories are interpreted as truncated Segal spaces, and the univalence condition corresponds to a completeness condition for such Segal spaces.) Most categories of interest are univalent, but not strict. For univalent categories, one can then show a univalence principle: paths $\mathcal{C} \rightsquigarrow \mathcal{D}$ correspond to adjoint equivalences $\mathcal{C} \simeq \mathcal{D}$; consequently, any construction on univalent categories is invariant under equivalence of categories.

In our work, we extend the completeness (or univalence) condition given in [1] for categories to other mathematical structures. Specifically, we define a notion of *theory* for mathematical structures. For any theory, we obtain, mechanically, a notion of *model*, *univalence* of models, and *equivalence* of models. We then prove a univalence principle for univalent models: we show that paths between univalent X and Y correspond to equivalences $X \simeq Y$. As before, this entails that any construction on univalent models in UF transports along equivalence of models. Our notion of theory encompasses many mathematical structures, such as structured sets, structured categories (e.g., monoidal categories), higher categories (e.g., bicategories, double bicategories), "enhanced" (higher) categories (e.g., involutive categories), and many more.

References

[1] Ahrens, B.; Kapulkin, K.; Shulman, M. Univalent categories and the Rezk completion. *Mathematical Structures in Computer Science* Volume 25 (2015), no. 05, 1010–1039

[2] Ahrens, B.; North, P. R.; Shulman, M.; Tsementzis, D. The Univalence Principle. https://arxiv.org/abs/2102.06275v3, 2022, to be published in Memoirs of the AMS

[3] Blanc, G. Équivalence naturelle et formules logiques en théorie des catégories. *Arch. Math. Logik Grundlag.* 19 (1978), no. 3-4, 131–137

[4] Coquand, T.; Danielsson, N. A. Isomorphism is equality, *Indagationes Mathematicae* Volume 24 (2013), no. 4, 1105 – 1120,

[5] Freyd, P. Properties invariant within equivalence types of categories, *Algebra, topology, and category theory (a collection of papers in honor of Samuel Eilenberg)*, 1976, pp. 55–61.

[6] Kapulkin, K.; Lumsdaine, P. L. The simplicial model of univalent foundations (after voevodsky), *Journal of the European Mathematical Society* Volume 23 (2021), no. 6, 2071–2126,

[7] Makkai, M. First order logic with dependent sorts, with applications to category theory, http://www.math.mcgill.ca/makkai/folds/foldsinpdf/FOLDS.pdf, 1995.

Spectra of Modelled Spaces à la Coste, Revisited*

Hisashi Aratake

Research Institute for Mathematical Sciences, Kyoto University

In this talk, we present the results of [1], which shed new light on Coste's theory of spectra [3].

Another construction of Coste spectra: Coste's theory of spectra generalizes various constructions of spectra of algebras, such as Zariski spectra of rings and prime spectra of distributive lattices, which give rise to dual adjunctions between categories of algebras and categories of "locally algebra-ed" spaces (or toposes). Coste's framework was greatly inspired by Cole's theory of topos-theoretic spectra [2], but the conditions involved in it are much more easily verified. A *Coste context*, which Coste originally called a *localisation triple*, consists of a triple (T_0, T, Λ) such that

- T_0 is a (for simplicity, single-sorted) cartesian $\mathcal{L}$-theory,
- Λ is a set of pairs $(\varphi(\boldsymbol{u}), \psi(\boldsymbol{u}, \boldsymbol{v}))$ of Horn $\mathcal{L}$-formulas with $T_0 \models \psi(\boldsymbol{u}, \boldsymbol{v}) \vdash \varphi(\boldsymbol{u})$, and
- T is a coherent $\mathcal{L}$-theory obtained by adding to T_0 some axioms of the form $\varphi(\boldsymbol{u}) \vdash \bigvee_i \exists \boldsymbol{v}_i \psi_i(\boldsymbol{u}, \boldsymbol{v}_i)$ for finitely many pairs $(\varphi(\boldsymbol{u}), \psi_i(\boldsymbol{u}, \boldsymbol{v}_i)) \in \Lambda$.

Λ determines a class of "admissible homomorphisms" between T_0-models. For example, in the context for Zariski spectra, admissible homomorphisms of rings are ring homomorphisms reflecting invertible elements. Additionally, a Coste context is said to be *spatial* if each $(\varphi(\boldsymbol{u}), \psi(\boldsymbol{u}, \boldsymbol{v})) \in \Lambda$ satisfies $T_0 \models \psi(\boldsymbol{u}, \boldsymbol{v}) \wedge \psi(\boldsymbol{u}, \boldsymbol{v'}) \vdash \boldsymbol{v} = \boldsymbol{v'}$. Most examples of Coste contexts are spatial in this sense, while a notable exception is the context for the étale topos of a ring. We write T_0-**Mod** for the category of T_0-models and $\mathbb{A}$-**ModSp** for the category of T-modelled spaces and "admissible morphisms." Coste argued that, for any spatial Coste context, the global section functor Γ has a right adjoint Spec

$$\mathbb{A}\text{-}\mathbf{ModSp} \underset{\mathrm{Spec}}{\overset{\Gamma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} T_0\text{-}\mathbf{Mod}^{\mathrm{op}} .$$

However, most of his proofs remain unpublished, and Osmond [6] has recently filled the details in the case of general (not necessarily spatial) Coste contexts, where $\mathbb{A}$-**ModSp** is replaced by the bicategory of T-modelled toposes and admissible morphisms.

Our main contributions are twofold: we provide an alternative construction of spectra of T_0-models for *spatial* Coste contexts, and we prove that it is equivalent to Coste's original construction. We outline it below: for a T_0-model A, using Λ, Coste defined a join-semilattice $\mathcal{V}_A$ which is (categorically equivalent to) a subcategory of the coslice category $A \backslash T_0$-**Mod**. Let A_λ denote the codomain of $\lambda \in \mathcal{V}_A$. Every $\lambda \colon A \to A_\lambda$ is an epimorphism in T_0-**Mod**. We define the underlying set of $\mathrm{Spec}(A)$ as the set of all the ideals $I \subseteq \mathcal{V}_A$ for which the colimit $A_I := \varinjlim_{\lambda \in I} A_\lambda$ is a T-model. It is equipped with the topology generated by the basic open sets $D_\lambda = \{ I \in \mathrm{Spec}(A) \mid \lambda \in I \}$ for $\lambda \in \mathcal{V}_A$. This construction is essentially the same as Coste's.

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For an open subset U of $\mathrm{Spec}(A)$, we define the sheaf S of T_0-models on $\mathrm{Spec}(A)$ by

$$SU := \left\{ s \in \prod_{I \in U} A_I \;\middle|\; \begin{array}{l} \forall J \in U,\ \exists \lambda \in J \text{ (i.e. } J \in D_\lambda),\ \exists a \in A_\lambda, \\ D_\lambda \subseteq U, \text{ and } \forall I \in D_\lambda,\ s_I = a_I. \end{array} \right\},$$

where a_I is the image of a under $A_\lambda \to A_I$. For $\lambda \in \mathcal{V}_A$, there exists a canonical homomorphism $A_\lambda \to S(D_\lambda)$, which is not necessarily an isomorphism. The construction of S directly generalizes the usual one of the structure sheaves of Zariski spectra of commutative rings. We then prove that the stalk S_I at I is isomorphic to A_I, and this observation leads to a direct proof of Coste adjunction. These results involve only elementary arguments on the locally finitely presentable category T_0-**Mod**. We also prove that the sheaf topos **Sh**(Spec(A)) equipped with the structure sheaf S is equivalent to Coste's construction as T-modelled toposes. Since this latter result employs more sophisticated methods in topos theory and categorical logic, we will not present any technical details in this talk.

Limits, colimits and spectra of modelled spaces: Cole and Coste considered more general constructions of spectra of modelled toposes, and it is natural to reconsider spectra of modelled spaces in our framework. Indeed, we extend the above construction to "relative spectra" of T_0-modelled spaces. Let T_0-**ModSp** be the category of T_0-modelled spaces.

Theorem For any T-modelled space (X, P), there exists an adjunction between the slice categories

$$\mathbb{A}\text{-}\mathbf{ModSp}/(X,P) \underset{\text{Spec}}{\overset{\text{forgetful}}{\rightleftarrows}} T_0\text{-}\mathbf{ModSp}/(X,P) \quad (\text{forgetful} \dashv \text{Spec}).$$

Our proof again relies only on elementary categorical arguments on T_0-**Mod**. Relativization of spectra is crucial to yield the "limit part" of the following theorem:

Theorem The categories T_0-**ModSp** and $\mathbb{A}$-**ModSp** have small limits and colimits.

In this sense, we would like to emphasize that a Coste context is an appropriate setting for treating limits and colimits of modelled spaces. Completeness and cocompleteness of the category of locally ringed spaces were proved in [5] and [4], respectively. As an interesting corollary of the above theorem, we can see that the category of ringed spaces whose stalks are fields is complete and cocomplete.

References

[1] H. Aratake, Limits, Colimits, and Spectra of Modelled Spaces. preprint (2022), arXiv:2203.10711

[2] J. C. Cole, The Bicategory of Topoi and Spectra. *Reprints in Theory and Applications of Categories* (2016), no. 25, 1–16.

[3] M. Coste, Localisation, Spectra and Sheaf Representation. In: Applications of Sheaves. *Lecture Notes in Mathematics* 753. Springer-Verlag, 1979. 212–238.

[4] M. Demazure and P. Gabriel, Introduction to Algebraic Geometry and Algebraic Groups. *North-Holland Mathematics Studies* 39. North-Holland, 1980.

[5] W. D. Gillam, Localization of Ringed Spaces. *Advances in Pure Mathematics* 1 (2011), no. 5, 250–263.

[6] A. Osmond, The General Construction of Spectra. preprint (2021), arXiv:2102.01259

The formal theory of relative monads*

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The theory of monads on categories admits a fruitful generalisation to the theory of monads in a 2-category [1]. In this way, many of the usual theorems regarding monads – for instance, that every adjunction induces a monad, and that every monad is realised by initial and terminal resolutions – may be proven just once, recovering as special cases the corresponding statements for ordinary monads, enriched monads, internal monads, and so on.

The notion of a relative monad is a generalisation of the notion of a monad, where the underlying functor is permitted to be an arbitrary functor, rather than an endofunctor [2, 3]. The theory of relative monads is in many respects similar to that of monads: for instance, every relative adjunction induces a relative monad, and every relative monad admits a Kleisli category and an Eilenberg–Moore category that induce the relative monad. However, to a significant extent, the theory of relative monads is much richer than the theory of monads [4, 3, 5].

In this talk, I will explain how the theory of relative monads may be carried out in a 2-dimensional setting, analogous to the formal theory of monads. In contrast to the theory of monads, for which it suffices to consider 2-categories, to capture relative monads requires a more expressive framework.

I will begin by explaining the subtleties behind formalising the notion of a relative monad compared to the notion of a monad, which will motivate the introduction of virtual equipments [6]. I will then discuss some of the fundamental results in the formal theory of relative monads in a virtual equipment. Particular attention will be paid to the aspects of the formal theory that is notably distinct from the formal theory of monads. I will conclude the talk by discussing some aspects of the theory of relative monads that have arisen in our work that are new even in classical setting of relative monads in **Cat**. This talk is based on a recent preprint [7].

References

[1] R. Street. The formal theory of monads. *Journal of Pure and Applied Algebra* 2.2 (1972). pp. 149–168.

[2] T. Altenkirch, J. Chapman and T. Uustalu. Monads need not be endofunctors. *International Conference on Foundations of Software Science and Computational Structures* (2010). pp. 297–311.

[3] T. Altenkirch, J. Chapman and T. Uustalu. Monads need not be endofunctors. *Logical Methods in Computer Science* 11 (2015).

[4] R. F. C. Walters. A categorical approach to universal algebra. *PhD thesis, The Australian National University* (1970).

[5] N. Arkor. Monadic and Higher-Order Structure. *PhD Thesis, University of Cambridge* (2022).

*Joint work with Dylan McDermott. Abstract submitted to CT2023.

[6] G. S. H. Cruttwell and M. A. Shulman. A unified framework for generalized multicategories. *Theory and Applications of Categories* 24.21 (2010). pp. 580–655.

[7] N. Arkor, D. McDermott. The formal theory of relative monads. *arXiv preprint: 2302.14014* (2023).

Cartesian cubical model categories*

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The category of Cartesian cubical sets is introduced and endowed with a Quillen model structure, using ideas coming from recent constructions of cubical systems of univalent type theory.

Recently, there has been renewed interest in the cubical approach to homotopy theory. This is due to connections with the formal system of (homotopy) type theory, which is being used for the purpose of computerized proof checking [AC13]. Unlike previous cubical models of homotopy theory such as [Jar02, Mal09], however, the cubes used for this purpose are generally assumed to be *closed under finite products*; we call such cube categories *Cartesian*. Among the advantages of this model, as proposed by F.W. Lawvere, is the tinyness of the geometric interval $\mathbb{I}$, which indeed plays a role in the current theory.

We define the *Cartesian cube category* $\Box$ to be the Lawvere theory of bipointed objects, the opposite of which is therefore the category of finite, strictly bipointed sets $\mathbb{B} = \Box^{\mathsf{op}}$. Thus $\Box$ is the free finite product category with a bipointed object $[0] \rightrightarrows [1]$. Our homotopy theory is based on the category of *Cartesian cubical sets*, which is the category of presheaves on $\Box$,

$$\mathsf{cSet} = \mathsf{Set}^{\Box^{\mathsf{op}}},$$

and thus consists of all *covariant* functors $\mathbb{B} \to \mathsf{cSet}$. Among these is an evident distinguished one, namely that which "forgets the points"; it is represented by the generating 1-cube [1],

$$\mathbb{I} = \mathbb{B}([1], -) : \mathbb{B} \to \mathsf{cSet}.$$

In cubical sets, the bipointed object $1 \rightrightarrows \mathbb{I}$ turns out to have the (non-algebraic) property that its two points have a trivial intersection.

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \\ 1 & \longrightarrow & \mathbb{I} \end{array}$$

We call such an object in a topos an *interval*, and this is the universal one.

For the purpose of homotopy theory, this interval provides a good cylinder $X + X \rightarrowtail \mathbb{I} \times X$ for every object X, as well as a good path object $X^{\mathbb{I}} \twoheadrightarrow X \times X$ for every *fibrant* object X. The notion of fibrancy here is given in terms of a Quillen model structure:

Theorem 1. *There is a Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on cSet, in which the cofibrations $\mathcal{C}$ are a subclass of monomorphisms determined axiomatically, the fibrations $\mathcal{F}$ are the maps $f : X \to Y$ for which the canonical map*

$$(f^{\mathbb{I}} \times \mathbb{I}, \mathsf{eval}) : X^{\mathbb{I}} \times \mathbb{I} \to (Y^{\mathbb{I}} \times \mathbb{I}) \times_Y X$$

has the right-lifting property against all cofibrations, and the weak equivalences $\mathcal{W}$ are the maps $f : X \to Y$ for which $K^f : K^Y \to K^X$ is bijective on connected components whenever K is fibrant.

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Although $\square$ is a strict test category in the sense of [Gro83], this model structure is not the test one determined by the method of [Cis06], nor is it Reedy [Ree74], although $\square$ is generalized Reedy in the sense of [BM08]. Instead, it is based on a new construction derived from the interpretation of type theory and making use of the *univalence axiom* of Voevodsky [CK21]. Our main goal is not merely to arrive at the above stated theorem, but to investigate the relationship between the model structure and certain features of the system of cubical type theory [CCHM], in which univalence is *constructively valid*. The resulting Quillen model category provides a natural model for the system of [Uni13], but *not* simply as a consequence of the powerful result of Shulman [Shu19], which applies only to ∞-toposes presented by *simplicial* model categories. Thus our investigation also explores the possibility of a cubical presentation of a higher topos.

References

[AC13] Awodey, S.; Coquand, T. Univalent foundations and the large-scale formalization of mathematics. *The Institute Letter*, Institute for Advanced Study, Summer, 2013.

[Jar02] Jardine, J. F. Cubical homotopy theory: a beginning. 2002. *(Unpublished)*

[Mal09] Maltsiniotis, G. La catégorie cubique avec connexions est une catégorie test stricte. *Homology, Homotopy and Applications*, 11(2):309–326, 2009.

[Gro83] Grothendieck, A. Pursuing stacks. 1983. *(Unpublished)*

[Cis06] Cisinski, D.-C. Les préfaisceaux comme modèles des types d'homotopie. *Astérisque*, 2006.

[Ree74] Reedy, C. Homotopy theory of model categories. 1974. *(Unpublished)*

[BM08] Berger, C.; Moerdijk, I. On an extension of the notion of Reedy category. *Mathematische Zeitschrift*, pages 1–28, 2008.

[CK21] LeFanu Lumsdaine, P.; Kapulkin, C. The simplicial model of univalent foundations (after Voevodsky). *Journal of the European Mathematical Society*, 23:2071–2126, 2021.

[CCHM] Cohen, C., Coquand, T., Huber, S., Mörtberg, A. Cubical type theory: A constructive interpretation of the univalence axiom. In Uustalu, T. (ed.), *21st International Conference on Types for Proofs and Programs (TYPES 2015)*, *Leibniz Int. Proc. Infor.*, vol. 69, 5:1–5:34, 2018.

[Shu19] Shulman, M. All $(\infty, 1)$-toposes have strict univalent universes, 2019. arXiv.1904.07004.

[Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. `https://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.

On the profinite fundamental group of a connected Grothendieck topos*

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There are several notions of finiteness relevant for topos theory [6]. A combination of two of them (local finiteness and decomposition-finiteness) is used here as a convenient concept of finiteness for a general Grothendieck topos. The subcategory of sums of finite objects is shown to be an atomic Grothendieck topos admitting a canonical point. The profinite automorphism group of this point may serve as fundamental group, at least in the connected case. The emerging interplay between finitely generated Grothendieck toposes and Galois categories is instructive. Our approach is closely related to Barr's abstract Galois theory [2, 3] and is also expressable by means of a certain universal factorisation of the global section functor, cf. Johnstone [4]. In a Grothendieck topos, local finiteness is equivalent to decidable Kuratowski-finiteness which is well-known in literature [1, 5]. An object is said to be decomposition-finite if it is a finite sum of connected objects. This added "geometric" finiteness changes quite a bit the baseline.

References

[1] O. Acuna-Ortega and F. E. J. Linton – *Finiteness and Decidability,* Lect. Notes Math. **753** (1979), 80–100.

[2] M. Barr – *Abstract Galois Theory,* J. Pure Appl. Alg. **19** (1980), 21–42.

[3] M. Barr – *Abstract Galois Theory II,* J. Pure Appl. Alg. **25** (1982), 227–247.

[4] P. T. Johnstone – *Factorization theorems for geometric morphisms II,* Lect. Notes Math. **915** (1982), 216-233.

[5] S. Henry – *On toposes generated by cardinal finite objects,* Math. Proc. Cambridge Philos. Soc. **165** (2018), 209–223.

[6] A. Kock, P. Lecouturier and C. J. Mikkelsen – *Some topos theoretic concepts of finiteness,* Lecture Notes Math. **445** (1975), 209–283.

*Joint work with Victor Iwaniack.

A skew approach to enrichment for Gray categories*

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The classic setting for enriched category theory concerns enrichment in symmetric monoidal closed categories [6]. Iterative enrichment in symmetric monoidal closed categories allows one to capture categories, 2-categories and Gray-categories [5], where the last of these are categories enriched in the Gray-tensor product. Each of these can be viewed as semistrict n-categories for $n = 1, 2, 3$.

One would like to carry on further up the dimensions, but it is known since Crans [3] that any symmetric monoidal closed structure on the category of Gray-categories must have undesirable limitations — for instance, none exists which models weak higher-dimensional transformations or interacts well with Lack's model structure on Gray-Cat [1].

In this talk, I will explain how Szlachányi's skew monoidal categories [7] enable us to overcome this obstruction. In particular I will describe closed skew monoidal structures on the category of Gray-categories capturing higher lax transformations and higher pseudo-transformations. This builds on the mapping space of Gohla [4]. In addition, I will discuss the interaction between these skew monoidal structures and the model structure on Gray-Cat, and what categories enriched in these skew structures — the resulting semi-strict 4-categories — look like.

This is joint work with Gabriele Lobbia and the results are based on our recent preprint [2].

References

[1] Bourke, J.; Gurski, N. A cocategorical obstruction to tensor products of Gray-categories. Theory and Applications of Categories, Vol. 30 (2015) No. 11, 387–409.

[2] Bourke, J.; Lobbia, G.. A skew approach to enrichment for Gray-categories (2022) Arxiv Preprint: arXiv:2212.12358.

[3] Crans, S. A tensor product for Gray-categories. Theory and Applications of Categories 5, 2 (1999),12–69.

[4] Gohla, B. Mapping Spaces of Gray-Categories. Theory and Applications of Categories, 29(5) (2014), 100–187.

[5] Gordon, R., Power, J., and Street, R. Coherence for tricategories. Memoirs of the American Mathematical Society 117, 1995, 81pp.

[6] Kelly, G. M. Basic Concepts of Enriched Category Theory. Cambridge University Press, 1982, 256pp.

[7] Szlachanyi, K. Skew-monoidal categories and bialgebroids. Advances in Mathematics 231, 3-4 (2012), 1694–1730.

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Nearly groupoids

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It is well known that internal categories in Mal'tsev varieties or categories are necessarily groupoids. In congruence modular varieties and their categorical versions, Gumm categories, this is no more the case [1].

Here we shall investigate what are the specific properties of internal categories in these varietal and categorical contexts. They are nearly groupoids with commutative monoids of endomorphisms, according to the following definition:
A nearly groupoid is a category in which any morphism is both monomorphic and epimorphic.

From [2], we shall also compare the structures of the fibers $Grd_Y\mathbb{E}$, the internal groupoids in $\mathbb{E}$ with Y as "object of objects", when $\mathbb{E}$ is a Mal'tsev or a Gumm category. They are both protomodular and naturally Mal'tsev categories, but of course $Grd_Y\mathbb{E}$ has a weaker structure when $\mathbb{E}$ is a Gumm category.

We shall also bring some precisisons about the structure of the fibers $Cat_Y\mathbb{E}$ when $\mathbb{E}$ is a Gumm category.

References

[1] Bourn, D; Gran, G. Categorical aspects of modularity. in: Galois theory, Hopf algebras, and Semi-abelian categories, G.Janelidze, B.Pareigis, W.Tholen editors, Fields Institute Communications **43** (2004), 77-100.

[2] Bourn, D. Fibration of points and congruence modularity, *Algebra Universalis* **52** (2005), 403-429.

Tambara modules as actions of a monoidal profunctor*

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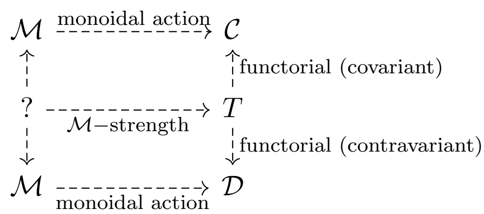
Tambara modules [3] are a notion of bimodule between categories equipped with an action of a monoidal category (often referred to as *actegories* [1]). While originally defined for applications in representation theory, Tambara modules have recently found use in the theory of modular data accessors; where they are central in a representation theorem which greatly facilitates their efficient machine encoding in kinded polymorphic languages [5, 2].

Being profunctors equipped with a strength in the action of a monoidal category, it is oft-said that Tambara modules are the corresponding notion of profunctor required when generalising from categories to actegories. Indeed in the seminal paper on the topic, Tambara refers to these modules as profunctors with a *two-sided action* [4]. However, in the usual definition there is no object which literally acts on the profunctor in any categorical sense, so we are led to ask if there is some way in which this analogy is made formal. In other words, is there any way for some Tambara module T, to complete the schematic below so that T's strength can be genuinely viewed as an action?



In this talk we will discuss how Tambara modules can be seen exactly as modules for certain monoid objects in a category of profunctors. The action of a monoidal category $\mathcal{M}$ on categories $\mathcal{C}$ and $\mathcal{D}$ extends to an action of $\mathbf{Psh}(\mathcal{M} \times \mathcal{M}^{\mathrm{op}})$ on $\mathbf{Psh}(\mathcal{C} \times \mathcal{D}^{\mathrm{op}})$, by an actegorical version of Day convolution [1, Example 3.2.8]. We hence obtain a notion of action of a monoid in $\mathbf{Psh}(\mathcal{M} \times \mathcal{M}^{\mathrm{op}})$ on an object of $\mathbf{Psh}(\mathcal{C} \times \mathcal{D}^{\mathrm{op}})$, which is equivalent to a generalised form of profunctorial strength. In particular we find that, restricting our attention to the hom-profunctor on $\mathcal{M}$, we recover the typical definition of Tambara module.

We can similarly define two-sided actions with respect to Day convolution, thus positioning these generalised Tambara modules into a double category $\mathbb{T}\mathbf{amb}$ of bimodules in the usual way.

This gives a particularly conceptual way to understand Pastro–Street theory (viz. [3, 2]) as simply dealing with the the constructions of the free and cofree modules over the hom-profunctor. Moreover this effortlessly extends to a more general theory of modules over arbitrary monoidal profunctors, for which the existing work continues to apply. The 'generalised Pastro–Street theory' also hints at a generalised class of optics where residual data is exchanged by a class of 'heteromorphisms' prescribed by the chosen monoidal profunctor.

Finally, realising that the actegories forming the boundaries between Tambara modules are themselves

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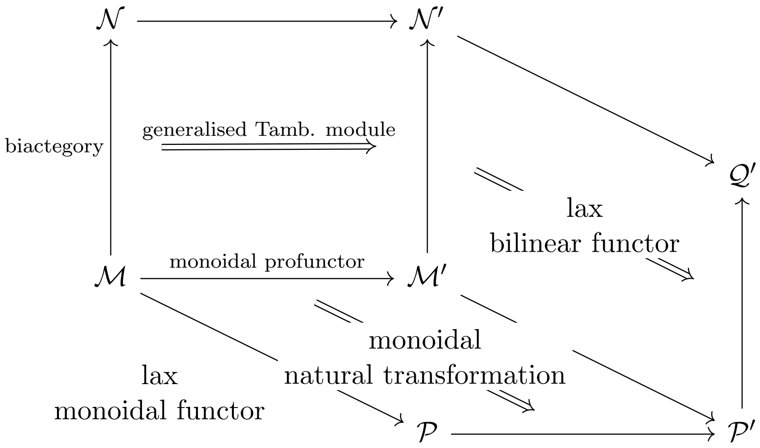
bimodules of monoidal categories, we see that the aforementioned double category $\mathbb{T}$**amb** appears as the double category of arrows in a triple category, $\mathfrak{Tamb}$:



Cubes in this triple category are a suitble notion of morphism of generalised Tambara modules.

References

[1] Capucci, M; Gavranović, G. Actegories for the Working Amthematician, *preprint* (arXiv:2203.16351)

[2] Clarke, B; Elkins, D; Gibbons, J; Loregian, F; Milewski, B; Pillmore, E; Román, M. Profunctor Optics, a Categorical Update, *preprint* (arXiv:2001.07488)

[3] Pastro, C; Street, R. Doubles for Monoidal Categories, *Theory and Application of Categories* 21 (2008), no. 2, 61–75

[4] Tambara, D. Distributors on a Tensor Category, *Hokkaido mathematical journal* 35 (2006), no. 2, 379–425.

[5] Pickering, M., J. Gibbons, and N. Wu. Profunctor Optics: Modular Data Accessors Art, Science, and Engineering of Programming 1, no. 2 (2017)

Green 2-functors as pseudomonoids for the Day convolution product*

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Ordinary Mackey functors, frequently used in representation theory and topology and valued in abelian groups, have been categorified in [1], which provides an axiomatic framework for the notions of induction and restriction in equivariant mathematics. The notion of a Mackey 2-functor $\mathcal{M}$ consists indeed of an algebraic gadget taking values on additive categories which encodes for every inclusion of groupoids $H \hookrightarrow G$ a restriction functor $\mathcal{M}(G) \to \mathcal{M}(H)$, as well as two isomorphic adjoints. A typical example is the one associating to every groupoid G the category of k-linear representation kG-Mod.

Ordinary Green functors are monoids in the category of Mackey functors, and can be seen as equivariant version of rings the same way Mackey functors can be seen as equivariant version of abelian groups. The notion of Green 2-functor is introduced in [2] and it consists of a particular Mackey 2-functor with a pseudomonoid structure in the appropriate category of 2-functors such that the external product structure associated to that of pseudomonoid "preserves" left and right adjoints in both variables. Equivalently, a Frobenius formula is satisfied.

The theory developed in [1] also provides a universal construction of a bicategory Mot of *Mackey 2-motives* through which it is possible to factor every Mackey 2-functor.

$$\begin{array}{ccc} \mathbb{G}^{\mathsf{op}} & \xrightarrow{\quad \mathcal{M} \quad} & \mathsf{ADD} \\ & {\scriptstyle \mathsf{mot}} \searrow \qquad \nearrow {\scriptstyle \widehat{\mathcal{M}}} & \\ & \mathsf{Mot} & \end{array}$$

This universal property provides a canonical biequivalence between the bicategory of Mackey 2-functors, which is a 2-full subcategory of $\mathsf{PsFun}(\mathbb{G}^{\mathsf{op}}, \mathsf{ADD})$, and the bicategory of all (additive) pseudofunctors in $\mathsf{PsFun}(\mathsf{Mot}, \mathsf{ADD})$.

The aim of our work is to characterize Green 2-functors as just pseudomonoids with respect to the Day convolution product in the bicategory of (additive) pseudofunctors $\mathsf{Mot} \to \mathsf{ADD}$. In order to do so, we need a monoidal structure on the category of 2-motives coming from the categorical product of $\mathbb{G}$, as well as the usual monoidal structure on ADD. Then, the Day convolution can be done thanks to the theory of bicoends introduced in [3].

References

[1] Balmer; Dell'Ambrogio. Mackey 2-functors and Mackey 2-motives. *European Mathematical Society (EMS)*, 2020. 227 pp.

[2] Dell'Ambrogio. Green 2-functors. *Transactions of the American Mathematical Society (TRAN)* 375 (2022), no. 4, 7783–7829.

[3] Corner. Day convolution for monoidal bicategories. *PhD Thesis* (2016).

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Principal bundles in sites and 2-sites and quotient stacks*

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Principal bundles over topological spaces are an important and useful notion in geometry, with links to cohomology theories. We will capture this notion in a categorical way and produce a new concept of principal bundle that makes sense in any site that has all pullbacks and a terminal object. We will then be able to further generalize this concept to dimension 2, where the setting will be a 2-site.

The topological group involved in the standard notion of principal bundle becomes, in a general site, a group object in the underlying category. And locally trivial morphisms are generalized considering pullbacks along the morphisms of a covering family for the Grothendieck topology. It will be important to notice that these generalized principal bundles are closed under pullbacks. And this will allow us to construct generalized quotient prestacks. We will also see a sketch of the proof of the fact that these new objects are indeed prestacks. Among generalized quotient prestacks, we will see the important particular case of classifying prestacks.

Furthermore, we will present a theorem that states that, if the site is subcanonical and the underlying category satisfies some mild conditions, our generalized quotient prestacks are stacks. We will see the key ideas behind the proof of this result, that will involve some important properties of the canonical Grothendieck topology on a category.

We will then move to dimension 2, introducing a notion of principal 2-bundles that makes sense in a 2-site. In this context the group object involved in the definition will become a 2-group in the underlying 2-category. Pullbacks will then be replaced by comma objects, and we will show that principal 2-bundles are closed under comma objects. Finally, we will see how much of our 1-dimensional theory of quotient stacks can be generalized to this 2-dimensional setting.

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A twisted Boardman–Vogt tensor product for operads*

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We define a generalisation of the Boardman–Vogt tensor product for operads, in which one operad acts on the other so that in the resulting theory, rather than the operations from the two operads commuting with each other, they interact via a parametrised interchange. We use a twisted version of the theory of commutativity of Garner–López Franco, and iterate the construction to produce operads for k-degenerate n-categories.

The aim is to provide a framework for studying degenerate weak n-categories in the Trimble framework [5]. A k-degenerate $(n+k)$-category has only one j-cell for all $0 \leq j < k$. We can then perform a "dimension shift" and regard the k-cells as 0-cells of an n-category, with k monoidal structures coming from composition along bounding j-cells for each $0 \leq j < k$ in the original $(n+k)$-category. Hence there is a slogan: *A k-degenerate $(n+k)$-category "is" a k-tuply monoidal n-category.*

Studying degenerate higher categories is interesting as it not only gives us a way to "test" definitions in low dimensions, but it also gives us insight into precisely where the obstructions are that cause weak higher categories to be genuinely different from strict ones. One of the challenges of studying degenerate higher categories is to construct a monad for them. The issue is that a free $(n+k)$-category on k-degenerate data is not itself k-degenerate, so the free $(n+k)$-category monad does not immediately restrict to one for k-degenerate structures. Rather, such a monad must be derived.

The framework for higher categories that we will use is a generalisation of Trimble's definition: we use iterated enrichment, where at each iteration the composition is weakened by the action of an operad. Given an operad E in a monoidal category $\mathcal{V}$ we write $\mathcal{V}\text{-}\mathsf{Cat}_E$ for the category of (small) categories enriched in $\mathcal{V}$ and weakened by E. A Trimble theory of higher categories is then given by for each $n \geq 0$

- a category $\mathcal{V}_n$ of n-categories and (strict) n-functors, and
- a contractible operad $E_n \in \mathcal{V}_n$, with
- $\mathcal{V}_{n+1} := \mathcal{V}_n\text{-}\mathsf{Cat}_{E_n}$.

The use of strict functors in this definition gives it strict interchange laws, making it less weak than a fully weak definition, but recent work [4, 1] has demonstrated that such a framework can still be weak enough to be equivalent to fully weak structures. Another feature is that while interchange is strict, it is parametrised by the operad actions, and invokes the action of one of the operads on the other. This is what gives rise to the "twist" we need in our generalised tensor product.

The idea of the Boardman–Vogt tensor product is to start with operads A and B in $\mathcal{V}$, then construct an operad $A \otimes B$ in $\mathcal{V}$, in which all operations of A commute with all operations of B in a form of interchange. However, for higher categories we want to start with an operad A in $\mathcal{V}$ and an operad B in $\mathcal{V}\text{-}\mathsf{Cat}_A$, which gives an action of A on B. We then wish to define an operad $A \ltimes B$ in $\mathcal{V}$ whose algebras are doubly-degenerate objects of $\mathcal{V}\text{-}\mathsf{Cat}_A\text{-}\mathsf{Cat}_B$. Thus we need to replace the commutativity condition in

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the tensor product with parametrised interchange, via the action of A on B; this is analogous to the move from direct to semi-direct products of groups, hence our choice of notation.

We use the general framework for commutativity of Garner–López Franco [3]. In that work, the Boardman–Vogt tensor product is constructed as an example of a universal commuting cocone; we will generalise this to a definition of "universal twisted commuting cocone". The generalisation is complicated by not only the action of A on B, but also by the need to reduce the dimensions of B, and also the need to introduce symmetric actions. (We hope to find a smoother way to address this in a future work.)

Main definition

Let A be an operad in $\mathcal{V}$ and B and operad in $\mathcal{V}\text{-}\mathsf{Cat}_A$. Then we define an operad $A \ltimes B$ as a universal twisted commuting cocone over A and B.

Theorem 1

The algebras for $A \ltimes B$ are the doubly-degenerate objects of $\mathcal{V}\text{-}\mathsf{Cat}_A\text{-}\mathsf{Cat}_B$.

This theorem provides the induction step for our eventual proof by induction. The iteration is in analogy with iterated loop spaces as follows. A k-fold loop space can be seen as analogous to a k-degenerate higher category. The little k-cubes operad C_n detects k-fold loop spaces, and the iterative nature can be seen by the decomposition of C_k as an iterated Boardman–Vogt tensor product of k copies of C_1 [2] that is $C_k \simeq C_1 \otimes \cdots \otimes C_1 \quad (k \text{ times})$. Our analogous result uses the twisted tensor product and is proved by induction:

Main Theorem

Consider a Trimble theory of higher categories parametrised by operads $P_j \in j\text{-}\mathsf{Cat}$, $j \geq 0$. Then the operad

$$P_n \ltimes \cdots \ltimes P_{n+k-1}$$

is an operad in n-Cat whose algebras are precisely the k-degenerate $(n+k)$-categories, expressed as n-categories with extra structure.

The eventual aim is to prove stabilisation for Trimble higher categories. While the technicalities of that are still some way off, the present work takes us significantly closer and presents a clear strategy for proceeding towards that goal.

References

[1] Cheng, E; Corner, A. Weak vertical composition. *Submitted 2023*, E-print 2212.11865.

[2] Dunn,G. Tensor products of operads and iterated loop spaces. *Journal of Pure and Applied Algebra* 50 (1988), 237–258.

[3] Garner, R; López Franco, I. Commutativity. *Journal of Pure and Applied Algebra* 220 (2016), no. 5, 1707–1751.

[4] Joyal, A; Kock, J. Weak units and homotopy 3-types. In Batanin, Davydov, Johnson, Lack, and Neeman, editors, *Categories in Algebra, Geometry and Mathematical Physics, proceedings of Streetfest* Volume 431 of *Contemporary Math.*, (2007), 257–276.

[5] Trimble, T. What are 'fundamental n-groupoids'?. Seminar at DPMMS, Cambridge, 24 August 1999.

A variant of a Dwyer-Kan theorem for model categories*

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If all objects of a simplicial combinatorial model category $\mathscr{A}$ are cofibrant, it is possible to construct the homotopy model structure on the category of small functors $\mathcal{S}^{\mathscr{A}}$, where the fibrant objects are the levelwise fibrant homotopy functors, i.e., functors preserving weak equivalences. When $\mathscr{A}$ fails to have all objects cofibrant, we construct the bifibrant-projective model structure on $\mathcal{S}^{\mathscr{A}}$ and argue that it is an adequate substitute for the homotopy model structure. Next, we present a generalization of a theorem of Dwyer and Kan, characterizing which functors $f\colon \mathscr{A} \to \mathscr{B}$ induce a Quillen equivalence $\mathcal{S}^{\mathscr{A}} \leftrightarrows \mathcal{S}^{\mathscr{B}}$ equipped with the bifibrant-projective model structures above. As an application to Goodwillie calculus, we discuss the Quillen equivalence between the category of small linear functors from simplicial sets to simplicial sets and the category of small linear functors from topological spaces to simplicial sets.

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Birkhoff subfibrations*

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The category $\mathsf{Mal}(\mathcal{E})$ of internal Mal'tsev algebras in an exact Mal'tsev category $\mathcal{E}$ with coequalizers is always a Birkhoff subcategory of $\mathcal{E}$ (see [3]). In particular, one can consider $\mathcal{E} = \mathcal{C}/B$, where $\mathcal{C}$ is a semi-abelian category and B an object of $\mathcal{C}$. The present work originates from the following question: how does the reflection

$$\mathsf{Mal}(\mathcal{C}/B) \underset{H}{\overset{I}{\underset{\perp}{\leftrightarrows}}} \mathcal{C}/B \qquad (1)$$

interact with the change-of-base functors $\beta^*\colon \mathcal{C}/B \to \mathcal{C}/B'$ for each $\beta\colon B' \to B$? First of all, does (1) extend to a regular epi reflective subfibration of the codomain fibration $\mathrm{Cod}\colon \mathsf{Arr}(\mathcal{C}) \to \mathcal{C}$? It turns out that this is the case precisely when $\mathcal{C}$ is a peri-abelian category (see [1]).

Moreover, in the latter situation, $\mathsf{Mal}(\mathcal{C}/B)$ is not only a Birkhoff subcategory of $\mathcal{C}/B$, which guarantees admissibility (with respect to regular epis – see [5]) in the sense of [4], but it is also admissible with respect to a larger class of morphisms. A new Galois structure is then available and, for example, one can interpret *crossed modules* as coverings in this context.

Actually, a further inspection reveals that most of the properties mentioned above rely on a single feature of the reflection (1) in the peri-abelian context. Namely, the fact that the kernels of the unit components for $B = 0$ are characteristic subobjects (see [2]). So the results apply to any such reflection. We will also give a characterization of regular epi reflective subfibrations in $\mathsf{Fib}(\mathcal{C})$ of the codomain fibration.

References

[1] D. Bourn, The cohomological comparison arising from the associated abelian object, preprint arXiv:1001.0905v2

[2] A. S. Cigoli, A. Montoli, Characteristic subobjects in semi-abelian categories, *Theory Appl. Categ.* 30 (2015) 206–228.

[3] Gran M. Central extensions and internal groupoids in Maltsev categories. *J. Pure Appl. Algebra* 155 (2001), 139–166.

[4] G. Janelidze, Pure Galois theory in categories, *J. Algebra* 132 (1990) 270–286.

[5] G. Janelidze, G. M. Kelly, Galois theory and a general notion of central extension, *J. Pure Appl. Algebra* 97 (1994) 135–161

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Biased elementary doctrines and quotient completions*

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Quotient completions are pervasive construction in mathematics and logic that have been deeply studied in category theory. A first explicit description of the free exact category of a left exact one was provided in [1]. Later in [2], the result was generalized for weakly left exact categories. In [7], Maietti and Rosolini introduced the elementary quotient completion in order to give an abstract description of the quotient construction in [5] for the Minimalist foundation [8]. In order to do that, they relativize the notions of equivalence relation and quotient for Lawvere's elementary doctrines, which are suitable functors of the form $P : \mathcal{C} \to \mathsf{Pos}$, from a category $\mathcal{C}$ with strict finite products to the category Pos of posets and order preserving functions [3, 4]. As shown in [7, 6], the elementary quotient completion generalizes the exact completion of a category with weak finite limits but strict finite products.

In this talk, we fill the gap between the elementary quotient completion and the exact completion of weakly left exact categories, extending the former construction to categories with weak finite products. Hence, we obtain a comprehensive generalization of the exact completion for weakly left exact categories.

In order to do that, we present the notion of *biased* elementary doctrine, which is a suitable functor $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{Pos}$ from a category with weak finite products. The structure of biased elementary doctrines is similar to the classical one, but the properties are restated taking into account a sort of *bias* due to the weak universal property of weak finite products. As expected, biased elementary doctrines generalize Lawvere's elementary doctrines.

For these structures we detect a class of elements, that we called *proof-irrelevant*, in the fibers of weak finite products that are used to obtain the two main constructions. The first one is the *strictification*, which associates to every biased elementary doctrine $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{Pos}$ an elementary doctrine $\mathsf{P}^s : \mathscr{C}_s{}^{op} \to \mathsf{Pos}$ on the finite product completion $\mathscr{C}_s$ of $\mathscr{C}$. The second is a quotient completion which extends both the elementary quotient completion and the exact completion of weakly left exact categories, even in case of weak finite products. For this construction we discuss a universal property similar to that in [2].

Our main example comes from the *intensional level* of the Minimalist foundation [8, 5]. The syntactic category of this theory, denoted by $\mathcal{CM}$, has strict finite products and weak pullbacks and is equipped with an elementary doctrine $G^{\mathbf{mTT}} : \mathcal{CM}^{op} \to \mathsf{Pos}$. For each object $A \in \mathcal{CM}$, the slice category $\mathcal{CM}/A$ inherits a functor, the *slice doctrine*, ${G^{\mathbf{mTT}}}_{/A} : \mathcal{CM}/A \to \mathsf{Pos}$, which has the structure of a biased elementary doctrine and its quotient completion provide a genuine example of elementary quotient completion that is not an exact completion.

References

[1] Carboni, A.; Magno, R. C.. The free exact category on a left exact one. *J. Austral. Math. Soc.* Ser. A 33.3 (1982), pp. 295–301.

*Abstract submitted to CT2023.

[2] Carboni, A.; Vitale, E. M.. Regular and exact completions. *J. Pure Appl. Algebra* 125.1-3 (1998), pp. 79–116.

[3] Lawvere, F. W.. Adjointness in foundations. *Dialectica* (1969), pp. 281–296.

[4] Lawvere, F. W.. *Equality in hyperdoctrines and comprehension schema as an adjoint functor. Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968).* Amer. Math. Soc., Providence, R.I., 1970, pp. 1–14

[5] Maietti, M. E.. A minimalist two-level foundation for constructive mathemathics. *Ann. Pure Appl. Logic* 160.3 (2009), pp. 319–354.

[6] Maietti, M. E.; Rosolini, G.. Elementary quotient completion. *Theory Appl. Categ.* 27 (2012), Paper No. 17, 463.

[7] Maietti, M. E.; Rosolini, G.. Quotient completion for the foundation of constructive mathematics. *Log. Univers.* 7.3 (2013), pp. 371–402.

[8] Maietti, M. E.; Sambin, G.. Toward a minimalist foundation for constructive mathematics. *From sets and types to topology and analysis.* Vol. 48. Oxford Logic Guides. Oxford Univ. Press, Oxford, 2005, pp. 91–114.

The right-connected completion of a double category*

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A double category is an internal category in the 2-category $\mathcal{C}\mathrm{AT}$, and is called *right-connected* if its identity-assigning map is right adjoint to its codomain-assigning map. Intuitively, a right-connected double category is one in which each vertical morphism has an *underlying* horizontal morphism. An important example of a right-connected double category arises from an *algebraic weak factorisation system* (L, R) on a category $\mathcal{C}$, where the vertical morphisms are the R-algebras and the horizontal morphisms come from $\mathcal{C}$. Riehl [5] showed that this example extends to a 2-functor from the 2-category $\mathcal{A}\mathrm{WFS}_{\mathrm{lax}}$ of algebraic weak factorisation systems to the 2-category $\mathcal{D}\mathrm{BL}$ of double categories, and its essential image was characterised by Bourke and Garner [2] to consist of those right-connected double categories which satisfy a certain monadicity condition. A natural question arises: can we construct an algebraic weak factorisation system from an arbitrary double category?

In this talk, I will introduce the *right-connected completion* $\Gamma(\mathbb{D})$ of a double category $\mathbb{D}$, and provide several instances where $\Gamma(\mathbb{D})$ satisfies the required monadicity condition to induce an algebraic weak factorisation system. In addition, I will exhibit many examples of the right-connected completion of well-known double categories, demonstrating why this completion is also interesting in its own right.

Two different approaches to constructing the right-connected completion will be established. The first approach involves characterising the internal nerve of a right-connected double category via a certain *relative left 2-adjoint*, which sends a category $\mathcal{C}$ to the free right-connected double category $\mathbb{R}\mathrm{c}(\mathcal{C})$ relative to the vertical double category $\mathbb{V}(\mathcal{C})$. This allows for an explicit description of $\Gamma(\mathbb{D})$ in terms of its nerve given by $\mathcal{D}\mathrm{BL}(\mathbb{R}\mathrm{c}(-), \mathbb{D}) \colon \Delta^{\mathrm{op}} \to \mathcal{C}\mathrm{AT}$. The second approach involves using comma objects in the slice 2-category $\mathcal{C}\mathrm{AT}/\mathcal{C}$ to construct the cofree left-adjoint-left-inverse on a split epimorphism in $\mathcal{C}\mathrm{AT}$, and applying this to the codomain-assigning map of a double category.

The right-connected completion characterises the 2-category $\mathcal{R}\mathrm{c}\mathcal{D}\mathrm{BL}$ of right-connected double categories as a *coreflective* sub-2-category of $\mathcal{D}\mathrm{BL}$, and the counit components $\Gamma(\mathbb{D}) \to \mathbb{D}$ are shown to satisfy a certain comonadicity condition under a mild assumption on $\mathbb{D}$. In this situation, we are able to view vertical morphisms in $\Gamma(\mathbb{D})$ as vertical morphisms in $\mathbb{D}$ equipped with additional *coalgebraic* structure. This highlights an interesting duality with algebraic weak factorisation systems, where the unit components of a *reflective* 2-adjunction between $\mathcal{C}\mathrm{AT}$ and $\mathcal{R}\mathrm{c}\mathcal{D}\mathrm{BL}$ satisfy an analogous monadicity condition, and thus allow the vertical morphisms in a right-connected double category to be seen as horizontal morphisms equipped with additional *algebraic* structure.

One of the main applications of the right-connected completion is to present a unified double-categorical framework for the study of *delta lenses* from computer science [4]. The double category of delta lenses [3] arises as the right-connected completion of the double category of categories, functors, and *cofunctors* [1]. This double category satisfies the monadicity condition characterising delta lenses as the R-algebras for an algebraic weak factorisation system on $\mathcal{C}\mathrm{AT}$, as well as the comonadicity condition characterising delta lenses as coalgebras for a comonad on a category of cofunctors.

*Abstract submitted to CT2023.

References

[1] Aguiar, M. Internal categories and quantum groups. PhD thesis, Cornell University (1997).

[2] Bourke, J; Garner, R. Algebraic weak factorisation systems I: Accessible AWFS. *Journal of Pure and Applied Algebra* 220 (2016), no. 1, 108–147.

[3] Clarke, B. The double category of lenses. PhD thesis, Macquarie University (2022).

[4] Johnson, M; Rosebrugh, R. Delta lenses and opfibrations. *Electronic Communications of the EASST* 57 (2013), 1–18.

[5] Riehl, E. Algebraic model structures. *New York Journal of Mathematics* 17 (2011), 173–231.

Weak Vertical Composition*

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We continue our study of semi-strict tricategories in which the only weakness is in vertical composition, that is, composition along bounding 1-cells. These tricategories can be conveniently constructed as categories enriched in $\mathbf{Bicat}_s$, the category of bicategories and strict functors, with monoidal structure given by Cartesian product. In previous work we showed that these were equivalent to braided monoidal categories, which we now extend to an equivalence of totalities.

In [2] we showed that any doubly-degenerate $\mathbf{Bicat}_s$-category X has an underlying braided monoidal category UX, and that given any braided monoidal category B there is a doubly-degenerate $\mathbf{Bicat}_s$-category ΣB such that $U\Sigma B$ is braided monoidal equivalent to B. This shows that weak vertical composition is "enough" to achieve braided monoidal categories in the doubly-degenerate case, a typical test case for whether a theory of tricategories is fully weak. For this construction we closely followed the work of [5] who treated the similar case of semi-strict tricategories in which the weakness lies in horizontal composition.

Both our work and that of [5] worked as an object-level correspondence, whereas here we extend this comparison to totalities. That is, we extend the object-level comparison of [2] to a comparison of 2-categories of doubly-degenerate $\mathbf{Bicat}_s$-categories and braided monoidal categories. The first task then is to assemble doubly-degenerate $\mathbf{Bicat}_s$-categories into a 2-category. In order to make an equivalence with the 2-category of braided monoidal categories we need to consider weak functors, so the first step is to make that definition. Note that as in [3] we do not simply take homomorphisms and transformations of tricategories as this gives the "wrong" structure in the doubly-degenerate case and in particular would not be expected to form a 2-category; *a priori* tricategories and their higher morphisms assemble into a tetracategory which does not form a coherent 2-dimensional structure. So we use a higher-dimensional generalisation of Lack's icons [4] to ensure a coherent 2-category totality and the "correct" totality for the doubly-degenerate structures.

In [2] we characterised doubly-degenerate $\mathbf{Bicat}_s$-categories as a particularly strict form of 2-monoidal category [1] in which one tensor product is weak but the other is strict, as is interchange. This suggests a characterisation of a weak functor as a weak monoidal functor with respect to each monoidal structure, together with some interaction axiom(s). To put this on a secure footing we will proceed abstractly via monads and distributive laws. First we construct $\mathbf{Bicat}_s$-categories as algebras for a 2-monad on the 2-category **Cat**-2-**Gph** of 2-graphs enriched in **Cat** (equivalently graphs enriched in **Cat**-**Gph**). The 2-monad in question is a composite of a 2-monad S for vertical composition and a 2-monad T for horizontal composition, composed via a strict distributive law coming from strict interchange.

We can then characterise strict TS-algebras via a T-algebra and S-algebra structure together with an interaction axiom. We go on to characterise a weak map of TS-algebras as a weak map with respect to the T-algebra structure and to the S-algebra structure, together with an interaction axiom. Transformations are just transformations of the T-structure and the S-structure, with no further interaction axiom required.

This gives us a 2-category TS-$\mathbf{Alg}_w$ and we will look at dd-$\mathbf{Bicat}_s$-$\mathbf{Cat}$, the full sub-2-category whose

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objects are the doubly-degenerate algebras. Using a weak Eckmann-Hilton argument we show that a weak map of doubly-degenerate TS-algebras in our case can be characterised as just a weak map of the S-structures interacting well with the Eckmann-Hilton-induced braiding; conversely such a weak map of doubly-degenerate S-algebras can be given the structure of a map of TS-algebras. We will characterise transformations similarly, and show that a transformation of the S-structures is automatically a transformation of the T-structures.

We are then ready to construct a biadjoint biequivalence by extending U to a 2-functor

$$\text{dd-}\mathbf{Bicat}_s\text{-}\mathbf{Cat} \to \mathbf{BrMonCat}.$$

Biessential surjectivity was shown in [2] and so in this work we prove local essential surjectivity on 1-cells, and local full and faithfulness on 2-cells.

The focus of this talk will be a discussion of how we can study weak maps of doubly-degenerate $\mathbf{Bicat}_s$-categories in the following ways:

1. as weak maps of TS-algebras;
2. as weak maps of both a T-algebra and an S-algebra, plus an interaction between them;
3. as a weak map of S-algebras, along with an Eckmann-Hilton argument.

While $\mathbf{Bicat}_s$-categories may appear to be composed of a peculiar mix of strictness and weakness, the strictness of horizontal composition yields dramatic technical advantages. This comparison at the level of totalities can also be considered as a blueprint for comparing Trimble theories of 3-categories, which we defer to future work.

References

[1] Aguiar, M; Mahajan, S. Monoidal functors, species and Hopf algebras. volume 29 of *CRM Monograph Series. American Mathematical Society, Providence, Rhode Island*, 2010. 784 pp.

[2] Cheng, E; Corner, A S. Weak vertical composition. *Submitted 2023, Eprint arXiv:2212.11865*, 2022.

[3] Cheng, E; Gurski, N. Iterated icons. *Theory and Applications of Categories* 29 (2014), no. 32, 929–977.

[4] Lack, S. Icons. *Applied Categorical Structures* 18 (2010), no. 3, 289–307.

[5] Joyal, A; Kock, J. Weak units and homotopy 3-types. *Contemporary Mathematics* 431, 257–276

Frobenius structures in $*$-autonomous categories*

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The set $[L, L]$ of join-preserving endomaps over a complete lattice L is a well-known quantale where the multiplication is the composition of maps. One may wonder under which condition this quantale is a Frobenius quantale. Following [1], a Frobenius quantale is a tuple $(Q, *, {}^{\perp}(-), (-)^{\perp})$ where ${}^{\perp}(-), (-)^{\perp} : Q \to Q^{\mathrm{op}}$, called the left and right linear negations, are two inverse antitone maps s.t. $y \backslash {}^{\perp}x = y^{\perp}/x$. In a series of successive works the following result is established:

Theorem 1 (See [9, 10, 6, 5]). *The quantale $([L, L], \circ)$ of sup-preserving endomaps of a complete lattice L is a Frobenius quantale if and only if L is a completely distributive lattice.*

Here[1] we study this theorem from a categorical perspective as suggested by the three following facts. First, the category **SLatt** of complete lattices and join-preserving maps is a $*$-autonomous category. Second, completely distributive lattices are exactly the nuclear objects – *ie* those objects L s.t. the canonical map $\mathtt{mix} : L^* \otimes L \to [L, L]$ is an isomorphism – in this category (see [4]). Third, the quantale $([L, L], \circ)$ is the canonical monoid of the internal hom of an object with itself in **SLatt**. Therefore we investigate whether the theorem holds in an arbitrary $*$-autonomous category.

To do so we give a definition of Frobenius structures in a symmetric monoidal category $(\mathcal{V}, \otimes, I, \sigma)$ mimicking the definition of Frobenius quantales. Other definitions have already been given (see [7, 8]), we use this one described here as it permits us to show some nice results. An important property of **SLatt** is that every object L comes with its dual L^{op}. The notion of dual pair abstracts this situation. A *dual pair* in a monoidal category $\mathcal{V}$ is a triple (A, B, ϵ), with $\epsilon : A \otimes B \to I$ in $\mathcal{V}$, yielding via Yoneda natural isomorphisms

$$\hom(X, B) \simeq \hom(A \otimes X, I) \qquad \text{and} \qquad \hom(X, A) \simeq \hom(X \otimes B, I)\,.$$

We informally say that (A, B) is a dual pair. Clearly, (A, A^*) is a dual pair in any $*$-autonomous category. This notion provides the framework by which to study objects that are dual to each other only up to isomorphism: for example $(A^* \otimes A, [A, A])$ is a dual pair in any $*$-autonomous category.

Given two dual pairs (A, B, ϵ) and (A', B', ϵ'), to each morphism $f : A \to A'$ there exists a unique morphism $\rho(f) : B' \to B$, the right adjoint of f, s.t. $\epsilon' \circ f \otimes B' = \epsilon \circ A \otimes \rho(f)$. Of course if the ambient monoidal category $\mathcal{V}$ is symmetric, $(B, A, \epsilon \circ \sigma)$ is also a dual pair and the notion of Galois connection is a special case of adjointness.

Moreover if A is endowed with an associative multiplication μ_A, then A acts on its dual B on the left and on the right. Those actions are the only maps s.t. $\epsilon \circ (A \otimes \alpha^{\ell}_A) = \epsilon \circ (\mu_A \otimes B)$ and $\epsilon \circ (\alpha^{\rho}_A \otimes A) = \epsilon \circ (B \otimes \mu_A)$.

In a quantale they are exactly the left and right implications $-\backslash-$ and $-/-$.

We define a *Frobenius structure* as a tuple $(A, B, \epsilon, \mu_A, l, r)$ where (A, B, ϵ) is a dual pair, (A, μ_A) is a semigroup and $l, r : A \to B$ are invertible maps forming a Galois connection s.t. $\alpha^{\ell}_A \circ (A \otimes r) = \alpha^{\rho}_A \circ (l \otimes A)$.

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[1] The main part of this work appears in the conference paper [2].

With this definition and a few basic properties of Frobenius structures, it is direct to derive the following:

Theorem 2. *If A is nuclear, then there is a map l such that $([A, A], \circ, [A, A]^*, l, l)$ is a Frobenius structure.*

Moreover, if the ambient category has an epi-mono factorization system, then the image of `mix` can always be endowed with a Frobenius structure. This result comes from a more general theorem: much like the image of the intuitionistic negation of a Heyting algebra gives rise to a Boolean algebra, the image of a morphism $f : A \to B$ may be endowed with a Frobenius structure:

Theorem 3. *Let $\mathcal{V}$ be a $*$-autonomous category with a factorization system. Let (A, μ_A) be a semigroup and (A, B) be a dual pair. Let $f : A \to B$ be a map, put $\psi_A = \epsilon \circ (A \otimes f)$ and suppose that $\psi_A = \psi_A \circ \sigma_{A,A}$. Factor f as $f = m \circ e$ with $e : A \to C$ epi and $m : C \to B$ mono. If C is a magma with multiplication μ_C and e is a magma homomorphism, then there exist maps $\psi_C : C \otimes C \to I$ and $g : C \to C^*$, transposing into each other, making (C, C^*, μ_C, g, g) into a Frobenius structure.*

The converse of Theorem 2 requires another condition to hold: the object must be pseudo-affine. An objet A of a monoidal category is *pseudo-affine* if the tensor unit I embeds into A as a retract. This condition is quite natural as it holds for every object in usual monoidal categories.

Theorem 4. *In a $*$-autonomous category, if A is a pseudo-affine object and the canonical monoid $([A, A], \circ)$ is part of a Frobenius structure, then A is nuclear.*

The pseudo-affine condition cannot be discarded. We use the category of P-Set described in [3] to construct a counter-example and demonstrate the following theorem:

Theorem 5. *For a well-chosen quantale $(Q, *)$, there exists an object A in the Q-Set category which is not nuclear but whose internal hom $[A, A]$ is part of a Frobenius structure.*

References

[1] de Lacroix, C.; Santocanale, L. Unitless Frobenius Quantales. *Applied Categorical Structures* 31 (2023).

[2] Lacroix, C. ; Santocanale, L. Frobenius Structures in Star-Autonomous Categories. *CSL* 31, (2023).

[3] Schalk, A. ; De Paiva, V. Poset-valued sets or how to build models for linear logics. *Theoretical Computer Science* 315 (2004), 83-107.

[4] Higgs, D. ; Rowe, K. Nuclearity in the category of complete semilattices. *J. Pure Appl. Algebra* 57 (1989), 67-78.

[5] Santocanale, L. Dualizing sup-preserving endomaps of a complete lattice. *ACT* (2020), 335-346.

[6] Santocanale, L. The Involutive Quantaloid of Completely Distributive Lattices. *RAMiCS* (2020), 286-301.

[7] Street, R. Frobenius monads and pseudomonoids. *J. Math. Phys* 45 (2004), 3930-3948.

[8] Egger, J. The Frobenius relations meet linear distributivity.. *Theory And Applications Of Categories* 24 (2010). pp. 25-38.

[9] Kruml, D. ; Paseka, J. Algebraic and categorical aspects of quantales. *Handbook Of Algebra* 5(2008). 323-362.

[10] Eklund, P.; Gutiérrez Garcia, J.; Höhle, U. ; Kortelainen, J.. Semigroups in complete lattices. *Springer*, (2018).

Triple delooping for multiplicative higher operads*

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Baez and Dolan [1] introduced a plus construction for operads, in order to give a definition of weak n-categories. Starting from the initial operad and iterating this construction, one gets the operad for monoids, then the operad for (one-coloured) non-symmetric operads. Iterating yet again this construction gives an operad for higher operads. Using homotopy theory for polynomial monads developed in [2], we proved a triple delooping for multiplicative higher operads analogous to the double delooping of Dwyer-Hess [3] and Turchin [4] concerning the space of long knots. In this talk, we will recall the concepts involved in this triple delooping and give an overview of the proof.

References

[1] Baez, J.; Dolan, J. Higher-dimensional algebra III: n-categories and the algebra of opetopes. *Advances in Mathematics* 135 (1998), no. 2, 145–206.

[2] Batanin, M.; De Leger, F. Polynomial monads and delooping of mapping spaces. *Journal of Noncommutative Geometry* 13 (2020), no. 4, 1521–1576.

[3] Dwyer, W.; Hess, K. Long knots and maps between operads. *Geometry & Topology* 16 (2012), no. 2, 919–955.

[4] Turchin, V. Delooping totalization of a multiplicative operad. *Journal of Homotopy and Related Structures* 9 (2014), no. 2, 349–418.

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A universal Kaluzhnin–Krasner embedding theorem*

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Given two groups A and B, the *Kaluzhnin–Krasner universal embedding theorem* states that the wreath product $A \wr B$ acts as a universal receptacle for extensions from A to B [2]. For a split extension, this embedding is compatible with the canonical splitting of the wreath product, which is further universal in a precise sense. This result was recently extended to Lie algebras [3] and cocommutative Hopf algebras [1].

In this talk we will explore the feasibility of adapting the theorem to other types of algebraic structures. By explaining the underlying unity of the three known cases, our analysis gives necessary and sufficient conditions for this to happen.

We will also see that the theorem cannot be adapted to a wide range of categories, such as loops, associative algebras, commutative algebras or Jordan algebras. Working over an infinite field, we may prove that amongst non-associative algebras, only Lie algebras admit a Kaluzhnin–Krasner theorem.

References

[1] Bartholdi, L., Siegenthaler, O. and Trimble, T.. Wreath products of cocommutative Hopf algebras (2014). arXiv:1407.3835.

[2] Krasner, M.; Kaloujnine L.. Produit complet de groupes de permutations et problème d'extension de groupes, I, II, III. *Acta Univ. Szeged*, 13 (1950), 208–230; 14 (1951), 39–66, 69–82.

[3] Petrogradsky, V. M.; Razmyslov, Yu. P.; Shishkin, E. O.. Wreath products and Kaluzhnin-Krasner embedding for Lie algebras. *Proc. Amer. Math. Soc.*, 135 (2007), no. 3, 625–636.

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Ultracategories Take 4: finally almost simple*

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The talk condenses some recent efforts in understanding the geometry of coherent topoi and its repercussion on coherent logic. After the contributions of Makkai, Marmolejo and Lurie, we investigate the existing approaches to the general concept of ultracategory in the quest of settling the most natural, concise, compact, and possibly correct definition. We start by showing that coherent topoi are right Kan injective with respect to flat embeddings of topoi. We recover the ultrastructure on their category of points as a consequence of this result. We speculate on possible notions of ultracategory in various arenas of formal model theory.

References

[1] Di Liberti, I. The geometry of coherent topoi and ultrastructures. *Journal* Arxiv preprint (2022), arXiv:2211.03104.

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Rational dagger categories

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The category of Hilbert spaces and bounded linear maps has many properties in common with the category of abelian groups. For example, they both have finite biproducts and finite equalisers, they are both enriched in the category of abelian groups, and they both have an orthogonal factorisation system. The notion of *abelian category* is a well-studied abstraction of categories that are similar to the category of abelian groups. Unfortunately, the category of Hilbert spaces and bounded linear maps is not an abelian category, despite satisfying almost all of the axioms.

Heunen's *Hilbert categories* were an initial attempt at defining an abelian-category-like abstraction that does include the category of Hilbert spaces [1]. Inspired by the work of Abramsky, Coecke, Harding and Selinger on a new category-theoretic approach to quantum mechanics, in a Hilbert category, adjoints of linear maps are encoded in the structure of an involutive contravariant endofunctor called the *dagger*, and products and equalisers are required to be, in a certain sense, compatible with the dagger. Unlike abelian categories, the extra structure of a symmetric monoidal product, and, in particular, one whose monoidal unit is simple, was needed to prove that the morphisms of a Hilbert category have additive inverses.

In this talk, I will introduce the notion of *rational dagger category*—a new dagger-category analogue of the notion of abelian category. They are so named because they are necessarily enriched in the category of rational vector spaces. The axioms are similar to those of a Hilbert category, except that a monoidal structure is not required, and only kernels rather than arbitrary equalisers are assumed to exist (the latter being derivable from the former with the other axioms). Starting with natural dagger-category variants of the axioms of an abelian category, an extra axiom, which amounts to requiring that all diagonal morphisms be normal monomorphisms, was found to be necessary to ensure the existence of additive inverses of morphisms. In the process of proving that these additive inverses exist, I discovered a simple characterisation of when an object in a semiadditive category is an internal group, namely, the codiagonal morphism on the object should be the cokernel of a split monomorphism.

Whilst the dagger category of Hilbert spaces and bounded linear maps is the motivating example of a rational dagger category, there are other interesting examples, such as the dagger category of matrices over a formally-complex involutive division ring and the dagger category of finite-dimensional inner-product spaces over a Baer-ordered involutive division ring. New notions of *dagger section* and *orthogonal product*, which are more-inclusive variants of the familiar notions of dagger monomorphism and dagger product, will also be introduced. These arise naturally when considering the family of all normal monomorphisms in dagger categories that are similar to the dagger category of Hilbert spaces. Phrasing the axioms for a rational dagger category in terms of dagger sections rather than dagger monomorphisms is necessary to include the above-mentioned matrix dagger categories as examples.

References

[1] Heunen, C. An embedding theorem for Hilbert categories. *Theory and Applications of Categories* 22 (2009), no. 13, 321–344.

Beck's monadicity in a 2-derivator*

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The theory of derivators was introduced independently by Grothendieck and Heller in the 1980s (see [2] and [3]) with the aim of formalising homotopy theory. They realised that a consistent amount of this theory (particularly homotopy limits and colimits) can be captured by working with collections of homotopy categories of diagram categories.

In recent years, Riehl and Verity started a program to develop $(\infty, 1)$-category theory in a model independent fashion using ∞-cosmoi, a well-behaved notion of $(\infty, 2)$-category. They noticed that working inside the homotopy 2-category of an ∞-cosmos is enough to recover a number of ∞-categorical notions, allowing even for internalisation of adjunctions from 2-categorical data (see [5]).

Inspired by these two lines of research, in my Master of Research thesis [1] I introduced a 2-dimensional version of derivator theory that has ∞-cosmology as a model. In this talk I will explain how one can adapt the ∞-cosmological argument provided in [4] to prove Beck's monadicity theorem inside a 2-derivator, extending the 1-categorical and the ∞-cosmological result to a more general context.

References

[1] Di Vittorio, N. 2-derivators. *Master of Research Thesis, available at* `https://doi.org/10.25949/19817653.v1`, 2020.

[2] Grothendieck, A. Pursuing Stacks. *Unpublished manuscript, LaTeXed version available at* `https://thescrivener.github.io/PursuingStacks/ps-a4paper.pdf`, 1983.

[3] Heller, A. Homotopy theories. *Memoirs of the American Mathematical Soc.* Vol. 383 (1988).

[4] Riehl, E.; Verity, D. Elements of ∞-Category Theory. *Online draft with several additional chapters available at* `https://elements-book.github.io/more-elements.pdf`, 2020. 748 pp.

[5] Riehl, E.; Verity, D. Homotopy coherent adjunctions and the formal theory of monads. *Advances in Mathematics* 286 (2016): 802-888.

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Ternary semidirect products in semi-abelian categories*

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One of the successes of the theory of semi-abelian categories is the link between the notions of split exact sequence, binary semidirect product, and internal action [1, 2, 3]. In the categories of groups and Lie algebras, a notion of n-semidirect products has been introduced by Carrasco and Cegarra [4, 5], which allows to construct the semidirect product of a sequence of objects $(A_i)_{i=0,\ldots,n}$, using a system of actions of A_j on A_i for $j > i$, and additional functions $A_k \times A_j \to A_i$ for $j > i > k$ satisfying certain axioms.

In this talk, we will study the 3-semidirect products in a semi-abelian category, using the definition of Cegarra and Carrasco; we show that such ternary semi-direct products may be seen as liftings of (split) short exact sequences to the category of C-actions, and give an internal version of the induced structure on the given objects. We will pay special attentions to the case of algebraically coherent categories, where the axioms can be simplified [6].

References

[1] Bourn, D. ; Janelidze, G. Protomodularity, descent, and semidirect products. *Theory And Applications Of Categories*, 4 (1998), no. 2, 37–46

[2] Borceux, F.; Janelidze, G. ; Kelly, G. Internal object actions. *Commentationes Mathematicae Universitatis Carolinae*, 46 (2005), no. 2, 235–255

[3] Borceux, F.; Janelidze, G. ; Kelly, G. On the representability of actions in a semi-abelian category. *Theory And Applications Of Categories*, 14 (2005), no. 11, 244–286

[4] Carrasco, P. ; Cegarra, A. Group-theoretic algebraic models for homotopy types. *Journal Of Pure And Applied Algebra.* 75 (1991), no. 3 , 195–235

[5] Carrasco, P. ; Cegarra, A. A Dold-Kan theorem for simplicial Lie algebras. *Theory And Applications Of Categories*, 32 (2017), no. 34, 1165–1212

[6] Cigoli, A.; Gray, J. ; Van der Linden, T. Algebraically coherent categories. *Theory And Applications Of Categories*, 30 (2015), no. 54, 1864-1905

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Categorical incarnations of infinite games*

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We present a new categorical approach to the study of infinite games in combinatorics. To this end, we define the categories $\mathbf{Game}_{\mathrm{A}}$ and $\mathbf{Game}_{\mathrm{B}}$ of infinite games which are shown to be complete, cocomplete, cartesian closed, regular and extensive.

We describe these categories in various equivalent forms, as they admit underlying functors to well-established categories, such as the category $\mathbf{CUltMet}_1$ of complete ultrametric spaces of diameter at most 1 with 1-Lipschitz mappings as morphisms, the category **Tree** of arborescences with homomorphisms of directed rooted trees as morphisms and the topos $\mathbf{Set}^{\mathbb{N}^{\mathrm{op}}}$.

As an application of this framework, we show how some classical topological games can be seen as functors into these ludic categories, so that a result of Scheepers about covering and tightness topological games can be seen as a consequence of the existence of natural transformations between the game functors.

References

[1] Duzi, M.; Szeptycki, P.; Tholen, W. Infinitely ludic categories. *Work in progress.*

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Grothendieck topologies and weak limits*

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The construction of the elementary quotient completion of an elementary doctrine is an excellent tool to produce models of constructive theories for mathematics, see [3, 4]. The construction freely adds quotients of (definable) equivalence relations to an elementary doctrine, which is an algebraic description of a logical theory with equality.

The elementary quotient completion extends the well-known categorical construction of the exact completion $\mathcal{A}_{\mathrm{ex/wlex}}$ of a given category $\mathcal{A}$ with weak limits [1] provided that products are strong. In fact, most of the works that characterise models as exact completions invoke that the given category $\mathcal{A}$ has strong finite products. This matches with the situation of an elementary doctrine, whose base category is required to have finite products. Indeed, equality involves considering pairs of elements.

On the other hand, the peculiarity of strong finite products with respect to weak limits is certainly apparent. In the work [2] for his PhD thesis, one of the collaborators determined a suitable set of conditions to present an extension of the notion of elementary doctrine with respect to a base category $\mathcal{B}$ with just weak finite products. It requires that equality behaves with some kind of bias with respect to a specific weak product diagram—hence the name ***biased elementary doctrine***. He also showed how the elementary quotient completion extends to the wider settings as a 2-functorial left adjoint.

We show how the two extensions refer to the same situation which involves the product completion $\mathcal{A}_{\mathrm{pr}} := (\mathrm{Fam}_{\mathrm{fin}}(\mathcal{A}^{\mathrm{op}}))^{\mathrm{op}}$ of a category $\mathcal{A}$. When $\mathcal{A}$ has weak finite limits there is a Grothendieck topology Θ where covers contain a diagram of weak binary products. This observation allows us to state our main results. Let $\mathcal{A}$ be a category with weak limits.

Theorem 1 *Let P be a doctrine on $\mathcal{A}_{\mathrm{pr}}$ which is a Θ-sheaf. Then P is elementary if, and only if, the restriction of P to $\mathcal{A}$ is biased.*

Theorem 2 *There is a full embedding $\mathcal{A}_{\mathrm{ex/wlex}} \hookrightarrow \mathrm{sh}(\mathcal{A}_{\mathrm{pr}}, \Theta)$ of the exact completion in the category of Θ-sheaves which is exact and preserves any local exponential which exists in $\mathcal{A}_{\mathrm{ex/wlex}}$.*

References

[1] Carboni, A. and Vitale, E.M., Regular and exact completions, *Journal of Pure and Applied Algebra*, vol. 125 (1998), pp. 79–117.

[2] Cioffo, C.J., Homotopy setoids and generalized quotient completion, PhD thesis, Università degli studi di Milano, 2022.

[3] M.E. Maietti and G. Rosolini, Elementary quotient completion, *Theory and Applications of Categories*, vol. 27 (2013), pp. 445–463.

[4] M.E. Maietti and G. Rosolini, Quotient completion for the foundation of constructive mathematics, *Logica Universalis*, vol. 7 (2013), no. 3, pp. 371–402.

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Bifunctor Theorem and strictification tensor product for double categories with lax double functors *

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Double categories have extensively been studied in recent years. Some of their positive aspects are: they feature inclusiveness of algebraic structures in a sense that makes them a more suitable framework to work in than bicategories, and also they provide a simplified way to studying monoidal products on bicategories.

In this talk we present constructions that we carried out in double categories generalizing some results on 2-categories from [3] (where strict 2-functors were considered) and [2] (where lax functors were considered).

To start with, we construct a double category $\mathrm{Lax}_{hop}(\mathbb{A},\mathbb{B})$ of lax double functors between double categories $\mathbb{A},\mathbb{B}$, horizontal oplax and vertical lax transformations, and modifications among the latter two. Mimicking the construction carried out in [1] for double categories and strict double functors, by viewing $[\![\mathbb{A},\mathbb{B}]\!] := \mathrm{Lax}_{hop}(\mathbb{A},\mathbb{B})$ formally as if it were an inner-hom for the category Dbl_{lx} of double categories and lax double functors, we construct a double category $\mathbb{A}\otimes\mathbb{B}$. Although the laxity of double functors prevents $[\![-,-]\!]$ from being a bifunctor, and $-\otimes-$ from being a (Gray type) monoidal product in Dbl_{lx}, we obtain the following results.

We characterize lax double quasi-functors in analogy to "quasi-functors of two variables" for 2-categories of Gray, and we introduce their double category $q\text{-}\mathrm{Lax}_{hop}(\mathbb{A}\times\mathbb{B},\mathbb{C})$. We extend the latter characterization to an isomorphism of double categories $\mathrm{Lax}_{hop}(\mathbb{A},[\![\mathbb{B},\mathbb{C}]\!]) \cong q\text{-}\mathrm{Lax}_{hop}(\mathbb{A}\times\mathbb{B},\mathbb{C})$. On the other hand, we construct a double functor $\mathcal{F} : q\text{-}\mathrm{Lax}_{hop}(\mathbb{A}\times\mathbb{B},\mathbb{C}) \to \mathrm{Lax}_{hop}(\mathbb{A}\times\mathbb{B},\mathbb{C})$ to the double category of lax double bifunctors. This is a double category version of the Bifunctor Theorem proved for 2-categories in [2]. We show when this double functor $\mathcal{F}$ restricts to a double equivalence. For a consequence we derive double functors known as *currying* and *uncurrying* functors in Computer Science, here in the context of double categories.

For the double category $\mathbb{A}\otimes\mathbb{B}$ we prove a universal property that it satisfies by constructing a double category isomorphism $q\text{-}\mathrm{Lax}_{hop}(\mathbb{A}\times\mathbb{B},\mathbb{C}) \cong \mathbb{D}bl_{hop}(\mathbb{A}\otimes\mathbb{B},\mathbb{C})$, where the right hand-side is the double category of strict double functors, horizontal oplax and vertical lax transformations, and modifications. Consequently, we obtain $\mathbb{D}bl_{hop}(\mathbb{A}\otimes\mathbb{B},\mathbb{C}) \cong \mathrm{Lax}_{hop}(\mathbb{A},[\![\mathbb{B},\mathbb{C}]\!])$. These two results reveal that lax double (quasi-)functors of the obvious form strictify when considered as double functors $\mathbb{A}\otimes\mathbb{B}\to\mathbb{C}$.

We finish by showing the application of the double functor $\mathcal{F}$ on monads by taking the domain double category $\mathbb{A}\times\mathbb{B}$ to be trivial. Namely, we obtain that when a lax double quasi-functor acts on the trivial double category it is a distributive law between double monads. In this way it turns out that $\mathcal{F}$ is a generalization to non-trivial double categories of the classical Beck's result, that a distributive law between two monads makes it possible for them to compose.

No prior knowledge of double categories is necessary, as we will introduce them in the talk.

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References

[1] (Böhm, G.). The Gray Monoidal Product of Double Categories, *Appl. Categ. Structures* 28 (2020), 477–515.

[2] (Faul, P.F.); (Manuell, G.); (Siqueira, J.). 2-Dimensional Bifunctor Theorems and Distributive laws, *Theory Appl. Categ.* 37 (2021), no. 34, 1149–1175.

[3] (Gray, J. W.). Formal category theory: adjointness for 2-categories, Lecture Notes in Mathematics **391**, *Springer-Verlag*, Berlin-New York, 1974, pp. 296.

Stabilized profunctors and stable species of structures*

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We shall examine the fundamental biequivalence between the compact closed bicategory $\mathbf{Prof}_G$ of profunctors (aka bimodules or distributors) between groupoids and natural transformations between them, and the 2-category $\mathbf{Cocont}_G$ of cocontinuous functors between categories of presheaves over groupoids (aka categories of groupoid actions) and natural transformations (aka equivariant functions) between them; as well as its extension to the biequivalence between the cartesian closed bicategory $\mathbf{Esp}_G$ of generalized species of structures between groupoids and natural transformations between them [4], and the 2-category $\mathbf{Ana}_G$ of analytic functors between presheaf categories over groupoids and quasi-cartesian natural transformations between them [6].

Our starting point is a basic representation theorem for presheaves over groupoids that leads to the consideration of groupoids with additional structure called *kits*. Kits have both combinatorial and logical character. From a combinatorial viewpoint, they serve to restrict presheaves to stabilized ones that give rise to *stabilized-presheaf categories*. From a logical perspective, we will consider a class of *Boolean kits*. These are drawn from Boolean algebras associated to groupoids by means of a general universal construction to be introduced and discussed. In this context, the dualities of profunctors and of Boolean algebras will be placed side by side to define a bicategory $\mathbf{StProf}_{BK}$ of *stabilized profunctors* between Boolean kits and natural transformations between them. We shall see that $\mathbf{StProf}_{BK}$ is $\star$-autonomous, with a projection onto $\mathbf{Prof}_G$ degenerating to its compact closed structure, and that it is biequivalent to the 2-category $\mathbf{Lin}_{BK}$ of *linear functors* (namely, those being left and right local adjoints) between stabilized-presheaf categories over Boolean kits and cartesian natural transformations between them.

The motivation for the above investigations are developments in category theory (analytic [8] and polynomial [5] functors), structural combinatorics (species of structures [7]), logical calculi (linear logic [6]), and the semantics of computation (stable domain theory [1, 9]). In these, symmetric-algebra (aka Fock-space or Lafont-exponential) structure plays a fundamental role and in the present context has led us to introduce a bicategorical model of classical differential linear logic on $\mathbf{StProf}_{BK}$ that may be seen as simultaneously extending and refining the model on $\mathbf{Prof}_G$ that underlies the bicategory $\mathbf{Esp}_G$ of generalized species of structures. We shall see that the induced cartesian closed coKleisli bicategory $\mathbf{StEsp}_{BK}$ of *stable species of structures* is biequivalent to the 2-category $\mathbf{StAna}_{BK}$ of *stable and analytic functors* (equivalently, epi-preserving finitary parametric right adjoints) between stabilized-presheaf categories over Boolean kits and cartesian natural transformations between them.

This is joint work with Zeinab Galal and Hugo Paquet [3].

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References

[1] Berry, G. Stable models of typed lambda-calculi. In *Automata, Languages and Programming* (1978), Lecture Notes in Computer Science 62, 72–89.

[2] Fiore, M. Analytic functors between presheaf categories over groupoids. *Theor. Comput. Sci.* 546 (2014), 120–131.

[3] Fiore, M; Galal, Z; Paquet, H. Stabilized profunctors and stable species of structures. *arXiv* (2023), `https://arxiv.org/abs/2303.04795`.

[4] Fiore, M; Gambino, N; Hyland, M; Winskel, G. The cartesian closed bicategory of generalised species of structures. *J. Lond. Math. Soc.* 77 (2008), no. 1, 203–220, 2008.

[5] Gambino, N; Kock, J. Polynomial functors and polynomial monads. *Mathematical Proceedings of the Cambridge Philosophical Society*, 154 (2013), no. 1, 153–192.

[6] Girard, J-Y. Normal functors, power series and λ-calculus. *Ann. Pure Appl. Logic* 37 (1988), no. 2, 129–177.

[7] Joyal, A. Une théorie combinatoire des séries formelles. *Adv. in Math.*, 42 (1981), no. 1, 1–82.

[8] Joyal, A. Foncteurs analytiques et espèces de structures. In *Combinatoire énumérative* (1986), Lecture Notes in Mathematics 1234, 126–159.

[9] Taylor, P. Quantitative domains, groupoids and linear logic. In *Category Theory and Computer Science* (1989), Lecture Notes in Computer Science 389, 155–181.

Slices of Higher Categories

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Globular operads are a kind of operad whose algebras share a strong formal similarity with higher categories. This approach to higher categories has been worked on extensively by Michael Batanin and by Tom Leinster, who defined fully weak n-categories as algebras for a specified globular operad [1], [4]. In a preprint of Michael Batanin [2], it is conjectured that it should possible to take 'slices' of globular operads. The k^{th} slice was said to be the symmetric operad obtained by considering only the k-dimensional data. Thus, given some notion of higher category, the slices of the corresponding globular operad should isolate the algebraic structure of those higher categories in each dimension. However, due to the gaps in knowledge surrounding globular operads at the time, it was not possible to formulate a definition of slices.

In this talk, I will show that using the theory of presentations developed in [3], we can define slices for any globular operad and that, up to isomorphism, the slices do not depend on the choice of presentation. We will also demonstrate how to build a presentation for the globular operad corresponding to any theory of algebraic higher category in such a way that the coherence theorem is satisfied automatically. This strategy will provide us with many concrete examples, and we will see that although weak higher categories are far more complex than their strict counterparts, the slices for fully weak structures are fairly simple objects, while the slices for stricter variations are typically more complex.

Batanin also hypothesised that slices could tell us when one theory of higher category is equivalent to another. This is significant because fully weak higher categories are often the most useful for applications to areas such as algebraic topology and homotopy theory, but become too complicated for practical use in dimensions greater than 2. A solution is to find a notion of semi-strict higher category that is just weak enough to be equivalent to the fully weak variety, while still being tractable enough to work with directly. We will consider two different theories of semi-strict higher category, namely n-categories with weak units and n-categories with weak interchange laws, and examine what their slices can tell us about their relation to fully weak n-categories. In the case of weak units, this can be done using the slices directly. In the case of weak interchange laws, this is accomplished by studying what the slices tell us about the geometric properties and graphical calculi of the associated string and surface diagrams.

References

[1] Batanin, M. Monoidal globular categories as a natural environment for the theory of weak n-categories. *Advances in Mathematics* 136 (1998), no 1, 39–103.

[2] Batanin, M. Computads and slices of operads. Preprint, available at link (2002).

[3] Griffiths, R. Presentations for Globular Operads. *Theory and Applications of categories* 39 (2023), 267-321.

[4] Leinster, T. Higher Operads, Higher Categories. *Cambridge University Press*, volume 298 of London Mathematical Society Lecture Note Series, 2004. 410 pp.

Edgewise subdivision, culf maps, and right fibrations*

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We will discuss a study of culf[1] maps between simplicial spaces, a generalization of discrete Conduché fibrations of categories. This type of map is especially important when the simplicial spaces involved are decomposition spaces of Gálvez–Kock–Tonks [3] (also known as 2-Segal spaces of Dyckerhoff–Kapranov [2]), as they are the class of maps for which the incidence coalgebra construction is functorial.

The edgewise subdivision of a simplicial space is a generalization of the twisted arrow category of a category. Our main result is an equivalence (of ∞-categories) between culf maps over a simplicial space X and right fibrations over the edgewise subdivision of X. This can be seen as a relative version of [1], where it is shown that a simplicial space X is a decomposition space if and only if its edgewise subdivision is a Segal space. This implies that the ∞-category of decomposition spaces is locally an ∞-topos, as each slice is a category of presheaves (this generalizes a theorem of Kock–Spivak [4] about discrete decomposition spaces).

An important application is to the theory of free decomposition spaces. Every presheaf $Z\colon \Delta_{\mathrm{int}}^{\mathrm{op}} \to \mathbf{Spaces}$ on the category of inert maps $\Delta_{\mathrm{int}} \subset \Delta$ freely generates a simplicial space (via left Kan extension), and this simplicial space is always a decomposition space. Moreover, this free functor exhibits an equivalence between the category of Δ_{int}-presheaves and the slice $\mathbf{Decomp}_{/B\mathbb{N}}$. This relies on the fact that the twisted arrow category of the monoid of natural numbers is Δ_{int}. It turns out that many important combinatorial examples of decomposition spaces arise from this construction.

References

[1] Bergner, J.E.; Osorno, A.M.; Ozornova, V.; Rovelli, M.; Scheimbauer, C.I. The edgewise subdivision criterion for 2-Segal objects. *Proc. Amer. Math. Soc.* 148 (2020), 71–82. doi:10.1090/proc/14679, arXiv:1807.05069.

[2] Dyckerhoff, T.; Kapranov, M. Higher Segal spaces. *Springer Cham*, 2019. xv+218 pp. doi:10.1007/978-3-030-27124-4, arXiv:1212.3563.

[3] Gálvez-Carrillo, I.; Kock, J.; Tonks, A. Decomposition spaces, incidence algebras and Möbius inversion I: Basic theory. *Adv. Math.* 331 (2018), 952–1015. doi:10.1016/j.aim.2018.03.016, arXiv:1512.07573.

[4] Kock, J.; Spivak, D.I. Decomposition-space slices are toposes. *Proc. Amer. Math. Soc.* 148 (2020), 2317–2329. doi:10.1090/proc/14834, arXiv:1807.06000.

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[1] standing for 'Conservative and Unique Lifting of Factorizations'

Some toposes over which essential implies locally connected*

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In this talk, we will discuss some of the ideas from the article [2].

In [1], Barr and Paré introduced the notion of a locally connected (or molecular) geometric morphism. Recall that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ between toposes is called *locally connected* if the inverse image functor $f^* : \mathcal{E} \to \mathcal{F}$ has an $\mathcal{E}$-indexed left adjoint.

The class of locally connected geometric morphisms is stable under composition and pullback, and whether a geometric morphism is locally connected or not can be checked after base change along an open surjection [3, Corollary C5.1.7]. Further, the topos of sheaves on a topological space is locally connected (over **Set**) if and only if X is locally connected. In this sense, local connectedness is a very natural geometric notion.

A more general notion is that of an essential geometric morphism. We say that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is *essential* if the inverse image functor f^* has a left adjoint, without the additional requirement that this left adjoint is $\mathcal{E}$-indexed. This more general class is not stable under base change anymore, so "being essential" is in this sense not a geometric property. An importance family of essential geometric morphisms are the geometric morphisms between presheaf toposes $\mathbf{PSh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{D})$ that are induced by functors $\mathcal{C} \to \mathcal{D}$. The latter geometric morphisms are often not locally connected, so in this situation the difference between essential and locally connected geometric morphisms is the most clear.

In the talk, we will discuss a question originally asked by Matías Menni, in his message "Essential vs Molecular" on the category theory mailing list (May 3, 2017). We will ask: what are the toposes $\mathcal{E}$ with the property that any essential geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is also locally connected? Toposes with this property will here be called EILC toposes ("Essential Implies Locally Connected"). It follows from the definition that **Set** is an EILC topos; the idea behind [2] and behind this talk is to construct more families of examples, and in this way hopefully get a step closer to a full characterization.

In particular, we will show that a topos of sheaves on a topological space X is EILC if the space X is Hausdorff, or more generally Jacobson. Jacobson étendues, i.e. étendues locally given by a Jacobson space, are also EILC. Finally, we can generalize our result to Jacobson étendues over a general EILC base topos that admits a natural number object. Other examples of EILC toposes are Boolean étendues and classifying toposes of compact groups (at the time of writing, the question whether every Boolean topos is EILC remains open). If time permits, we will also discuss the weaker notion of CILC toposes, which are the toposes $\mathcal{E}$ such that any geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is locally connected as soon as the inverse image functor f^* is cartesian closed.

References

[1] Barr, M.; Paré, R.. Molecular toposes. *J. Pure Appl. Algebra* 17 (1980), no. 2, 127–152.

*Abstract submitted to CT2023.

[2] Hemelaer, J.. Some toposes over which essential implies locally connected. *Cahiers de Topologie et Géométrie Différentielle Catégoriques* LXIII (2022), no. 4, 425–451.

[3] Johnstone, P.T.. Sketches of an Elephant: A Topos Theory Compendium. *The Clarendon Press, Oxford University Press*, 2002. 1284 pp.

A categorical framework for congruence of bisimilarity [*]

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Motivation and summary of contribution In research on programming languages, languages are often modelled mathematically using a set of techniques called **operational semantics**. The general idea of operational semantics consists in viewing programs as elements of an inductively-generated object, and evaluation steps as some sort of binary relation, so that programs and evaluation steps form a graph. Let us call this graph the **evaluation graph**.

An important research area about programming languages is **behavioural equivalences**: its goal is to delineate conditions under which certain program transformations, typically optimisations, may be performed without changing observable behaviour, in a suitable sense. We say that the program and its transform are **observationally equivalent**.

The difficulty is that, in most cases, observational equivalence is hard to establish. A standard idea to overcome this problem is to design a different equivalence relation, which entails observational equivalence but is easier to establish. A typical kind of such alternative equivalence relation is **bisimilarity**. In order to establish that two given programs are bisimilar, one merely needs to exhibit a relation between their reachable vertices in the evaluation graph, such that for each related pair, one program simulates the other's behaviour. Such a relation is called a **bisimulation**.

In this work, we focus on a particular property of bisimilarity, which is crucial for showing that it entails observational equivalence: the fact that bisimilarity is a **congruence**, which means that it is preserved by all constructions of the considered language.

This is far from obvious in general. E.g., it famously fails for Milner's π-calculus, a simple language of concurrent programs. Perhaps more significantly, it is famously hard to prove for pure λ-calculus. Indeed, for a long time, the only known proof relied on a detour through some different mathematical model. It was only with Howe's work [2] that a direct, syntactic proof was found.

The issue we want to address in this work is that, although congruence of bisimilarity has been proved for a wide variety of languages, there is no general, unifying theory.

We are not the first to work on this issue. We notably know of syntactic (Howe, 1996; Bernstein, 1998) and semantic (Turi and Plotkin, 1997) frameworks, which prove congruence of bisimilarity for quite a few applications. Our work improves on these frameworks in two important ways:

- We cover languages whose evaluation graph relies on inductively-defined operations on programs, typically but not exclusively **capture-avoiding substitution**.
- We cover languages whose evaluation graph may have its edges labelled by programs.

Both features are actually useful in applications, as illustrated on our running example, a λ-calculus with delimited continuations (Biernacki and Lenglet, 2012). Our framework improves on recent work (2022) by

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going beyond capture-avoiding substitution and allowing programs as labels. However, it leaves room for further improvement. Notably, we do not cover Lenglet and Schmitt's congruence results for higher-order process calculi [3].

Overview of framework Our framework comprises four main components: (1) a format for specifying syntax with inductively-defined operations; (2) a format for specifying evaluation; (3) a definition of bisimilarity, for any language complying with (1) and (2); and (4) sufficient conditions for bisimilarity to be a congruence. Let us describe this in a bit more detail.

(1) For us, a **syntax** is a finitary endofunctor Σ_0 on a presheaf category $\widehat{\mathbb{V}}$, for some small category $\mathbb{V}$ of **vertex types**. For inductively-defined operations, we need finer data: we consider a bifunctor $\Gamma \colon \widehat{\mathbb{V}}^2 \to \widehat{\mathbb{V}}$, required to be cocontinuous (resp. finitary) in its first (resp. second) argument. Typically, for capture-avoiding substitution, $\widehat{\mathbb{V}}$ would be equipped with monoidal structure, and $\Gamma(X, Y)$ would be $X \otimes Y$. We omit the proper format for lack of space, but it suffices to generate Γ-algebra structure on the initial algebra $\Sigma_0^*(\emptyset)$, i.e., a morphism $\Gamma(\Sigma_0^*(\emptyset), \Sigma_0^*(\emptyset)) \to \Sigma_0^*(\emptyset)$, which is suitably compatible with the Σ_0-algebra structure.

(2) For specifying evaluation, we first need a suitable notion of graph. For this, we postulate a small category $\mathbb{E}$ of **edge types**. For each edge type α, we furthermore (functorially) postulate source s_α, labels $l_1^\alpha, \ldots, l_{n_\alpha}^\alpha$, and target t_α vertex types $\mathbb{V}$. These data induce a functor $\Delta \colon \widehat{\mathbb{V}} \to \widehat{\mathbb{E}}$ defined by $\Delta(V)(\alpha) = V(s_\alpha) \prod_i V(l_i^\alpha) V(t_\alpha)$, and our **graphs** are objects of the oplax limit of Δ, i.e., triples (V, E, ∂) with $\partial \colon E \to \Delta(V)$. We then restrict attention to **algebraic graphs**, i.e., graphs whose vertex object V is equipped with suitably compatible Σ_0- and Γ-algebra structures. Finally, evaluation is specified by an endofunctor Σ_1 on algebraic graphs, required to fix the vertex object. In applications, the initial Σ_1-algebra, say $\mathbf{Z}$, is the usual, syntactic evaluation graph.

(3) For any algebraic graph $G = (V, E, \partial)$, we define bisimilarity to be the largest **enhanced bisimulation** relation $\sim^G \rightarrowtail V^2$, where **enhanced** means that it is closed under inductively-defined operations, i.e., $\Gamma(\sim^G, V) \subseteq \sim^G$ over V^2, and **bisimulation** is straightforwardly defined by analogy with applications.

(4) Finally, we isolate a sufficient condition on Σ_1 for bisimilarity over $\mathbf{Z}$ to be a congruence, i.e., $\Sigma_0(\sim^{\mathbf{Z}}) \subseteq \sim^{\mathbf{Z}}$. The useful form of this condition appears when Σ_1 **familial** in the sense of Diers (1978), Carboni and Johnstone (1995), or Weber (2001). Indeed, in this case, we may decompose Σ_1 into a family of "abstract transition rules". Furthermore, we can extract from any such transition rule a characteristic morphism which we call its **border arity**, and our sufficient condition boils down to the border arity of each rule being a left map in a suitable, cofibrantly generated weak factorisation system. In concrete applications, this instantiates to a classical acyclicity criterion (Howe, 1996; Bernstein, 1998).

The talk is based on the preprint [1].

References

[1] Tom Hirschowitz and Ambroise Lafont. A more general categorical framework for congruence of applicative bisimilarity. Preprint, 2023. URL: `https://hal.science/hal-03999584`.

[2] Douglas J. Howe. Proving congruence of bisimulation in functional programming languages. *Information and Computation*, 124(2):103–112, 1996. `doi:10.1006/inco.1996.0008`.

[3] Sergueï Lenglet and Alan Schmitt. Howe's method for contextual semantics. In Luca Aceto and David de Frutos-Escrig, editors, *Proc. 26th International Conference on Concurrency Theory*, volume 42 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 212–225. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015. `doi:10.4230/LIPIcs.CONCUR.2015.212`.

On the commutativity of products with coequalisers

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This talk presents the work of [1, 2], which develops a categorical–algebraic analysis of the property that finite products commute with arbitrary coequalisers ($*$). This property is closely related to several categorical/universal–algebraic conditions of independent interest, notably, the property of a category to be unital or factor-permutable. Moreover, one may consider categories satisfying ($*$) as a common generalisation of both (regular) unital and factor-permutable categories with coequalisers, one which is well–suited for the notion of (Huq-)central morphism and commutative object. In a pointed category $\mathbb{C}$ satisfying ($*$), the full subcategory of commutative objects $\mathrm{Com}(\mathbb{C})$ is equivalent the category of internal commutative monoids in $\mathbb{C}$. Under a suitable condition in $\mathbb{C}$, the inclusion $\mathrm{Com}(\mathbb{C}) \to \mathbb{C}$ admits a finite product preserve left adjoint given by "abelianization".

References

[1] Hoefnagel, M. Products and coequalisers in pointed categories. *Theory and Applications of Categories* 34 (2019), no. 43, 1386—1400.

[2] Hoefnagel, M. Centrality and the commutativity of finite products with coequalisers. *Theory and Applications of Categories* 39 (2023), no. 13, 423–443.

Barr-coexactness for representable spaces*

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The recent PhD thesis [1] presents a very detailed study of various aspects of compact ordered spaces [7] and their duality theory. Among the many interesting results, it is shown there that the dual of the category of compact ordered spaces and homomorphisms is a variety, generalising this way the corresponding well-known result for compact Hausdorff spaces.

Our interest in these structures stems from our study of Stone-type dualities [5], where we extended the context from order to metric structures (and, more general, quantale-enriched structures) which, for instance, allows us to view duality for compact Hausdorff spaces as an enriched version of Stone duality. This leads us in particular to the study of enriched compact Hausdorff spaces which constitute a natural generalisation of Nachbin's ordered compact spaces (see [9]); due to the analogy with [4] we refer to these structures as representable spaces.

In this talk we present some improvement of duality results in [5], in particular how to restrict results for categories of distributors to categories of maps (functors) [6]. Secondly, we investigate those properties (coexactness, local copresentability) of the category of representable spaces which expose the algebraic nature of the dual of this category. Finally, we also apply the techniques of [3, 2, 8] in this context.

This talk is based on joint work with Marco Abbadini, Pedro Nora, Carlos Fitas and Maria Manuel Clementino.

References

[1] Marco Abbadini. *On the axiomatisability of the dual of compact ordered spaces*. PhD thesis, Università degli Studi di Milano, 2021.

[2] Jiří Adámek and M. Cristina Pedicchio. A remark on topological spaces, grids, and topological systems. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 38(3):217–226, 1997.

[3] Michael Barr and M. Cristina Pedicchio. $\mathsf{Top}^{\mathrm{op}}$ is a quasi-variety. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 36(1):3–10, 1995.

[4] Claudio Hermida. Representable multicategories. *Advances in Mathematics*, 151(2):164–225, May 2000.

[5] Dirk Hofmann and Pedro Nora. Enriched Stone-type dualities. *Advances in Mathematics*, 330:307–360, May 2018.

[6] Dirk Hofmann and Pedro Nora. Duality theory for enriched Priestley spaces. *Journal of Pure and Applied Algebra*, 227(3):107231, March 2023.

*Abstract submitted to CT2023.

[7] Leopoldo Nachbin. *Topology and Order*, volume 4 of *Van Nostrand Mathematical Studies*. D. Van Nostrand, Princeton, N.J.-Toronto, Ont.-London, 1965. Translated from the Portuguese by Lulu Bechtolsheim.

[8] M. Cristina Pedicchio and Richard J. Wood. Groupoidal completely distributive lattices. *Journal of Pure and Applied Algebra*, 143(1-3):339–350, November 1999.

[9] Walter Tholen. Ordered topological structures. *Topology and its Applications*, 156(12):2148–2157, July 2009.

Internal Parameterization of Hyperconnected Quotients

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This talk is about my paper [1], especially on the following three points.

1. Lawvere's open problem on quotient toposes
2. Main theorem: Internal Parameterization of Hyperconnected Quotients
3. Key notion, *a local state classifier*, which is just a colimit of all monomorphisms!

1 Lawvere's open problem on quotient toposes

Lawvere listed open problems in topos theory in [2]. The first problem is as follows:

> "Is there a Grothendieck topos for which the number of these quotients is not small? At the other extreme, could they be parameterized internally, as subtoposes are?"

He asks whether the number of *quotients* of a Grothendieck topos is small. Here, a quotient of a topos $\mathcal{E}$ refers to a (suitable equivalence class of) *connected geometric morphism* from $\mathcal{E}$, i.e., a geometric morphism whose inverse image part is fully faithful.

Furthermore, for the case where the number of quotients is small, Lawvere further requires an *internal parameterization* of them. The word "internal parametrization" here means a bijective correspondence between quotients and "internal structures." Recall the case of subtoposes that Lawvere mentions in the quote. The internal parameterization of subtoposes is the bijective correspondence between subtoposes of a topos $\mathcal{E}$ (i.e., geometric embedding *into* $\mathcal{E}$) and Lawvere-Tierney topologies *in* $\mathcal{E}$. Since a Lawvere-Tierney topology is defined as an internal structure (namely, internal semilattice idempotent homomorphism on the subobject classifier), this bijective correspondence is worth being called the internal parametrization of subtoposes. Lawvere seeks a similar internal parameterization for quotient toposes.

There are several motivations for obtaining an internal parametrization of quotients. First, it makes it possible to classify all quotients just by studying a specific object in the topos without dealing with vast amounts of data about the entire category. Also, correspondence with an internal structure provides a new perspective on quotients and may lead to a new operation on the class of quotients.

2 Main theorem: Internal Parameterization of Hyperconnected Quotients

The main result is giving an internal parametrization of *hyperconnected quotients*. (Here, a hyperconnected geometric morphism from a topos $\mathcal{E}$ is referred to as a hyperconnected quotient of $\mathcal{E}$, emphasizing the aspect as a quotient of the topos $\mathcal{E}$.) In detail, we introduce the notion of *a local state classifier* and prove the following main theorem.

Theorem. *Let $\mathcal{E}$ be a topos with a local state classifier Ξ (for example, an arbitrary Grothendieck topos).*

Then the following three concepts correspond bijectively.

1. *hyperconnected quotients of* $\mathcal{E}$
2. *internal filters of* Ξ
3. *internal semilattice homomorphisms* $\Xi \to \Omega$

Our result gives a partial solution to Lawvere's open problem in two ways. First, since a hyperconnected quotient is a particular case of a quotient, it is a solution for the subclass of quotients. The second, somewhat nontrivial, is that our result solves the case of Boolean toposes. For a Boolean Grothendieck topos, whose quotients are automatically hyperconnected, we establish the internal parameterization of all quotients.

3 The key notion, *a local state classifier*

The key notion to achieving the main theorem is *a local state classifier* Ξ. The first thing to emphasize in our context is its theoretical necessity. As mentioned above, a local state classifier plays a central role throughout our theory, like a subobject classifier in the case of subtoposes.

Despite its theoretical importance, the definition of a local state classifier is unexpectedly simple: it is just a colimit of all monomorphisms!

$$\Xi = \mathrm{colim}(\mathcal{C}_{\mathrm{mono}} \to \mathcal{C})$$

At first glance, this definition might seem odd. In my talk, I will explain as intuitively as possible how studying hyperconnected quotients leads us to this simple definition and tell some facts about its existence and properties.

A local state classifier is often given by a familiar concept. For example, the local state classifier of the topos of directed graphs is the bouquet with 2 edges.

$$\circlearrowleft \bullet \circlearrowright$$

That of a group action topos $\mathrm{Set}^{G^{\mathrm{op}}}$ is the set of all subgroups $\mathrm{SubGrp}(G)$ of G equipped with the conjugate action. That of the topos of sheaves over a topological space is the terminal sheaf. These explicit descriptions enable us to connect the classifications of hyperconnected quotients and existing mathematical concepts.

References

[1] Hora, R. Internal Parameterization of Hyperconnected Quotients. *arXiv preprint arXiv:2302.06851*, 2023.

[2] Lawvere, F.W. open problems in topos theory. `https://www.acsu.buffalo.edu/~wlawvere/Openproblemstopos.htm` 2016.

Spectra and the Small Object Argument for Cones*

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Given a set I of morphisms in a category satisfying some assumptions, Quillen's small object argument [1] can in particular be used to find, for each object, a weakly initial morphism out of this object into an I-injective object. In this talk I am going to tell you about our recent work [2] that looks at the case in which I is a set of cones instead of a set of morphisms.

This small object argument for cones can be used in algebraic geometry to construct spectra. The procedure is as follows. The functor $\Gamma\colon \mathrm{LRSp} \to \mathrm{CRing}^{\mathrm{op}}$ that maps a locally ringed space $(X, \mathcal{F})$ to a ring of global sections $\mathcal{F}(X)$ has a right adjoint $\mathrm{Spec}\colon \mathrm{CRing}^{\mathrm{op}} \to \mathrm{LRSp}$ that assigns to a commutative ring its spectrum. This observation is often used in order to verify whether some particular construction is "Spec-like" by checking whether there exists such an analogous adjunction in the context one is working in. The problem of existence of a right adjoint to Γ can easily be reduced to the problem of existence of a right adjoint to the embedding $\mathrm{LRSp} \to \mathrm{RSp}$ of locally ringed spaces into ringed spaces. In our work we consider the analogue of the aforementioned embedding in a quite general setting, formally show that its right adjoint exists, and then deduce the construction of the right adjoint. This general approach encompasses many particular well-known "Spec-like" constructions and might be useful for thinking about new "Spec-like" constructions.

References

[1] Quillen, D.G. Homotopical Algebra. Lecture Notes in Mathematics, No. 43 Springer-Verlag, Berlin-New York, 1967. iv+156 pp. (not consecutively paged).

[2] Jurka, J.; Perutka, T.; Vokřínek, L. Constructing spectra using cone injectivity. 2022. arXiv:2201.03516.

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Measuring how much a model is not positively closed *

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Positive model theory is the study of a certain fragment of full first-order logic (called coherent/$\aleph_0$-geometric/positive/h-inductive): the one dealing with containments between definable sets corresponding to positive existential formulas, i.e. with theories whose axioms are of the form $\forall \vec{x}(\varphi(\vec{x}) \to \psi(\vec{x}))$ where φ and ψ are positive existential (built up from atomic formulas using $\top$, $\bot$, $\wedge$, $\vee$ and $\exists$). This fragment captures all first-order theories at the price of extending the signature (so that the homomorphisms become the same as the elementary maps); a process called Morleyization.

When one works with positive formulas the natural notion of an elementary embedding is what is called an immersion: a homomorphism preserving and reflecting the validity of positive existential formulas (at any given tuple from the domain). As mentioned above, if T is a Morleyized first-order theory then all maps between its models are immersions. Given an arbitrary coherent theory T, one of the central notions is that of a positively closed model; a model such that any homomorphism out of it is an immersion. (See e.g. [2].)

Following [4] one can identify theories (of a certain complexity) with categories (having certain exactness properties), by constructing their so-called syntactic category. In particular coherent theories, interpretations/models and homomorphisms are the same as coherent categories (the ones having finite limits, pullback-stable image factorization and pullback-stable finite unions), coherent functors and natural transformations. This allows for a translation between logical and category theoretic notions. E.g. immersions correspond to those natural transformations whose naturality squares at monomorphisms are pullbacks (called elementary in [1]).

In this talk I would like to study a functor which takes models to distributive lattices (the left Kan-extension of $Sub_{\mathcal{C}} : \mathcal{C}^{op} \to \mathbf{DLat}$ along the Yoneda-embedding, where $\mathcal{C}$ is a coherent category and $Sub_{\mathcal{C}}$ maps objects to their subobject lattices and arrows $(f : x \to y)$ to the pulling back homomorphism $(f^* : Sub_{\mathcal{C}}(y) \to Sub_{\mathcal{C}}(x))$). It yields the 2-element Boolean-algebra iff it yields a Boolean-algebra iff the model is positively closed. The length of a surjective chain out of M gives a lower bound on the Krull-dimension of the associated lattice. The results are (mostly) contained in [3].

References

[1] Barr, M. Representation of categories. *Journal of Pure and Applied Algebra* 41 (1986), pp. 113-139.

[2] Belkasmi, M. Positive model theory and amalgamations. *Notre Dame Journal of Formal Logic* 55 (2014). pp. 205-230.

[3] Kanalas, K. Positive model theory of interpretations. ArXiv:2211.14600, (2022).

[4] Makkai, M.; Reyes, G. E. First Order Categorical Logic. *Springer Berlin, Heidelberg*, 1977. 318 pp.

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Birkhoff's variety theorem for relative algebraic theories*

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Introduction

An algebraic theory, sometimes called an equational theory, is a theory defined by finitary operations and equations, such as groups and rings. It is well known that algebraic theories correspond to finitary monads on **Set**, and moreover, S-sorted algebraic theories correspond to finitary monads on $\mathbf{Set}^S$ for a given set S.

It is natural to speculate that finitary monads over general categories would extend such classical algebraic theories and give "algebraic theories" over those categories rather than $\mathbf{Set}^S$. In fact, Ford, Milius, and Schröder showed in [1] that on a category of models of a relational Horn theory (which is a special case of locally presentable categories), finitary monads give "algebraic theories" over that category. Moreover, Adámek showed in [2] that finitary monads on **Pos** (the category of posets) correspond to so-called ordered algebraic theories. We substantially generalize their results to locally finitely presentable categories. That is, we show that for a given locally finitely presentable category $\mathscr{A}$, finitary monads on $\mathscr{A}$ correspond to "algebraic theories over $\mathscr{A}$", which are called *$\mathscr{A}$-relative algebraic theories.*

In $\mathscr{A}$-relative algebraic theories, roughly speaking, ordinary (total) operators are replaced with partial operators whose domain is identified by a "formula over $\mathscr{A}$", and ordinary equations are replaced with logical sequents whose precondition is over $\mathscr{A}$, which are called *$\mathscr{A}$-relative judgements.*

	S-sorted algebraic theory	$\mathscr{A}$-relative algebraic theory
Base category	$\mathbf{Set}^S$	$\mathscr{A}$
Operator	$s_1 \times \cdots \times s_n \xrightarrow{\omega} s$	$(x_1{:}s_1, \ldots, x_n{:}s_n).\varphi \xrightarrow{\omega} s$
Axiom	equation $\tau = \tau'$	$\mathscr{A}$-relative judgement $\varphi \overset{\vec{x}}{\longmapsto} \psi$

In order to treat a logical language intrinsic to a locally finitely presentable category $\mathscr{A}$, we use neither Cartesian theories nor essentially algebraic theories but partial Horn theories [3] even though all of these theories characterize locally finitely presentable categories. This is because partial operations appear centrally in $\mathscr{A}$-relative algebraic theories and because we want to treat relation symbols explicitly, as in **Pos**.

Generalized Birkhoff's theorem

One of the famous results in classical **Set**-based algebraic theories is the following Birkhoff's variety theorem (also called HSP-theorem):

*Abstract submitted to CT2023.

Theorem. *Let (Ω, E) be a single-sorted algebraic theory (in our term,* **Set***-relative algebraic theory). Then the following are equivalent for a full subcategory $\mathscr{E} \subset Alg(\Omega, E)$.*

1. *$\mathscr{E}$ is an equational class, i.e., there exists a set of equations E' satisfying $\mathscr{E} = Alg(\Omega, E + E')$.*
2. *$\mathscr{E} \subset Alg(\Omega, E)$ is closed under:*
   - *products,*
   - *subobjects,*
   - *quotients, i.e., if $p : A \to B$ is a surjective morphism in $Alg(\Omega, E)$ and A belongs to $\mathscr{E}$, then B also belongs to $\mathscr{E}$.*

We generalize the above theorem to $\mathscr{A}$-relative algebraic theories. That is, for a given $\mathscr{A}$-relative algebraic theory, we characterize subclasses defined by $\mathscr{A}$-relative judgements via closedness conditions. In this generalization, we replace subobjects with Σ-*closed subobjects*, quotients with U-*retracts*, and add the closedness condition under filtered colimits. Note that the closedness condition under filtered colimits can not be removed even in the case of $\mathscr{A} := \mathbf{Set}^S$. The classical Birkhoff's theorem does not depend on syntax because surjections can be characterized by regular epimorphisms and all the conditions are purely category-theoretic. In contrast, our theorem depends on syntax because the concept of Σ-*closed subobjects* depends on the choice of syntax for $\mathscr{A}$. This is a notable feature of our generalization of Birkhoff's theorem.

References

[1] Ford, C; Milius, S; Schröder, L. Monads on categories of relational structures. arXiv preprint arXiv:2107.03880.

[2] Adámek, J; Ford, C; Milius, S; Schröder, L. Finitary monads on the category of posets. *Mathematical Structures in Computer Science* 31 (2021), no. 7, 799–821.

[3] Palmgren, E; Vickers, S. J. Partial Horn logic and cartesian categories. *Annals of Pure and Applied Logic* 145 (2007), no. 3, 314–353.

[4] Kawase, Y. Birkhoff's variety theorem for relative algebraic theories. in preparation.

A universal property for $(\infty, 2)$-categories of lax algebras and lax morphisms*

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Given a monad T in a 2-category, its Eilenberg–Moore object — of algebras — can be characterised as a certain weighted limit of the corresponding 2-functor from the generic monad of [1]. This is used by [7, 6] to define Eilenberg–Moore objects of homotopy-coherent monads, or $(\infty, 1)$-monads, in $(\infty, 2)$-categories. If T is a 2-monad (in the 3-category of 2-categories), its 2-category of strong algebras and strong morphisms admits the same characterisation.

For such a 2-monad, one can also talk of *lax* morphisms, and even of lax algebras. We will explain that, in order to understand these in a universal fashion, it is necessary to generalise the study to lax monads[2, 5], whose classifying Gray-category is considered independently in [4]. We then obtain a characterisation of 2-categories of lax algebras as a weighted limit, which can be used to define $(\infty, 2)$-categories of lax algebras over an $(\infty, 2)$-monad. Time allowing, we will sketch the ideas of coherence through codescent objects, that allow one to study lax morphisms (between strong algebras) in terms of strong ones as in [3].

References

[1] Auderset, C. Adjonctions et monades au niveau des 2-catégories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques* Volume 15 (1974), issue 1, 3–20.

[2] Bunge, M. Coherent extensions and relational algebras. *Transactions of the American Mathematical Society* Volume 197 (1974), 355–390.

[3] Lack, S. Codescent objects and coherence. *Journal of Pure and Applied Algebra* Volume 175 (2002), issues 1–3, 223–241.

[4] MacDonald, J. L.; Scull, L. General 2-Dimensional Adjunctions, Universal Monads and Simplicial Structures. *arXiv preprint* (2021), arXiv:2103.02001.

[5] MacDonald, J.L.; Stone, A. Soft adjunction between 2-categories. *Journal of Pure and Applied Algebra* Volume 60 (1989), issue 2, 155–203.

[6] Riehl, E.; Verity, D. Homotopy coherent adjunctions and the formal theory of monads. *Advances in Mathematics* Volume 286 (2016), 802–888.

[7] Riehl, E.; Verity, D. Elements of ∞-Category Theory. *Cambridge University Press*, 2022. 760 pp.

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Signs in objective linear algebra[*]

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Standard objective linear algebra works with slice categories instead of vector spaces and with colimit-preserving functors instead of linear maps. (Such functors are represented by spans, so that matrix multiplication becomes pullback composition of spans.) This is useful in algebraic combinatorics, although the amount of linear algebra that can be carried out in this setting is quite limited. One serious limitation is the absence of negatives. In this talk, I will explain how this can be overcome, outlining an objective theory of signs in linear algebra. It turns out one can maintain a nice topos flavour by not having the signs directly on the objects but rather on 'states' (for a monoidal structure which is not the cartesian product). By using groupoid coefficients instead of set coefficients, the signs can be encoded as homotopies, and some of the sign rules can be derived rather than stipulated. I will illustrate some of the features of the theory with an objective treatment of exterior powers.

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The Tangent Categories of Algebras over an Operad *

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Tangent categories, first introduced by Rosický [1] and recently revisited and generalized by Cockett and Cruttwell [2], provide a categorical axiomatization of the tangent bundle. There are many interesting examples and applications of tangent categories in a variety of areas such as differential geometry, algebraic geometry [3], algebra, and even computer science [8], [5]. In this talk, we will show how to expand the theory of tangent categories into a new direction: the theory of operads (cf. [7]). The main result we provide is that both the category of algebras of an operad and its opposite category are tangent categories. The tangent bundle for the category of algebras is given by the semi-direct product, while the tangent bundle for the opposite category of algebras is constructed using the module of Kähler differentials, and these tangent bundles are in fact adjoints of one another. To prove these results, we first show that the category of algebras of a coCartesian differential monad [4] is a tangent category. We then show that the monad associated to any operad is a coCartesian differential monad. This also implies that we can construct Cartesian differential categories (cf. [6]) from operads. Therefore, operads provide a bountiful source of examples of tangent categories and Cartesian differential categories, which both recapture previously known examples (e.g., affine schemes) and also yields new interesting examples (e.g., Lie algebras, Poisson algebras, Zinbiel algebras, etc.). We also discuss how certain basic tangent category notions recapture well-known concepts in the theory of operads. In this regard, we will show that vector fields correspond to derivations of operadic algebras and differential objects to modules over the initial algebra.

The paper is available at https://arxiv.org/abs/2303.05434.

References

[1] (Rosický, J.). Abstract Tangent Functors. *Diagrammes* 12 (1984), JR1–JR11.

[2] (Cockett, R.), (Cruttwell, G.). Differential Structure, Tangent Structure, and SDG. *Applied Categorical Structures* 22 (2014), no. 2, 331–417.

[3] (Cruttwell, G.); (Lemay, J.-S. P.) Tangent categories as a bridge between differential geometry and algebraic geometry. *arXiv preprint arXiv:2301.05542* (2023).

[4] (Ikonicoff, S.); (Lemay, J.-S. P.). Cartesian Differential Comonads and New Models of Cartesian Differential Categories. *arXiv preprint arXiv:2108.04304* (2021).

[5] (Manzyuk, O.). Tangent bundles in differential lambda-categories. *arXiv preprint arXiv:1202.0411* (2012).

[6] (Blute, R.); (Cockett, R.); (Seely, R. A. G.). Cartesian differential categories. *Theory and Applications of Categories* 22 (2009), no. 23, 622–672.

*Joint work with Sacha Ikonicoff and Jean-Simon Pacaud Lemay. Abstract submitted to CT2023.

[7] (Loday, J.); (Vallette, B.). Algebraic operads. *Springer, Heidelberg*, 2012. Number of pages xxiv+634.

[8] (Cockett, R.); (Lemay, J.-S. P.); (Lucyshyn-Wright, R.). Tangent Categories from the Coalgebras of Differential Categories. *Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik*, 2020. Number of pages 17:1–17:17.

Magnitude homology*

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Magnitude is a numerical invariant of enriched categories. Magnitude homology is a categorification of magnitude, first introduced by Hepworth and Willerton for graphs (seen as categories enriched in $(\mathbb{N}, \geq, +)$) and extended to enriched categories by Shulman and myself [5, 9].

Magnitude homology generalizes the ordinary homology of categories (which in turn includes group homology), and is most novel in the case of metric spaces. There, it provides an $\mathbb{R}^+$-graded homology theory of metric spaces. Work of many authors [1, 2, 3, 4, 6, 7, 10, 11] (and see [8] for more) has shown how magnitude homology conveys information about convexity and curvature in metric spaces. For example, while topological homology detects the *existence* of holes, magnitude homology detects the *size* of holes.

I will give a survey, including some results from ongoing joint work with Adrián Doña Mateo.

References

[1] Asao, Y. Magnitude homology of geodesic metric spaces with an upper curvature bound. *Algebraic & Geometric Topology* 21 (2021), 647–664.

[2] Asao, Y. Magnitude homology and path homology. *Bulletin of the London Mathematical Society* 55 (2023), no. 1, 375–398.

[3] Gomi, K. Magnitude homology of geodesic space. arXiv:1902.07044, 2019.

[4] Gomi, K. Smoothness filtration of the magnitude complex. *Forum Mathematicum* 32 (2020), 625–639.

[5] Hepworth, R; Willerton, S. Categorifying the magnitude of a graph. *Homology, Homotopy and Applications* 19 (2017), no. 2, 31–60.

[6] Jubin, B. On the magnitude homology of metric spaces. arXiv:1803.05062, 2018.

[7] Kaneta, R; Yoshinaga, M. Magnitude homology of metric spaces and order complexes. *Bulletin of the London Mathematical Society* 53 (2021), no. 3, 893–905.

[8] Leinster, T. Magnitude: a bibliography. Available at https://www.maths.ed.ac.uk/~tl/magbib, 2023.

[9] Leinster, T; Shulman, M. Magnitude homology of enriched categories and metric spaces. *Algebraic & Geometric Topology* 21 (2021), 2175–2221.

[10] Roff, E. The size and shape of things: magnitude, diversity, homology. PhD thesis, University of Edinburgh, 2022.

[11] Tajima, Y; Yoshinaga, M. Causal order complex and magnitude homotopy type of metric spaces. arXiv:2302.09752, 2023.

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Torsion theories in simplicial groups and homology*

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Torsion theories, who had been firstly used in abelian categories, had been studied in recent years in the context of semi-abelian categories (for instance in [1], [2]). The aim of this talk is to present examples of torsion theories in the semi-abelian category *Simp*(*Grp*) of simplicial groups. These torsion theories constitute a linearly ordered lattice $\mu(Grp)$ and we can find relations of torsion theories with the homological aspects of simplicial groups (see [3], [4]). First, since each torsion theory $(\mathcal{T}, \mathcal{F})$ consists in a torsion category $\mathcal{T}$ and a torsion-free $\mathcal{F}$, we can notice that each torsion-free category of the lattice $\mu(Grp)$ is a category of homotopy n-types of simplicial groups. Moreover, a lattice of torsion theories induces for each simplicial group X a lattice of "torsion subjects":

$$0 \leq \dots \quad \leq \ \mu_{n\geq}(X) \ \leq \ \mu_{\geq n}(X) \ \leq \dots \leq \ \mu_{\geq 2}(X) \ \leq \ \mu_{1\geq}(X) \ \leq \ \mu_{\geq 1}(X) \ \leq \ \mu_{0\geq}(X) \ \leq X\,.$$

Then, we can described the $n+1$th homotopy groups of X as the quotient of the torsion subobjects:

$$\mu_{\geq n+1}(X)/\mu_{\leq n+1}(X) \cong K(\pi_{n+1}(X), n+1)\,,$$

where $K(\pi_{n+1}(X), n+1)$ is the $n+1$th Eilenberg-Mac Lane simplicial group of the abelian group $\pi_{n+1}(X)$.

The torsion theories of $\mu(Grp)$ generalise the well-known torsion theories for Whitehead's crossed modules/internal groupoids; and allow to introduce examples of torsion theories in the categories of Conduché's 2-crossed modules and Ashley's crossed complexes in groups.

References

[1] Bourn D. ; Gran M. Torsion theories in homological categories. *J. Algebra* 305 (2006), 18-47.

[2] Everaert T.; Gran M. Homology of n-fold groupoids. *Th. Appl. Categ.* 23 (2010), 2, 22-41.

[3] López Cafaggi, G. Torsion theories of simplicial groups with truncated Moore complex . *Cah. Topol. Geom . Diff. Categ.* LXIII (2022), 4, 343-386.

[4] López Cafaggi, G. Torsion theories and TTF theories in Birkhoff subcategories of simplicial groups. Preprint available at *arXiv:2207.13816*, (2022)

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Spanning the tale of “Monades et descente”

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In the classical context of [4, 5]: assuming that C has pullbacks, if $\mathcal{A} : \mathsf{C}^{\mathrm{op}} \to \mathsf{Cat}$ is an indexed category, the descent category $\mathrm{Desc}_{\mathcal{A}}(p)$ is the category of actions of the internal groupoid/equivalence induced by the kernel pair of p.

The celebrated Bénabou-Roubaud Theorem [1] shows that $\mathrm{Desc}_{\mathcal{A}}(p)$ is equivalent to the category of algebras induced by $\mathcal{A}(p)! \dashv \mathcal{A}(p)$ in the classical context of [4], under the so-called Beck-Chevalley condition.

In [2], we started investigating whether commuting properties of 2-dimensional limits are useful in proving classical and new theorems of Grothendieck descent theory. Exploiting this perspective, we were able to give a generalization of the Bénabou-Roubaud Theorem in terms of commuting properties of bilimits in [2, Theorem 7.4 and Theorem 8.5].

Further investigation of commuting properties yields, in particular, to the main result of [3] which can be seen as a counterpart to the Bénabou-Roubaud Theorem, giving a characterization of monadic functors.

In this talk, we present some aspects of this work, especially emphasizing ongoing work.

References

[1] Bénabou, J.; Roubaud, J. Monades et descente. C. R. Acad. Sci. Paris Sér. A-B 270 (1970) A96–A98.

[2] Lucatelli Nunes, F. Pseudo-Kan extensions and descent theory. *Theory Appl. Categ.* 33 (2018), no. 15, 390–444.

[3] Lucatelli Nunes, F. Semantic Factorization and Descent. *Appl. Categ. Struct.* 30 (2022), no. 6, 1393–1433.

[4] Janelidze, G.; Tholen, W. Facets of Descent I. *Appl. Categ. Struct.* 2 (1994), no.3, 245-281.

[5] Janelidze, G.; Tholen, W. Facets of descent II. *Applied Categ. Structures* 5 (1997), no. 3, 229-248.

Norms on Categories*

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When working in some area of mathematics with a categorical approach, one often has to limit the class of morphisms under consideration to guarantee nice categorical properties. A good example of this phenomenon is the category $\underline{\text{MET}}$ of metric spaces, which is usually endowed with contractions as morphisms to get limits and colimits to behave like they should. This is in contrast to the standard theory of metric spaces which uses all kinds of maps, including non-continuous ones like ε-isometries.

The solution we propose is to endow the category $\underline{C}$ under investigation with an additional structure, a seminorm, defined as a map $\|\,.\,\|$ from the set of morphisms $\underline{C}_1$ to the interval $[0, \infty]$ such that

$$\|\mathrm{id}_X\| = 0 \quad \text{and} \quad \|f\,;g\| \leq \|f\| + \|g\|$$

for any $X \in \underline{C}_0$ and $f, g \in \underline{C}_1$, where $f\,;g = g \circ f$ denotes composition of morphisms. Alternatively, this can be phrased by saying that $\|\,.\,\|$ is a lax functor to the one-object 2-category $(*, [0, \infty], +, \geq)$. A seminorm is further called a norm if the following Cantor-Schröder-Bernstein style property

$$\|f\| = \|g\| = 0 \implies X \text{ isomorphic to } Y \text{ witnessed by some } f' \text{ with } \|f'\| = 0,$$

whenever $f \colon X \to Y$ and $g \colon Y \to X$, and the approximation property

$$(\forall \varepsilon > 0 \colon \exists (f \colon X \to Y) \colon \|f\| \leq \varepsilon) \implies (\exists (f \colon X \to Y) \colon \|f\| = 0)$$

hold. We will provide numerous examples of norms and seminorms spanning a multitude of disparate areas of mathematics; these include set theory, functional analysis, measure theory, topology, and metric space theory—putting special emphasis on the latter. In the examples it will turn out that a seminorm is a norm when restricting to a full subcategory of objects that are "compact" in the sense of the respective mathematical theory.

The easiest example of this phenomenon is given by the assignment

$$\|f\|_{\mathrm{inj}} \coloneqq \sup_{x \in X} \log(\#\{\, y \mid f(x) = f(y) \,\})$$

for a morphism $f \colon X \to Y$ in $\underline{\text{SET}}$. This norm measures the deviation from being injective and the Cantor-Schröder-Bernstein style property is exactly the Cantor-Schröder-Bernstein theorem.

Another motivation for our approach is to provide a framework for systematic and convenient metrization of families of equivalence classes of spaces, like the Gromov-Hausdorff space, moduli spaces, and representation spaces. The problems with doing metrizations in practice are often that they become very technical, involve arbitrary choices, and basic properties like the triangle inequality or completeness become hard to check. A category theoretical approach is natural considering the example of moduli spaces or representation spaces: representatives of the point (i.e. equivalence classes of spaces) are objects of a category and morphisms are comparison maps.

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In the final step we use the ind-completion to get a norm for “non-compact” objects. This is achieved by fixing a directed set $I = (I, \leq)$ and an order preserving function $F\colon I \to [0,1]$, thought of as the distribution of a probability measure, and defining a norm using a Choquet style integral

$$\int f(i)\,\mathrm{d}\dot{F} \coloneqq \int 1 - F(\sup\{\, i \mid f(i) \leq t \,\})\,\mathrm{d}t \qquad \text{where}$$
$$f(i) \coloneqq \inf\{\, \|g\| \mid \iota_{ij}(g) = \mathrm{pr}_i\, f, g \in \underline{C}[X_i, Y_j] \,\}$$

for inductive systems $(X_i)_{i\in I}$, $(Y_j)_{j\in I}$ and a map $f \in \underline{\textit{ind-C}} = \lim_{i\in I} \operatorname{colim}_{j\in I} \underline{C}[X_i, Y_j]$, where ι_{ij} is the universal map $\underline{C}[X_i, Y_j] \to \operatorname{colim}_{j\in I} \underline{C}[X_i, Y_j]$. In the example $(\underline{\textsc{Met}}, \|\,.\,\|_{\mathrm{dil}})$ of metric spaces this corresponds to the pointed Gromov-Hausdorff distance.

Large parts of this work are laid out in [2] and—in an extended and improved fashion—in [1], wherein references to related approaches are found.

References

[1] Insall, M.; Luckhardt, D. Norms on Categories and Analogs of the Schröder-Bernstein Theorem. Version 2. arXiv: 2105.06832 [math.CT]. May 14, 2021.

[2] Insall, M.; Luckhardt, D. Norms on Categories and Analogs of the Schröder-Bernstein Theorem. In: Proceedings of the Contemporary Mathematics in Kielce 2020 (February 24–27, 2021). October 7, 2021.

Structure–semantics and an axiomatics of enriched algebra for a subcategory of arities*

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We develop a general axiomatic framework for enriched structure–semantics adjunctions and monad–theory equivalences for a *subcategory of arities* $\mathcal{J}$ in a $\mathcal{V}$-category $\mathcal{C}$, i.e., for a full and dense sub-$\mathcal{V}$-category $\mathcal{J} \hookrightarrow \mathcal{C}$. Not only do we establish a simultaneous generalization of the monad–theory equivalences previously developed in the settings of Lawvere [1], Linton [2], Dubuc [3], Borceux-Day [4], Power [5], Nishizawa-Power [6], Lack-Rosický [7], Lucyshyn-Wright [8], and Bourke-Garner [9], but also we establish a structure–semantics theorem that generalizes those in [1, 2, 3, 4] while applying also to the settings [5, 6, 7, 8, 9] for which such a result has not previously been developed. Furthermore, we employ our axiomatic framework to establish broad new classes of examples of monad–theory equivalences and structure–semantics adjunctions for subcategories of arities enriched in *locally bounded* closed categories $\mathcal{V}$ [10].

Our axiomatics for enriched algebra begins with an arbitrary symmetric monoidal closed category $\mathcal{V}$ with equalizers, a $\mathcal{V}$-category $\mathcal{C}$, and a subcategory of arities $\mathcal{J} \hookrightarrow \mathcal{C}$, which is *not* assumed to be small. In this setting, we consider *$\mathcal{J}$-theories* and, more generally, *$\mathcal{J}$-pretheories*, in the terminology of Bourke and Garner [9]. We axiomatically demand that free algebras for all $\mathcal{J}$-theories (resp. $\mathcal{J}$-pretheories) exist, calling the subcategory of arities $\mathcal{J} \hookrightarrow \mathcal{C}$ *amenable* (resp. *strongly amenable*) in this case. We show that these axioms entail the existence of a structure–semantics adjunction for $\mathcal{J}$-theories (resp. $\mathcal{J}$-pretheories) that restricts to a dual equivalence between $\mathcal{J}$-theories and *strictly $\mathcal{J}$-algebraic $\mathcal{V}$-categories over $\mathcal{C}$*, and we establish an intrinsic characterization theorem for such $\mathcal{V}$-categories over $\mathcal{C}$. Consequently, we also obtain generalizations of certain results that Bourke and Garner [9] had proved in the special case of small subcategories of arities in the locally presentable setting, namely (1) an equivalence between $\mathcal{J}$-theories and *$\mathcal{J}$-nervous $\mathcal{V}$-monads*, and (2) an adjunction between $\mathcal{J}$-pretheories and $\mathcal{V}$-monads on $\mathcal{C}$.

Among amenable subcategories of arities one finds the identity functor $\mathcal{C} \hookrightarrow \mathcal{C}$ on an arbitrary $\mathcal{V}$-category $\mathcal{C}$ (for an arbitrary base $\mathcal{V}$), so that we recover Dubuc's structure–semantics for arbitrary $\mathcal{V}$-monads [3, 11]. More generally, we show that every *eleutheric* subcategory of arities [8, 12, 13] is amenable, so that the settings of [1, 2, 3, 4, 5, 6, 7, 8] are thus accommodated in full generality. On the other hand, it follows from the work of Bourke and Garner [9] that if $\mathcal{V}$ and $\mathcal{C}$ are locally presentable then every *small* subcategory of arities in $\mathcal{C}$ is strongly amenable. Generalizing this, we prove that if $\mathcal{V}$ is *locally bounded* and $\mathcal{C}$ is the $\mathcal{V}$-category of models for a Φ-limit theory [10], then every small subcategory of arities in $\mathcal{C}$ is strongly amenable. In particular, every small subcategory of arities in a locally bounded closed category $\mathcal{V}$ is strongly amenable. Thus we obtain wide classes of new examples of strongly amenable subcategories of arities in various convenient closed categories in topology and analysis, which need not be locally presentable, so that all of the above results are applicable in such settings.

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References

[1] Lawvere, F. W.; Functorial semantics of algebraic theories, Ph.D. thesis, Columbia University, New York, 1963, Available in: *Repr. Theory Appl. Categ.* 5 (2004).

[2] Linton, F. E. J.; Some aspects of equational categories, Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965), Springer, New York, 1966, pp. 84–94.

[3] Dubuc, E. J.; Enriched semantics-structure (meta) adjointness, *Rev. Un. Mat. Argentina* 25 (1970/71), 5–26.

[4] Borceux, F.; Day, B.; Universal algebra in a closed category, *J. Pure Appl. Algebra* 16 (1980), no. 2, 133–147.

[5] Power, J.; Enriched Lawvere theories, *Theory Appl. Categ.* 6 (1999), 83–93.

[6] Nishizawa, K.; Power, J.; Lawvere theories enriched over a general base, *J. Pure Appl. Algebra* 213 (2009), no. 3, 377–386.

[7] Lack, S.; Rosický, J.; Notions of Lawvere theory, *Appl. Categ. Structures* 19 (2011), no. 1, 363–391.

[8] Lucyshyn-Wright, R. B. B.; Enriched algebraic theories and monads for a system of arities, *Theory Appl. Categ.* 31 (2016), No. 5, 101–137.

[9] Bourke, J.; Garner, R.; Monads and theories, *Adv. Math.* 351 (2019), 1024–1071.

[10] Kelly, G. M.; Basic concepts of enriched category theory, *Cambridge University Press*, 1982. 245 pp. Available in: *Repr. Theory Appl. Categ.* 10 (2005).

[11] Dubuc, E. J.; Kan extensions in enriched category theory, *Lecture Notes in Mathematics*, Vol. 145, Springer-Verlag, Berlin-New York, 1970.

[12] Lucyshyn-Wright, R. B. B.; Parker, J.; Presentations and algebraic colimits of enriched monads for a subcategory of arities, *Theory Appl. Categ.* 38 (2022), No. 38, 1434–1484.

[13] Lucyshyn-Wright, R. B. B.; Parker, J.; Diagrammatic presentations of enriched monads and varieties for a subcategory of arities, Preprint, arXiv:2207.05184, 2022.

Coinductive equivalences in algebraic weak ω-categories*

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There are many different approaches to weak higher-dimensional categories. One proposed by Leinster [1], based on an idea of Batanin's [2], defines weak ω-categories as the algebras for a particular monad on the category of globular sets. Intuitively, this monad encodes only the existence part of a pasting theorem for globular pasting diagrams. In this talk, I will make precise (and sketch a proof of) the uniqueness part, and discuss some of its applications.

References

[1] Leinster, T. Higher operads, higher categories. Volume 298 of London Mathematical Society Lecture Note Series. *Cambridge University Press*, 2004. 380pp.

[2] Batanin, M. A. Monoidal globular categories as a natural environment for the theory of weak n-categories. *Advances in Mathematics* 136(1)(1998), 39–103.

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The representing localic groupoid of a geometric theory*

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In [1] Joyal and Tierney famously proved that every Grothendieck topos can be represented as a topos of equivariant sheaves on a localic groupoid. This provides a way to understand toposes in terms of 'locales plus automorphisms'. However, for non-experts it is not always obvious from the original proof exactly how to find such a localic groupoid for concrete examples. Given a geometric theory $\mathbb{T}$, we give an explicit presentation of a representing localic groupoid $G^{\mathbb{T}}$ for the classifying topos of $\mathbb{T}$. This can be compared to the topological groupoids of Forssell in [3].

Explicitly, if $\mathbb{T}$ is a geometric theory (with any functions expressed as functional relations), then the locale of objects $G_0^{\mathbb{T}}$ of the classifying groupoid is the classifying locale the propositional geometric theory $P[\mathbb{T}]$ defined as follows.

- For each sort X of $\mathbb{T}$, there is a basic proposition $[n \sim^X m]$ for each $n, m \in \mathbb{N}$ together with axioms making $\sim^X$ act like a partial equivalence relation on $\mathbb{N}$.
- For each relation symbol $R \subseteq X^1 \times \cdots \times X^k$ of $\mathbb{T}$, and for each $n_1, \ldots, n_k \in \mathbb{N}$ and $m_1, \ldots, m_k \in \mathbb{N}$, we have a basic proposition $[(n_1, \ldots, n_k) \in R]$ and axioms ensuring that R respects the equivalence relations $\sim^{X^i}$.
- For each axiom $\varphi \vdash_{x_1,\ldots,x_k} \psi$ of $\mathbb{T}$, we add an axiom

  $$\bigwedge_{i=1}^{k} [n_i \sim^{X^i} n_i] \wedge \varphi_{n_1,\ldots,n_k} \vdash \psi_{n_1,\ldots,n_k}$$

  for each $n_1, \ldots, n_k \in \mathbb{N}$, where $\varphi_{n_1,\ldots,n_k}$ and $\psi_{n_1,\ldots,n_k}$ are obtained from φ and ψ by replacing each free variable x_i by a (fixed) natural number n_i, each quantified formula $\exists x \colon X.\ \chi(x, \ldots)$ by a join $\bigvee_{n_x \in \mathbb{N}} \chi(n_x, \ldots)$, each subformula of the form $(y_1, \ldots, y_\ell) \in R$ with $[(y_1, \ldots, y_\ell) \in R]$, and each subformula of the form $x =_X y$ with $[x \sim^X y]$.

The locale of morphisms $G_1^{\mathbb{T}}$ is similarly the classifying locale of a propositional geometric theory $P[\mathbb{T}_{\cong}]$, where $\mathbb{T}_{\cong}$ is a naturally-defined geometric theory of isomorphisms between two $\mathbb{T}$-models.

Objects of the classifying topos $\mathbf{Set}[\mathbb{T}]$ then correspond to equivariant étale bundles over this groupoid. In particular, the generic $\mathbb{T}$-model in the classifying topos gives a certain family of equivariant étale bundles over $G^{\mathbb{T}}$ (one bundle for each sort).

In fact, this family of étale bundles on $G^{\mathbb{T}}$ is universal, not only in the bicategory of étale-complete localic groupoids (those coming from toposes), but in a much larger bicategory of localic groupoids and internal anafunctors. (See also [2] which describes how geometric morphisms between toposes correspond to anafunctors between their representing groupoids, but restricts to open localic groupoids and does not discuss the 2-morphisms.)

This presentation recovers the construction of [3] under countability restrictions on $\mathbb{T}$. We can also

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derive a presentation for the localic reflection of $\mathbf{Set}[\mathbb{T}]$ from the fact that this is the locale of connected components of $G^{\mathbb{T}}$ (given by the coequaliser of the domain and codomain morphisms).

Our approach suggests the possibility of analogous 'classifying groupoids' that are not directly related topos theory. For example, suppose $\mathbb{T}$ is a *dual geometric* theory, where the language has finite joins, arbitrary meets, existential quantification and equality. Then there is localic groupoid which classifies families of proper separated bundles modelling the theory $\mathbb{T}$.

References

[1] Joyal, A; Tierney, M. An extension of the Galois theory of Grothendieck. *Mem. Amer. Math. Soc.* 51 (1984), no. 309, v–71.

[2] Moerdijk, I. The Classifying Topos of a Continuous Groupoid I. *Trans. Amer. Math. Soc.* 310 (1988), no. 2, 629–668.

[3] Forssell, H. Topological representation of geometric theories. *Math. Log. Quart.* 6 (2012), no. 58, 380–393.

Intuitionistic compact ordered spaces*

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A *compact ordered space* is a compact space X equipped with an order $\leq$ which is closed as a subset of X^2. A *Priestley space* is a compact ordered space which embeds in a power of $\{0,1\}$ (the Priestley separation axiom). The Priestley dual of a Priestley space X is the set of continuous order-preserving maps $X \to \{0,1\}$ and each order-preserving map $\{0,1\}^n \to \{0,1\}$ gives rise to an n-ary operation on this dual. It turns out that the maps $\{0,1\}^n \to \{0,1\}$ correspond to the n-ary operations in the theory of distributive lattices (i.e., the category of order-preserving maps between powers of $\{0,1\}$ is the Lawvere theory of distributive lattices), and this yields *Priestley duality*: a contravariant equivalence between the category of Priestley spaces and the category of distributive lattices.

It was shown in [1, 2] that $\{0,1\}$ can be replaced with the interval $[0,1]$ to yield a variety of algebras dual to the category $\mathbf{KH}_{\leq}$ of compact ordered spaces. This variety is $\aleph_1$-ary instead of finitary.

The goal of this talk is to explain how several properties of Priestley duality generalize to this $[0,1]$-based duality for $\mathbf{KH}_{\leq}$. The Katětov-Tong theorem [4, 5] plays an important role similar to the Priestley separation axiom. In particular, it will be explained how propositional and first-order intuitionistic logic fit in this context by giving several equivalent definitions of *intuitionistic compact ordered spaces* generalizing Esakia spaces. They are the compact ordered spaces such that for all open subset U, the least downward closed subset $\downarrow U$ containing U is open, and we denote their category by $\mathbf{IKH}_{\leq}$. We will conclude by giving a generalization of the open mapping theorem of [3], which translates topologically Pitts' uniform interpolation, to compact ordered spaces: the right adjoint of the forgetful functor $\mathbf{IKH}_{\leq} \to \mathbf{KH}_{\leq}$ sends projection maps to open maps.

References

[1] Abbadini, M. On the Axiomatisability of the Dual of Compact Ordered Spaces. PhD Thesis. Università degli studi di Milano, Italy, 2021.

[2] Abbadini, M.; Reggio, L. On the Axiomatisability of the Dual of Compact Ordered Spaces. *Applied Categorical Structures* 28 (2020), no. 6, 921–934.

[3] van Gool, S.; Reggio, L. An open mapping theorem for finitely copresented Esakia spaces. *Topology and its Applications* 240 (2018), 69–77.

[4] Katětov, M. On Real-Valued Functions in Topological Spaces. *Fundamenta Mathematicae* 38 (1951), 85–91.

[5] Tong, H. Some characterizations of normal and perfectly normal spaces. *Duke Mathematical Journal* 19 (1952), no. 2, 289–292.

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Normality for monoid monomorphisms*

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In the category of monoids we characterize monomorphisms that are normal, in an appropriate sense, to internal reflexive relations, preorders or equivalence relations. The zero-classes of such internal relations are first described in terms of convenient syntactic relations associated to them and then through the adjunctions associated with the corresponding normalization functors. The largest categorical equivalences induced by these adjunctions provides equivalences between the categories of relations generated by their zero-classes and the ones of monomorphisms that we suggest to call *normal with respect to* the internal relations considered. This idea, although being transverse to the literature in the field, has not in our opinion been presented and explored in full generality. The existence of adjoints to the normalization functors permits developing a theory of normal monomorphisms, thus extending many results from groups and protomodular categories to monoids and unital categories. .

References

[1] N. Martins-Ferreira and M. Sobral, *On the normality of monoid monomorphisms*, Preprint (2022). https://arxiv.org/abs/2210.03490

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A 'Basis Theorem' for 2-rigs and Rig Geometry*

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To approach some of the semi-combinatorial examples of Rig Geometry [1] in a way analogous to Grothendieck's algebraic geometry, preliminary algebraic work is needed to understand the relevant sites of definition. Let **Rig** be the category of rigs and let 2 be the distributive lattice with two elements, so that 2/**Rig** is the category of rigs with idempotent addition. We prove, for 2/**Rig**, an analogue of Hilbert's Basis Theorem, and apply it to different coextensive categories of rigs (not necessarily with idempotent addition) to prove that finitely presentable objects have finite direct product decompositions. As corollaries we deduce moleculartity of the associated Gaeta and Zariski toposes. (Most details may be found in [2].)

References

[1] F. W. Lawvere. Core varieties, extensivity, and rig geometry *TAC* Volume 20 (2008), no. 14, pp 497-503.

[2] M. Menni. A 'Basis Theorem' for 2-rigs and Rig Geometry. *Cah. Topol. Géom. Différ. Catég.* LXII (2021), no. 4, 451–490.

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Nerves of enriched categories via necklaces*

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In [4], Dugger and Spivak gave a very elegant description of the left adjoint $\mathfrak{C}$ to the homotopy coherent nerve functor $N^{hc} : S\,\mathrm{Cat} \to \mathrm{SSet}$ [3] by means of necklaces. A necklace is a finite sequence of simplices which are glued at their endpoints. The main goal of this talk is to present a general construction of nerves of enriched categories using necklaces. Apart from the homotopy coherent nerve, this construction will encompass various other interesting examples like the dg-nerve [2][10], the Duskin nerve [5] (for strict 2-categories) and the nerve for cubical categories [11][7]. Consider a suitable monoidal category $\mathcal{W}$ and a strong monoidal functor from the category of necklaces to $\mathcal{W}$:

$$D : \mathcal{N}ec \to \mathcal{W}$$

I will explain how D generates a nerve functor

$$N^D : \mathcal{W}\,\mathrm{Cat} \to \mathrm{SSet} \tag{$*$}$$

from $\mathcal{W}$-categories to simplicial sets, and describe its left-adjoint. This approach allows to simplify some arguments, like showing that the nerve $N^D(\mathcal{C})$ of some $\mathcal{W}$-category $\mathcal{C}$ is a quasi-category, or showing that two nerves are homotopically equivalent. As an application, I will give a simple explicit description of the left-adjoint of the dg-nerve $N^{dg} : dg\,\mathrm{Cat} \to \mathrm{SSet}$, in analogy to Dugger and Spivak's description of the categorification functor $\mathfrak{C}$.

The above procedure presents itself more naturally in the context of *templicial objects* $S_\otimes\mathcal{V}$. These were introduced in joint work with Wendy Lowen [9] for a suitable (possibly non-cartesian) monoidal category $\mathcal{V}$. Inspired by Leinster's homotopy monoids [8], templicial objects are certain colax monoidal functors. They may be viewed as simplicial objects internalized to $\mathcal{V}$, and have an associated notion of *quasi-categories in* $\mathcal{V}$. When $\mathcal{V} = \mathrm{Set}$, this recovers the usual simplicial sets and quasi-categories. If $\mathcal{W}$ is appropriately enriched over $\mathcal{V}$, every nerve N^D as in $(*)$ can be lifted to an enriched version $N^D_\mathcal{V} : \mathcal{W}\,\mathrm{Cat} \to S_\otimes\mathcal{V}$. I will explain how this works and hone in on the special case where $\mathcal{W} = S\mathcal{V}$ is the category of simplicial objects in $\mathcal{V}$, which yields an enriched version of the homotopy coherent nerve: $N^{hc}_\mathcal{V} : S\mathcal{V}\text{-}\,\mathrm{Cat} \to S_\otimes\mathcal{V}$. In work in progress, we wish to show that this functor is a Quillen equivalence, generalizing the classical equivalence between the model categories for simplicial categories [1] and quasi-categories [6].

References

[1] Bergner, J. A model category structure on the category of simplicial categories. *Trans. Amer. Math. Soc.* 359 (2007), no. 5, 2043–2058.

[2] Block, J.; Smith, A. M. The higher Riemann-Hilbert correspondence. *Adv. Math.* 252 (2014), 382–405.

[3] Cordier, J.-M. Sur la notion de diagramme homotopiquement cohérent. *Cahiers de Topologie et Géom. Différentielle Catégoriques* 23 (1982), no. 1, 93–112.

*Abstract submitted to CT2023.

[4] Dugger, D.; Spivak D. I. Rigidification of quasi-categories. *Algebr. Geom. Topol.* 11 (2011), no. 1, 225–261.

[5] Duskin, J. Simplicial matrices and the nerves of weak n-categories I: nerves of bicategories. *Theory Appl. Categ.* 9 (2001), 198–308.

[6] Joyal, A. The theory of quasi-categories and its applications. *unpublished*, 2008. 348 pp.

[7] Le Grignou, B. Cubical model categories and quasi-categories. *Theory Appl. Categ.* 35 (2020), no. 18, 622–664.

[8] Leinster, T. Homotopy algebras for operads. arXiv:math/0002180v1 (2000), 102 pp.

[9] Lowen, W.; Mertens, A. Enriched quasi-categories and the templicial homotopy coherent nerve. arXiv:2302.02484v1 (2023), 44 pp.

[10] Lurie, J. Higher algebra. *unpublished*, 2017. 1553 pp.

[11] Rivera, M.; Zeinalian, M. Cubical rigidification, the cobar construction and the based loop space. *Algebr. Geom. Topol.* 18 (2018), no. 7, 3789–3820.

2-classifiers via dense generators and the case of stacks*

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In order to generalize the fundamental notion of subobject classifier to dimension 2, we need to upgrade the concept of monomorphism to the one of discrete opfibration. Indeed, in dimension 1, we want to classify morphisms with fibres of dimension 0, i.e. essentially with fibres that are empty or singletons. Whereas in dimension 2, we want to classify morphisms with fibres of dimension 1, i.e. essentially with fibres that are general sets. Wishing to keep as classifier the inclusion of the verum inside generalized truth values, this passage corresponds to upgrading the classification process from one regulated by pullbacks to one regulated by comma objects. Exactly as $\mathcal{S}et$ is the archetypal example of elementary topos, the 2-category $\mathcal{CAT}$ becomes the archetypal example of elementary 2-topos, classifying the discrete opfibrations with small fibres via the construction of the category of elements.

Our main result is that, for nice enough 2-categories, the study of the 2-classifiers can be reduced to dense generators. In particular, a morphism is classified precisely when, expressing its codomain as relatively absolute 2-colimit of the dense generators, every change of base of it along the components of the universal cocylinder is classified. This brings many advantages; for example, the 2-classifier in $\mathcal{CAT}$ and hence the study of the construction of the category of elements is reduced to the classification of functors over the terminal category 1, which is trivial. Indeed a functor $\mathcal{C} \to 1$ is classified by the map $1 \to \mathcal{CAT}$ that picks $\mathcal{C}$.

To prove the reduction to dense generators, we have the idea to use a preservation of 2-colimits result for the 2-functor of pullback along a discrete opfibration. But in order to apply such idea, we first need to generalize the calculus of colimits in 1-dimensional slices to dimension 2. In dimension 1, a colimit in a slice is the same thing as the map from the colimit of the domains that is induced by the universal property of the colimit. In dimension 2, we can achieve an analogue of this by reducing weighted 2-colimits to oplax normal conical ones. The price to pay is that we need to work with lax slices and use $\mathcal{F}$-category theory. And as a consequence, we need to consider 2-functors of pullback along a discrete opfibration between lax slices.

We apply the reduction of the study of 2-classifiers to dense generators to obtain a 2-classifier in 2-presheaves. This involves a 2-dimensional analogue of sieves. Considering then a suitable 2-dimensional analogue of closed sieves, we restrict the 2-classifier in 2-presheaves to a 2-classifier in stacks.

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Galois structures in preordered groups*

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A *preordered group* $(G, \leq)$ is a (not necessarily abelian) group $G = (G, +, 0)$ endowed with a preorder relation $\leq$ which is compatible with the addition $+$ of the group G: for any a, b, c, d in G, $a \leq c$ and $b \leq d$ implies that $a + b \leq c + d$. Given two preordered groups $(G, \leq_G)$ and $(H, \leq_H)$, a morphism f from $(G, \leq_G)$ to $(H, \leq_H)$ is a *morphism of preordered groups* when $f\colon G \to H$ is a preorder preserving group morphism. The fundamental properties of the category $\mathsf{PreOrdGrp}$ of preordered groups were studied in [3] by Clementino, Martins-Ferreira and Montoli. Among other things, they recall that the category $\mathsf{PreOrdGrp}$ of preordered groups is isomorphic to the category whose objects are pairs (G, M), where G is a group and M a submonoid of G closed under conjugation in G (that is, $g+m-g \in M$ for any $g \in G$ and $m \in M$), and whose arrows $f\colon (G, M) \to (H, N)$ are group morphisms $f\colon G \to H$ satisfying the condition $f(M) \subseteq N$. The submonoid M in a given preordered group (G, M) is called the *positive cone* of G and is written P_G. Two important results of the article [3] are the fact that $\mathsf{PreOrdGrp}$ is a *normal category* [9] and that the *effective descent morphisms* [7] in this context exactly coincide with the normal epimorphisms.

In this talk, we present two different *Galois structures* [6] in $\mathsf{PreOrdGrp}$. We first of all consider the reflector $I\colon \mathsf{PreOrdGrp} \to \mathsf{ParOrdGrp}$ to the full subcategory $\mathsf{ParOrdGrp}$ of *partially ordered groups*. The objects of $\mathsf{ParOrdGrp}$ are given by the preordered groups whose preorder is antisymmetric or, equivalently, by the pairs (G, P_G) for which the positive cone P_G is a *reduced monoid* (in the sense that the only element of P_G having its inverse in P_G is the neutral element 0). The functor I is defined, for any preordered group (G, P_G), by $I(G, P_G) = (G/N_G, P_G/N_G)$ where N_G is the normal subgroup $N_G = \{x \in G \mid x \in P_G \text{ and } -x \in P_G\}$. It is *semi-left-exact* [2], which means that the *absolute* Galois structure induced by I is *admissible*. In this context it is then possible to give an explicit description of both *trivial* and *central extensions*. A morphism $f\colon (G, P_G) \to (H, P_H)$ in $\mathsf{PreOrdGrp}$ is a trivial extension if and only if the restriction $\phi\colon N_G \to N_H$ of $f\colon G \to H$ to N_G is a group isomorphism, and it is a central extension if and only if its kernel $\mathsf{Ker}(f)$ lies in $\mathsf{ParOrdGrp}$. Remark that the class $\mathscr{M}^*$ of central extensions with respect to this absolute Galois structure is also part of a *monotone-light* factorization system $(\mathscr{E}', \mathscr{M}^*)$ in the sense of [1]. The article [4] proves (among other things) these statements, which can be extended to the more general setting of *V-groups* (for V a commutative, unital and *integral* quantale) [10].

We next present a generalization (to preordered groups) of a well-known Galois theory existing in the category Grp of groups. If we consider the abelianization functor $ab\colon \mathsf{Grp} \to \mathsf{Ab}$ to the category Ab of abelian groups, as well as the two classes $\mathcal{E}_{ab}$ and $\mathcal{Z}_{ab}$ of surjective group homomorphisms in Grp and Ab, respectively, we indeed obtain an *admissible Galois structure* $\Gamma_{ab} = (\mathsf{Grp}, \mathsf{Ab}, ab, u, \mathcal{E}_{ab}, \mathcal{Z}_{ab})$, where u denotes the inclusion functor $\mathsf{Ab} \hookrightarrow \mathsf{Grp}$. In this situation, it then turns out that the *Γ_{ab}-central* and *Γ_{ab}-normal extensions* both coincide with the *algebraically central extensions*, i.e. with the surjective group homomorphisms $f\colon G \twoheadrightarrow H$ whose kernel $\mathsf{Ker}(f)$ is in the center $Z(G) = \{x \in G \mid x + y = y + x \ \forall y \in G\}$ of G.

It turns out that the subcategory $\mathsf{Ab}(\mathsf{PreOrdGrp})$ of abelian objects in $\mathsf{PreOrdGrp}$ is the full subcategory whose objects are preordered groups $(G, \leq)$ for which G is an abelian group and $\leq$ an equivalence relation on G or, alternatively, pairs (G, P_G) such that $G \in \mathsf{Ab}$ and $P_G \in \mathsf{Ab}$. Accordingly, the appropriate generalization to preordered

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groups of the abelianization functor $ab\colon \mathsf{Grp} \to \mathsf{Ab}$ is a functor $F\colon \mathsf{PreOrdGrp} \to \mathsf{Ab}(\mathsf{PreOrdGrp})$ whose definition will be explained in the talk. We will see that this functor F gives rise to an admissible Galois structure Γ, and that the Γ-central and Γ-normal extensions both coincide with the regular epimorphisms $f\colon (G, P_G) \twoheadrightarrow (H, P_H)$ in PreOrdGrp such that

- $\mathsf{Ker}(f) \subseteq Z(G)$;
- $y - x \in P_G$ and $-x + y \in P_G$, for any pair $(x, y) \in \mathsf{Eq}(f) \cap (P_G \times P_G)$,

where $\mathsf{Eq}(f)$ is the kernel pair of f. The proofs of these assertions make extensive use of the results in [11], as well as of the *relative modularity* of the category PreOrdGrp. Note that these results are interesting also because PreOrdGrp is not even a *subtractive* category in the sense of [8]. This part of the presentation is based on the preprint [5] written in collaboration with Marino Gran.

References

[1] Carboni A.; Janelidze G.; Kelly G. M.; Paré R. On localization and stabilization for factorization systems. *Appl. Categor. Struct.* 5 (1997), 1–58.

[2] Cassidy C.; Hébert M.; Kelly G. M. Reflective subcategories, localizations and factorization systems. *J. Austral. Math. Soc.* 38 (1985), 287–329.

[3] Clementino M. M.; Martins-Ferreira N.; Montoli A. On the categorical behaviour of preordered groups. *J. Pure Appl. Algebra* 223 (2019), 4226–4245.

[4] Gran M.; Michel A. Torsion theories and coverings of preordered groups. *Algebra Univers.* 82 (2021), no. 22.

[5] Gran M.; Michel A. Central extensions of preordered groups. *Preprint* (2023).

[6] Janelidze G. Pure Galois theory in categories. *J. Algebra* 132 (1990), no. 2, 270–286.

[7] Janelidze G.; Sobral M.; Tholen W. Beyond Barr exactness: effective descent morphisms, in: M. C. Pedicchio, W. Tholen (Eds.), Categorical Foundations, in: Encyclopedia of Mathematics and Its Applications. *Cambridge University Press*, 2004. 47 pp.

[8] Janelidze Z. Subtractive categories. *Appl. Categor. Struct.* 13 (2005), 343–350.

[9] Janelidze Z. The pointed subobject functor, 3×3 lemmas, and subtractivity of spans. *Theory Appl. Categ.* 23 (2010), no. 11, 221–242.

[10] Michel A. Torsion theories and coverings of V-groups. *Appl. Categor. Struct.* 30 (2022), 659–684.

[11] Montoli A.; Rodelo D.; Van der Linden T. A Galois theory for monoids. *Theory Appl. Categ.* 29 (2014), no. 7, 198–214.

Graph models for E_n-operads*

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The "complete graph operad" was introduced by Clemens Berger quite a while ago, and was soon followed by the introduction of a variant by Brun, Fiedorowicz and Vogt. However, the relation of these operads to each other and to other well-known operads left something to be clarified, and the literature contains several gaps. In this talk, I will show that these operads are equivalent to each other and to the little n-cubes operad.

References

[1] Beuckelmann, A.; Moerdijk, I. A small catalogue of E_n-operads. *arxiv* Volume (2022)

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Non-pointed abelian categories*

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Abelian categories have proved to be an extremely useful notion in many fields of mathematics. They can be defined by the so-called "Tierney Equation"

$$\text{Abelian} = \text{Additive} + \text{(Barr-)Exact},$$

which makes it clear how exactness in the sense of Barr plays a fundamental role in the context of additive categories.

Various notions have been introduced as weakenings of abelianness, often by weakening one or both parts of Tierney's equation. In [6], starting from a characterization of the categories of affine spaces as certain slices of categories of modules, Carboni introduced *modular categories* as a non-pointed version of additive categories. In fact, a pointed category $\mathcal{C}$ is modular if and only if it is additive.

After Carboni's result, Bourn studied the categories $\mathcal{P}t_B(\mathcal{C}) = 1\backslash(\mathcal{C}/B)$, where B is any object of $\mathcal{C}$. In fact these categories may be seen as the fibers of the codomain functor $\mathcal{P}t(\mathcal{C}) \to \mathcal{C}$, which is also known as the fibration of points when $\mathcal{C}$ has pullbacks. There has been extensive interest in properties relating to this functor. An example of such a property is *protomodularity* [3], where the change-of-base functors between the fibers are required to be conservative. By contrast with modularity, protomodularity, in the pointed context, is weaker than additivity, since it includes categories such as (non-abelian) groups. Other properties can be considered as non-pointed versions of additivity; for example, a category is:

- *naturally Mal'tsev* [9] if and only if all fibers $\mathcal{P}t_B(\mathcal{C})$ are additive;
- *penessentially affine* [5] if all change-of-base functors of the fibration of points are fully faithful and create subobjects;
- *essentially affine* [3] if and only if all change-of-base functors of the fibration of points are equivalences.

All these notions coincide with additivity for pointed categories, and any of these is strictly stronger than the previous one in the list.

Protomodularity plays a key role in the definition of semi-abelian [8] and homological categories [1]. In fact, homological categories are defined by weakening both summands of Tierney's Equation, since they are only regular, instead of exact, and pointed protomodular, instead of additive. Therefore, additive regular categories, as for instance the category of torsion free abelian groups, are examples of homological categories. In this context, many of the homological lemmas (such as the Five Lemma, the Snake Lemma, the Nine Lemma), as well as the Noether Isomorphisms Theorems, are still valid.

In this talk, we are going to consider a new point of view on a possible notion of non-pointed abelian category. We rely on a more classical description of an abelian category; namely, an abelian category is a pointed category with finite limits and colimits where all monomorphisms and epimorphisms are normal.

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We will study, in a regular category, the following simple property, which may be seen as a generalization of the fact that every monomorphism is the kernel of its cokernel:

(P) For every span $Z \xleftarrow{p} X \xrightarrow{m} Y$ where p is a regular epimorphism and m is a monomorphism, their pushout exists and is also a pullback.

We investigate the consequences of (P), also in conjunction with protomodularity, and prove that categories satisfying these requirements share with abelian categories the property that every monomorphism is Bourn-normal [4]. This observation shows that such categories are, in particular, naturally Mal'tsev. We then prove that, for a quasi-pointed, regular protomodular category, (P) also implies (Barr-)exactness. As a consequence, we get that this property characterizes abelian categories among the homological ones.

We then show that exact protomodular categories satisfying (P) may be seen as a new non-pointed version of abelian categories, which turns out to be weaker than another possible one, namely that of exact essentially affine categories. In fact, for exact categories, property (P) characterizes penessentially affine categories among the protomodular ones. Furthermore, we show that a category which is regular protomodular, satisfies (P), and whose dual is also protomodular, is essentially affine.

An interesting example of a category with property (P) is the category of abelian extensions of an object in a semi-abelian category. By exploiting an observation from this example in the particular case of groups, we can provide two different characterizations of strongly semi-abelian categories. The second one is given by means of a variant of the axiom of normality of unions, which was introduced in [2] in relation to the representability of internal actions, and is also related to the existence of normalizers in a semi-abelian category [7].

References

[1] Borceux, F; Bourn, D. Mal'cev, protomodular, homological and semi-abelian categories. Mathematics and its Applications 566, *Kluwer Academic Publishers*, 2004.

[2] Borceux, F; Janelidze, G; Kelly, G.M. On the representability of actions in a semi-abelian category, *Theory Appl. Categ.* 14 (2005), 244–286.

[3] Bourn, D. Normalization equivalence, kernel equivalence and affine categories, in Category theory (Como, 1990), Lecture Notes in Mathematics 1488, *Springer*, 1991, 43–62.

[4] Bourn, D. Normal subobjects and abelian objects in protomodular categories, *J. Algebra* 228 (2000), 143–164.

[5] Bourn, D. Abelian groupoids and non-pointed additive categories, *Theory Appl. Categ.* 20 (2008), 48–73.

[6] Carboni, A. Categories of affine spaces, *J. Pure Appl. Algebra* 61 (1989), 243–250.

[7] Gray, G.R.A. Normalizers, centralizers and action representability in semi-abelian categories, *Appl. Categ. Struct.* 22 (2014), 981–1007.

[8] Janelidze, G; Márki, L; Tholen, W. Semi-abelian categories, *J. Pure Appl. Algebra* 168 (2002), 367–386.

[9] Johnstone, P.T. Affine categories and naturally Mal'cev categories, *J. Pure Appl. Algebra* 61 (1989), 251–256.

New methods for contructing model categories*

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Model categories provide a good environment to do homotopy theory. A model category consists of a bicomplete category together with three classes of morphisms (*weak equivalences*, *cofibrations*, and *fibrations*) satisfying a list of axioms. While weak equivalences are the main players in a model category and encode how two objects should be thought of as being "the same", the additional data of cofibrations and fibrations typically facilitates computations of homotopy limits and colimits, and of derived functors.

However, because of their robust structure, model categories are usually hard to construct. To address this question, several methods have been developed in the literature [1, 2, 3, 4, 8, 10, 11]. In recent work [7], we develop yet new techniques for constructing model structures from given classes of cofibrations, fibrant objects, and weak equivalences between them. The requirement that one only needs to provide a class of weak equivalences between fibrant objects both simplifies the conditions to check and seems more natural in practice: often, the fibrant objects are the "well-behaved" objects in a model category and so the weak equivalences should only be expected to exhibit a good behavior between these objects. As a straightforward consequence of our result, we obtain a more general version of the usual right-induction theorem along an adjunction, where fibrations and weak equivalences are now only right-induced between fibrant objects; we refer to such an induced model structure as *fibrantly-induced*.

As applications of these new methods, we construct several model structures on the category DblCat of double categories. One of the applications shows that, while a certain right-induced model structure on DblCat does not exist [5], the fibrantly-induced one does. Another application of interest provides a model structure on DblCat which is Quillen equivalent to Lack's model structure on the category 2Cat of 2-categories through the *square* (also called *quintet*) functor $\mathrm{Sq}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$. In particular, this proves the strict analogue of a conjecture by Gaitsgory-Rozenblyum [6] formulated in the ∞-setting.

References

[1] Beke, T. Sheafifiable homotopy model categories. *Mathematical Proceedings of the Cambridge Philosophical Society* 129 (2000), no. 3, 447–475.

[2] Cisinski, D.-C. Les préfaisceaux comme modèles des types d'homotopie. *Société mathématique de France*, 2006. xxiv+394 pp.

[3] Cole, M. Mixing model structures. *Topology and its Applications* 153 (2006), no. 7, 1016–1032.

[4] Crans, S. E. Quillen closed model structures for sheaves. *Journal of Pure and Applied Algebra* 101 (1995), no. 1, 35–57.

[5] Fiore, T. M.; Paoli, S.; Pronk, D. Model structures on the category of small double categories. *Algebraic & Geometric Topology* 8 (2008), no. 4, 1855–1959.

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[6] Gaitsgory, D.; Rozenblyum, N. A Study in Derived Algebraic Geometry, Volume I: Correspondences and duality. *American Mathematical Society*, 2017. xl+531 pp.

[7] Guetta, L.; Moser, L.; Sarazola, M.; Verdugo, P. Fibrantly induced model structures. Preprint on arXiv:2301.07801 (2023).

[8] Hess, K.; Kędziorek, M.; Riehl, E.; Shipley, B. A necessary and sufficient condition for induced model structures. *Journal of Topology* 10 (2017), no. 2, 324–369.

[9] Lack, S. A Quillen model structure for 2-categories. *K-Theory* 26 (2002), no. 2, 171–205.

[10] Olschok, M. Left determined model structures for locally presentable categories. *Applied Categorical Structures* 19 (2011), no. 6, 901–938.

[11] Stanculescu, A. E. Constructing model categories with prescribed fibrant objects. *Theory and Applications of Categories* 29 (2014), no. 23, 635–653.

The Para Construction as a Distributive Law*

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The Para construction takes a monoidal category $\mathcal{M}$ and gives a category $\mathbf{Para}(\mathcal{M})$ where a morphism $A \to B$ is a pair $(P,\, f : P \otimes A \to B)$ of a "parameter space" P and a parameterized map f in $\mathcal{M}$. This construction formalizes the idea of separating inputs into special "control variables" or "parameters" which will be set separately from the other inputs to a process. The Para construction has played an important role in categorical accounts of deep learning — where it was first described by Fong, Spivak, and Tuyeras — open games, and cybernetics.

The Para construction has been generalized in a number of ways. First, it can take an action of a monoidal category $\mathcal{M}$ on a category $\mathcal{C}$ (an "actegory"). And second, the resulting category can be seen as the shadow of a bicategory where 2-cells are reparameterizations. In this talk, we will see a further generalization of the scope of the Para construction — we will take an actegory $\odot : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ and produce a double category $\mathbf{Para}(\odot)$ whose vertical morphisms are parameterized by objects of $\mathcal{M}$ and whose horizontal morphisms are those of $\mathcal{C}$. We will, indeed, go even further and show that for any "dependent actegory" — a pseudomonad in spans of categories whose left leg is a (cloven) cartesian fibration — we can perform the Para construction to get a double category. As an example of this added generality, we see that the double category of spans in a category $\mathcal{C}$ with pullbacks arises as the Para construction applied to the dependent actegory of "dependent sums" in $\mathcal{C}$: the pseudomonad with underlying span $\mathcal{C} \xleftarrow{\partial_1} \mathcal{C}^{\downarrow} \xrightarrow{\partial_0} \mathcal{C}$ whose left leg is a cartesian fibration precisely when $\mathcal{C}$ has pullbacks.

We will show that in this guise, the Para construction arises as a (pseudo)distributive law between the action double category of the actegory and the double category of arrows of $\mathcal{C}$, each seen as (pseudo)monads in a 2-double category of spans. Our construction is abstract and applies in any suitably complete 2-category $\mathbb{K}$, in particular in the 2-category of double categories with vertical transformations. This lets us construct a triple category $\mathbf{Para}(\mathbf{Arena})$ whose morphisms are lenses, charts, and parameterized lenses respectively. The cubes in this triple category give representable behaviors of Capucci-Gavranovic-Hedges-Rischel cybernetic systems [1], and one of the resulting face double categories is a variant of Shapiro and Spivak's double category $\mathbb{O}\mathbf{rg}$ [4].

Our proof is abstract and follows from Gambino and Lobbia's formal theory of pseudomonads in Gray categories [3]. Let $\mathbf{fSpan}(\mathbb{K})$ be the (Gray-categorical strictification of) the tricategory of spans in $\mathbb{K}$ whose left legs are cloven cartesian fibrations. We show that there is a fully faithful Gray-functor $\mathbf{fSpan}(\mathbb{K}) \to \mathbf{Psm}(\mathbf{fSpan}(\mathbb{K}))$ sending $\mathcal{C} \in \mathbb{K}$ to the "double category of squares" $\mathcal{C} \xleftarrow{\partial_0} \mathcal{C}^{\downarrow} \xrightarrow{\partial_1} \mathcal{C}$ considered as a pseudomonad in $\mathbf{fSpan}(\mathbb{K})$. In other words, a monad morphism in the Gray category of pseudomonads of spans in $\mathbb{K}$ between "double categories of squares" is precisely a span whose left leg is a cloven cartesian fibration. Applying the functor $\mathbf{Psm}$ again gives us a Gray functor $\mathbf{Psm}(\mathbf{fSpan}(\mathbb{K})) \to \mathbf{PsmPsm}(\mathbf{fSpan}(\mathbb{K}))$, which sends any dependent actegory in $\mathbb{K}$ to a distributive law over the "double category of squares" of its object of objects.

*Joint work with Matteo Capucci. Abstract submitted to CT2023.

References

[1] Capucci, M; Gavranovic, B; Hedges, J.; Richsel, E. Towards foundations of categorical cybernetics. *Proceedings of Applied Category Theory 2021, Electronic Proceedings in Theoretical Computer Science* (2022), no. 372, 235–248.

[2] Fong, B.; Spivak D.I.; Tuyeras, R. Backprop as functor: a compositional perspective on supervised learning. *Thirty-Fourth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)* 34 (2019), no. 1, 1–33.

[3] Gambino, N.; Lobbia, G. On the formal theory of pseudomonads and pseudodistributive laws. *Theory and Applications of Categories* 37 (2021), no. 2, 14–56.

[4] Shapiro, B.; Spivak D.I. Dynamic categories, dynamic operads: from deep learning to prediction markets. *Proceedings of Applied Category Theory 2022, Electronic Proceedings in Theoretical Computer Science* (forthcoming)

Cartesian Closed Double Categories*

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In this talk, we present two approaches to cartesian closed double categories generalizing equivalent 1-categorical definitions, and give examples to show that the two differ in the double category case. Recall that one can define a cartesian closed 1-category via a pointwise or a 2-variable adjunction, and arrive at equivalent definitions. The pointwise approach for double categories, previously considered in [1], requires the lax functor $(-) \times Y$ on $\mathbb{D}$ to have a right adjoint $(-)^Y$, for every object Y, while the other supposes that the exponentials are given by a bifunctor $\mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathbb{D}$ also involving vertical (i.e., loose) morphisms of $\mathbb{D}$. Examples include the double categories $\mathbb{C}\mathrm{at}, \mathbb{P}\mathrm{os}, \mathbb{T}\mathrm{op}, \mathbb{L}\mathrm{oc}$ and $\mathbb{Q}\mathrm{uant}$, whose objects are small categories, posets, topological spaces, locales, and commutative quantales, respectively; as well as, the double categories $\mathbb{S}\mathrm{pan}(\mathcal{D})$ and Q-$\mathbb{R}\mathrm{el}$, whose vertical morphisms are spans in a category $\mathcal{D}$ with pullback and relations valued in a locale Q, respectively. We are restricting to lax functors, since the right adjoints in many of our examples are not pseudo even when $(-) \times Y$ is.

References

[1] Niefield, S. Exponentiability in double categories and the glueing construction, Theory and Applications of Categories 35 (2020), 1208–1226.

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Codescent and bicolimits of pseudo-algebras *

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This work provides a categorification of two classical results of monad theory. For a monad on a cocomplete category, it is well known that existence of colimits in the category of algebras reduces to existence of coequalizers, since the latter can be used in a first step to construct coproducts of algebras, and secondly, arbitrary colimits as quotients of such coproducts. In the case of a finitary monad (or more generally a monad with rank) on a complete and cocomplete category, such coequalizers can always be forced to exist thanks to a transfinite process, ensuring the cocompletness of categories of algebras of finitary monads.

In 2-category theory, 2-dimensional analogs of monads and their algebras decline in several flavours of strictness, and one may ask what are their corresponding cocompleteness results. It is known from [BKP89] that, in the case of a finitary 2-monad on a 2-complete and 2-cocomplete 2-category, both the 2-categories of *strict algebras* with either *strict morphisms* or *pseudomorphisms* are known to have all bicolimits: this result first construct 2-colimits for strict morphisms, then uses strictification techniques to extract those bicolimits from existing stricter 2-colimits of strict algebras and strict morphisms. However, this cannot be used for pseudo-algebras, since not all pseudoalgebras of a 2-monad can be strictified into strict algebras, even for well-behaved, finitary 2-monads. As a consequence, a bicocompleteness result for pseudoalgebras was still lacking to our knowledge. This work, which sits in the more general context of *pseudomonads*, proposes a direct proof of the bicocompleteness of the 2-category of pseudo-algebras of bifinitary pseudomonads. Our strategy, inspired by the classics of 1-dimensional monad theory as [BW00], reduces existence of arbitrary bicolimits to existence of bicolimits of more specific shape, which can be more directly proven to exist for pseudo-algebras.

In 1-dimension, it is known that cocompleteness of categories of algebras depends on the sole existence of coequalizers, and that those latter exist in the case of a finitary monad thanks to a famous, yet arcane strategy relying on a transfinite induction. In 2-dimension, though several shapes of bicolimits could provide generalizations of coequalizers of parallel pairs, we claim that in our context their correct analogs are *bicoequalizers of codescent objects* in the sense of [LCMV02] (also known as *codescent objects of coherence data*). Those latter encode coherence data akin to those of internal categories and 2-congruences. The role of codescent objects in the theory of 2-monads has been established for some time: many results of monad theory involving instances of reflexives and split coequalizers categorify into pseudocoequalizing statements relative to some codescent objects in the 2-dimensional context. In this talk, we prove that codescent objects are, more generally, useful to generate arbitrary bicolimits. As well as colimits can be constructed from coproducts and coequalizer in 1-category theory, we prove that one can construct weighted bicolimits from oplax bicolimits and bicoequalizers of codescent objects. Our argument relies on the more recent notion of *marked bicolimits* (aka, *σ-bicolimits*), intermediate between bicolimits and oplax bicolimits, which allows to turn weighted bicolimits into conical ones. Then, for a given functor over a marked 2-category, we construct a certain codescent object from its oplax bicolimit, whose higher data encode the maps we are going to invert in the marked bicolimit, which is exhibited as the bicoequalizer of this codescent diagram.

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Then we apply this result in the context of pseudo-algebras of pseudomonads. We prove that, for a pseudomonad on a bicomplete 2-category, one can construct the oplax bicolimit of a diagram of pseudoalgebras as the bicoequalizer of a certain codescent diagram whose data consist of oplax bicolimits of the underlying objects, categorifying a famous result from [Lin69]. As a consequence, the sole existence of bicoequalizers of codescent objects in the 2-category of pseudo-algebras becomes sufficient to ensure existence of oplax bicolimits, and hence, following our previous observation, of all bicolimits.

Finally, we establish existence of bicoequalizers of codescent diagrams in the 2-category of pseudo-algebras for a bifinitary pseudomonad on a bicomplete and bicocomplete category. In a construction close in spirit to [Bor94][Theorem 4.3.6], we construct, for a codescent diagram of pseudo-algebras, a transfinite sequence of codescent diagrams in the underlying category, each step measuring how much the diagram at this step fails to coincide with the bar construction corresponding to a would-be pseudo-algebra structure. However a filteredness argument ensures that this construction stabilises at ω, and the pseudo-algebra thus obtained provides a single-object 2-dimensional solution set for the 2-functor sending a pseudo-algebra to the category of weighted pseudococones over the codescent diagram of pseudoalgebras. A useful 2-dimensional birepresentability theorem from [BG88] then ensures representability of this 2-functor, and hence the existence of a bicoequalizer for the given codescent diagram of pseudo-algebras. This ensures from what precedes that the 2-category of pseudo-algebras of a bifinitary pseudomonad on a bicomplete and bicocomplete 2-category is always bicocomplete.

Our main theorem happens to apply to a wide class of examples. First, to the 2-category **Lex** of small lex categories. We then come to the the powerful paradigm of [GL12]'s *lex colimits*, which formalizes various exactness properties as pseudo-algebras structure for pseudo-monads on **Lex** corresponding to free cocompletion under suited classes of finite weights. Those pseudomonads being proven in [DLO22] to be finitary, they fall under our theorem, proving bicocompleteness of several examples as **Reg**, the 2-category of small regular categories, **Coh** the 2-category of small coherent categories, **Ext** the 2-category of small extensive categories, or $\mathbf{Pretop}_\omega$, the 2-category of small finitary pretopoi.

References

[BG88] Renato Betti and Marco Grandis. Complete theories in 2-categories. *Cahiers de topologie et géométrie différentielle catégoriques*, 29(1):9–57, 1988.

[BKP89] R. Blackwell, G. M. Kelly, and A. J. Power. Two-dimensional monad theory. *Journal of pure and applied algebra*, 59(1):1–41, 1989.

[Bor94] F. Borceux. *Handbook of Categorical Algebra: Volume 2, Categories and Structures*, volume 2. Cambridge University Press, 1994.

[BW00] M. Barr and C. Wells. *Toposes, triples, and theories*. Springer-Verlag, 2000.

[DLO22] I. Di Liberti and A. Osmond. Bi-accessible and bipresentable 2-categories, 2022.

[GL12] R. Garner and S. Lack. Lex colimits. *Journal of Pure and Applied Algebra*, 216(6):1372–1396, 2012.

[LCMV02] I.J. Le Creurer, F. Marmolejo, and E.M. Vitale. Beck's theorem for pseudo-monads. *Journal of Pure and Applied Algebra*, 173(3):293–313, 2002.

[Lin69] F.E.J. Linton. Coequalizers in categories of algebras. In *Seminar on triples and categorical homology theory*, pages 75–90. Springer, 1969.

Weak units in double categories
Abstract submitted to CT2023.

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Weak units are an important notion in higher category theory and it was conjectured by Simpson that they should be enough to obtain models of higher categories (which are otherwise strict) satisfying the homotopy hypothesis. Although this conjecture is still open for general dimension n, we prove a novel link between weak units and the notion of weak globularity. The latter was introduced by Paoli and Pronk for $n = 2$ [4] and by Paoli for $n > 3$ [2] as a new paradigm to weaken higher categorical structures, leading to a model of higher categories, the weakly globular n-fold categories, satisfying the homotopy hypothesis.

We show [3] in the case $n = 2$ how spaces of weak units are precisely encoded by the weak globularity condition in weakly globular double categories. We prove this by establishing a direct comparison between weakly globular double categories and Fair 2-categories. The latter, introduced by J. Kock [1], model weak 2-categories with strictly associative compositions and weak unit laws. This model has some features in common with the simplicial models, but with the simplicial delta replaced by the category 'fat delta'. The proof of this direct comparison is highly non-trivial and involves new results in the combinatorics of the fat delta, which are of independent interest. Further, the techniques used are very general and do not involve the well-known equivalence between bicategories and several models of weak 2-categories. These techniques have good potential for higher dimensional generalizations to tackle Simpson's conjecture for general n.

References

[1] Kock, J. Weak identity arrows in higher categories. *Int. Math. Res. Pap.* 2006, 1–54.

[2] Paoli, S. *Simplicial methods for higher categories: Segal-type models of weak n-categories*, Algebra and Applications 26, Springer, 2019.

[3] Paoli, S. Weakly globular double categories and weak units, preprint (2022), arXiv:2008.11180v3

[4] Paoli, S. ; Pronk, D. A double categorical model of weak 2-categories. *Theory Appl. Categ.* 27 (2013), 933–980.

Diagrammatic presentations of enriched monads and the axiomatics of enriched algebra*

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A theme of recent interest in enriched category theory has been the study of enriched monads, theories, and pretheories defined relative to a *subcategory of arities* $\mathscr{J}$ in a $\mathscr{V}$-category $\mathscr{C}$ [1, 2, 3], and in particular *presentations* of these enriched structures by generalized operations and equations. Work of Kelly, Power, and Lack [4, 5] provides a framework for presentations of enriched α-ary monads on a locally α-presentable $\mathscr{V}$-category $\mathscr{C}$ over a locally α-presentable closed category $\mathscr{V}$ for a regular cardinal α, where the *arities* of the operations are the α-presentable objects of $\mathscr{C}$. Bourke and Garner [2] employ arbitrary small subcategories of arities in locally presentable $\mathscr{V}$-categories in the case where $\mathscr{V}$ is locally presentable, but in this case the arities are still α-presentable for some α. The Kelly-Power-Lack approach to presentations has recently been generalized by the authors [3] to apply to small *eleutheric* subcategories of arities in *locally bounded* $\mathscr{V}$-categories [6] over a *locally bounded* closed category $\mathscr{V}$ [7], thus removing the assumption of local presentability. Neither of the frameworks in [2] and [3] subsumes the other, and one may argue that none of the above frameworks entirely achieves the practical objective of presenting enriched monads directly in terms of individual operations, instead requiring the user to construct a signature internal to $\mathscr{C}$ or a pretheory enriched in $\mathscr{V}$.

In this talk, generalizing previous work of the authors [3, 8], we establish a common extension of the above frameworks for presentations of enriched monads, and on this basis we introduce a flexible formalism for directly describing enriched algebraic structure borne by an object of a $\mathscr{V}$-category $\mathscr{C}$ in terms of what we call *parametrized* $\mathscr{J}$*-ary operations* and *diagrammatic equations*, for a suitable subcategory of arities $\mathscr{J}$ in $\mathscr{C}$. We introduce the notion of *diagrammatic* $\mathscr{J}$*-presentation*, and we show that every such presentation presents a $\mathscr{J}$*-nervous* $\mathscr{V}$-monad on $\mathscr{C}$ whose algebras may be equivalently described as objects of $\mathscr{C}$ equipped with specified parametrized operations, satisfying specified diagrammatic equations. We also show that every $\mathscr{J}$-nervous $\mathscr{V}$-monad admits such a presentation, by showing that the category of such $\mathscr{V}$-monads is monadic over a certain category of $\mathscr{J}$*-signatures*. We work in an axiomatic setting, developed by the authors in [9], which is based primarily on the assumption that free algebras for $\mathscr{J}$-pretheories exist; we say that a subcategory of arities $\mathscr{J}$ that satisfies this assumption is *strongly amenable*. We show that the strong amenability of a subcategory of arities $\mathscr{J}$ is in fact equivalent to the requirement that $\mathscr{J}$ *supports presentations* in an axiomatic sense. We show that our results on presentations of enriched monads are applicable in a wide variety of new contexts in which $\mathscr{V}$ need not be locally presentable, such as in locally bounded closed categories $\mathscr{V}$ and various categories $\mathscr{C}$ enriched over such $\mathscr{V}$. In particular, among locally bounded closed categories $\mathscr{V}$ one finds various convenient categories in topology, analysis, and geometry, and in this context *every* small full sub-$\mathscr{V}$-category $\mathscr{J} \hookrightarrow \mathscr{V}$ that contains the unit object is a strongly amenable subcategory of arities. We also discuss examples of diagrammatic $\mathscr{J}$-presentations in these settings to illustrate their wide applicability and ease of construction.

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References

[1] Lucyshyn-Wright, R.B.B. Enriched algebraic theories and monads for a system of arities. *Theory and Applications of Categories* 31 (2016), no. 5, 101–137.

[2] Bourke, J.; Garner, R. Monads and theories. *Advances in Mathematics* 351 (2019), 1024–1071.

[3] Lucyshyn-Wright, R.B.B.; Parker, J. Presentations and algebraic colimits of enriched monads for a subcategory of arities. *Theory and Applications of Categories* 38 (2022), no. 38, 1434–1484.

[4] Kelly, G.M.; Power, A.J. Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads. *Journal of Pure and Applied Algebra* 89 (1993), no. 1–2, 163–179.

[5] Kelly, G.M.; Lack, S. Finite-product-preserving functors, Kan extensions and strongly-finitary 2-monads. *Applied Categorical Structures* 1 (1993), no. 1, 85–94.

[6] Lucyshyn-Wright, R.B.B.; Parker, J. Locally bounded enriched categories. *Theory and Applications of Categories* 38 (2022), no. 18, 684–736.

[7] Kelly, G.M. Basic concepts of enriched category theory. *Cambridge University Press*, 1982. 245 pp.

[8] Lucyshyn-Wright, R.B.B.; Parker, J. Diagrammatic presentations of enriched monads and varieties for a subcategory of arities. Preprint (2022), arXiv:2207.05184.

[9] Lucyshyn-Wright, R.B.B.; Parker, J. Enriched structure–semantics adjunctions and monad–theory equivalences for subcategories of arities. In preparation (2023).

Effective descent for enriched categories *

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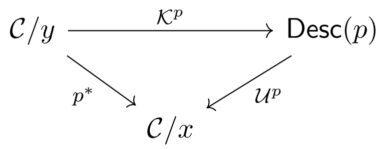
Janelidze-Galois theory [BJ01] and Grothendieck descent theory [JT97] rely on the study of effective descent morphisms, requiring some knowledge of such morphisms in the category of interest, and are the main motivation to undertake the study of finding sufficient conditions for, or even characterising, effective descent morphisms; see [JST03, JT94] for introductions to the subject.

Let $\mathcal{C}$ be a category with pullbacks. For all morphisms $p\colon x \to y$, we consider the change-of-base functors along p:

$$p^*\colon \mathcal{C}/y \to \mathcal{C}/x$$

Thanks to the Bénabou-Roubaud theorem [BR70], the *descent category* for p with respect to the basic bifibration, denoted $\mathsf{Desc}(p)$, is equivalent to the Eilenberg-Moore category for the monad induced by the adjunction $p_! \dashv p^*$. In this setting, this allows us to say that the morphism p is *effective for descent* if the comparison functor $\mathcal{K}^p$ in the Eilenberg-Moore factorisation, given below, is an equivalence.



Despite this simplification, the characterisation of effective descent morphisms in a given category $\mathcal{C}$ is a notoriously difficult problem in general; for instance, see the characterisation in [RT94] and a subsequent reformulation [CH02] for the case $\mathcal{C} = \mathsf{Top}$.

From the perspective of internal structures, we have the work of Le Creurer [Cre99], in which he studies effective descent morphisms for essentially algebraic structures internal to a category $\mathcal{B}$ with finite limits. In particular, the author provides sufficient conditions for a morphism to be effective for descent in $\mathcal{C} = \mathsf{Cat}(\mathcal{B})$.

Based on Le Creurer's results, Lucatelli Nunes, via his study on effective descent for bilimits of categories, provides sufficient conditions for effective descent morphisms in $\mathcal{C} = \mathcal{V}\text{-}\mathsf{Cat}$ via a suitable pseudo-pullback relating enriched and internal $\mathcal{V}$-categories for suitable categories $\mathcal{V}$ (see [Luc18, Lemma 9.10, Theorem 9.11]).

The aim of this talk is to present the central contribution of [Pre23]; namely, we show how we can extend Lucatelli Nunes's result to all categories $\mathcal{V}$ with finite limits. We highlight the use of the following three tools, which are the skeleton of the argument: (1) the properties of familial 2-functors studied in [Web07], in particular, of the endo-2-functor $\mathsf{Fam}\colon \mathsf{CAT} \to \mathsf{CAT}$ which takes every category to its free cocompletion

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under coproducts, (2) results given in [Luc18, Theorem 9.2 and Corollary 9.5] regarding effective descent morphisms for bilimits, and (3) preservation of pseudo-pullbacks via enrichment.

References

[BJ01] Borceux, F.; Janelidze, G. Galois Theories. *Cambridge University Press*, 2001. 341+xiii pp.

[BR70] Bénabou, J.; Roubaud, J. Monades et descente. *Comptes Rendus Hebdomadaires des Séances de L'academie des Sciences* 270 (1970), Série A, 96–98.

[CH02] Clementino, M. M.; Hofmann, D. Triquotient maps via ultrafilter convergence. *Proceedings of the American Mathematical Society* 130 (2002), no. 11, 3423–3431.

[Cre99] Le Creurer, I. Descent of Internal Categories. PhD thesis. *Université Catholique de Louvain* (1999).

[JT94] Janelidze, G.; Tholen, W. Facets of Descent, I. *Applied Categorical Structures* 2 (1994), no. 3, 245–281.

[Luc18] Lucatelli Nunes, F. Pseudo-Kan Extensions and Descent Theory. *Theory and Applications of Categories* 33 (2018), no. 15, 390–448.

[JST03] Janelidze, G.; Sobral, M.; Tholen, W. Beyond Barr Exactness: Effective Descent Morphisms. In *Categorical Foundations: Special Topics in Order, Topology, Algebra and Sheaf Theory* (2003), vol. 57 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, pp. 359–406.

[JT97] Janelidze, G.; Tholen, W. Facets of Descent, II. *Applied Categorical Structures* 5 (1997), no. 3, 229–248.

[Pre23] Prezado, R. On effective descent $\mathcal{V}$-functors. *In preparation.*

[RT94] Reiterman, J.; Tholen, W. Effective descent maps of topological spaces. *Topology and its Applications* 57 (1994), 53–69.

[Web07] Weber, M. Familial 2-functors and parametric right adjoints. *Theory and Applications of Categories* 18 (2007), no. 22, 665–732.

Double Fibrations*

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In this talk I will present the work published in [2]. We introduce a notion of double fibration that forms a common generalization for recent work on monoidal fibrations by Moeller and Vasilakopoulou [4] and Shulman [6], discrete double fibrations by Lambert [3], and various notions of fibrations between 2-categories and bicategories, as in [1] for instance.

Moeller and Vasilakopoulou's monoidal fibrations are pseudo monoids in a 2-category of fibrations and Lambert's discrete double fibrations are category objects in a 2-category of discrete fibrations. Hence, we were led to introduce a double fibration as a particular kind of pseudo category structure in a suitable category of fibrations. We show that we can equivalently view double fibration as a double functors between (pseudo) double categories with certain properties; namely, the ones that make it an internal fibration (as defined in [8]) in a suitable 2-category of double categories.

The papers just mentioned also define the indexed version of monoidal fibrations and discrete double fibrations reespectively. We also generalize this aspect by introducing a generalization of the double category of elements construction (double Grothendieck construction) given by Paré in [7] to obtain a representation theorem establishing a correspondence between double fibrations and Span(Cat)-valued double pseudo-functors as indexing functors, or "indexed double categories" (for a suitable double 2-category Span(Cat)). This directly generalizes the result for discrete double fibrations given by Lambert in [3]. When considering monoidal categories as a special kind of double categories, our representation theorem also induces the equivalence between monoidal fibrations and monoidal indexed categories given in [4, 6]. Finally, the "double Grothendieck constructions" introduced by David Jaz Myers in [5] can be seen as instances of our construction.

Aside from introducing these concepts and the ways they generalize existing concepts, I will also give various examples of double fibrations.

References

[1] Buckley, Mitchell. Fibred 2-Categories and Bicategories. *Journal of Pure and Applied Algebra* 218 (2014), no. 6, 1034–1074.

[2] Cruttwell, Geoff; Lambert, Michael; Pronk, Dorette; Szyld, Martin. Double Fibrations. *Theory and Applications of Categories* 38 (2022), no. 35, 1326–1394.

[3] Lambert, Michael. Discrete Double Fibrations. *Theory and Applications of Categories* 37 (2021), no. 22, 671–708.

[4] Moeller, Joe; Vasilakopoulou, Christina. Monoidal Grothendieck Construction. *Theory and Applications of Categories* 35 (2020), no. 31, 1159–1207.

*Joint work with Geoff Cruttwell, Michael Lambert and Martin Szyld. Abstract submitted to CT2023.

[5] Myers, David Jaz; Double Categories of Open Dynamical Systems. *Electronic Proceedings in Theoretical Computer Science* 333 (2021), 154–167.

[6] Shulman, Michael. Framed Bicategories and Monoidal Fibrations. *Theory and Applications of Categories* 20 (2008), no. 18, 650–738.

[7] Paré, Robert. Yoneda Theory for Double Categories. *Theory and Applications of Categories* 25 (2011), no. 17, 436–489.

[8] Street, R. Fibrations and Yoneda's Lemma in a 2-Category. In: Kelly, G.M. (eds) Category Seminar. *Lecture Notes in Mathematics* 420, Springer, 1974.

On the Categorical Theory of Substitution with Variable Binding*

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Variable binding and substitution are ubiquitous in mathematics and computer science. For instance, in symbolic formalisms for calculus, logic, and computation they respectively occur in the Leibniz product rule, universal instantiation, and reduction of lambda calculus as follows:

$$\begin{aligned} \partial_x \left(f[x] \cdot g[x]\right)\big|_t \;&=\; \partial_x\big(f[x]\big)\big|_t \cdot g[t] \;+\; f[t] \cdot \partial_x\big(g[x]\big)\big|_t \\ \forall\, x.\, P[x] \;&\Longrightarrow\; P[t] \\ \big(\lambda\, x.\, M[x]\big)\, t \;&\overset{\beta}{\longrightarrow}\; M[t] \end{aligned}$$

In these cases, as it occurs generally, one observes an operator $(\partial, \forall, \lambda)$ *binding* a variable (x) in its argument $\big(f[x] \cdot g[x],\, f[x],\, g[x],\, P[x],\, M[x]\big)$ and a term $\big(g[t],\, f[t],\, P[t],\, M[t]\big)$ resulting from the *substitution* of a term (t) for a free variable (x) in a term $\big(g[x],\, f[x],\, P[x],\, M[x]\big)$. Such operators are referred to as variable binding and such substitutions as capture avoiding; the latter to emphasise that the mechanism needs to prevent the inadvertent binding of free variables. Both require particular care when defined formally.

An algebraic categorical theory of variable-binding operators and capture-avoiding substitution was established by Fiore et al [1, 2]. They introduce binding signatures extending the signatures of universal algebra, characterised their syntactic models as free algebras of binding-signature endofunctors over variables, defined capture-avoiding substitution by parameterised structural recursion proving its equational laws, and universally characterised the resulting structures as initial algebraic models with substitution. The aim of this work is to revisit the part of this theory concerned with *partial* (or single-variable) substitution providing a streamlined perspective on it.

We start by generalising the notion of structural recursion with parameters for free algebras to be defined over an arbitrary adjunction, rather than simply the tensor-hom adjunction. We then use it to construct a partial substitution operation on free algebras of binding-signature endofunctors over variables and prove its equational laws. The upshot of the work is a new direct proof of the aforementioned initiality result of Fiore et al [1] for partial substitution structure. This entirely relies on the universal property of free algebras and is thereby suitable for formalisation with computation in proof assistants.

In the talk, we will develop the categorical theory necessary for the above. In particular, we study the properties of the canonical *symmetric monad* on the category of finite cardinals and functions (see Grandis [3]) and the induced *strong* symmetric monad over the associated covariant presheaf category. Furthermore, we consider distributive laws between such monads and binding-signature endofunctors. A central observation in this context is a generalisation of parameterised initiality stating that for endofunctors Σ, Σ', F on the same category, where F is a left adjoint, and for a natural transformation $\psi : F\Sigma \to \Sigma' F$, if $\alpha : \Sigma(A) \to A$ is an initial Σ-algebra then for all Σ'-algebras $\beta : \Sigma'(B) \to B$ there exists

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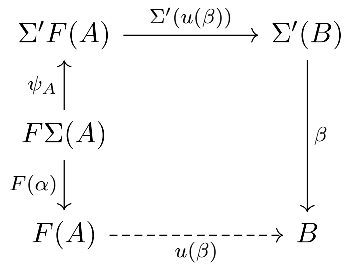
a unique $u(\beta) : F(A) \to B$ making the following diagram commute:



As work in progress, we are considering the analogous development for the linear case, as needed for instance in the theory of symmetric operads and linear lambda calculus, for which only the theory of simultaneous (or multi) substitution has been developed so far (see Tanaka [4]).

References

[1] Fiore, M; Plotkin, G; Turi, D. Abstract syntax and variable binding (extended abstract). *Proceedings of 14th Symposium on Logic in Computer Science* (1999), 193–202.

[2] Fiore, M. Second-order and dependently-sorted abstract syntax. *Proceedings of 23rd Symposium on Logic in Computer Science* (2008), 57–68.

[3] Grandis, M. Finite sets and symmetric simplicial sets. *Theory Appl. Categ.* 8 (2001), 244–252.

[4] Tanaka, M. Abstract syntax and variable binding for linear binders. *Mathematical Foundations of Computer Science 2000: 25th International Symposium* (2000), 670–679.

Limits in (∞, n)-Categories*

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(∞, n)-categories naturally generalize both $(\infty, 1)$-categories and n-categories and have found a variety of applications in mathematical physics, representation theory and derived algebraic geometry. In order to be able to use (∞, n)-categories as an effective framework for such mathematical objects we need a working category theory and in particular a definition of limits and colimits.

If an (∞, n)-category is given as a strictly enriched category, then we could employ the theory of enriched limits to resolve this issue, however, many (∞, n)-categories of interest do not naturally permit such strict description. Fortunately, for the case $n = 1$, a good notion of limits was obtained via the theory of *right fibrations*. Concretely, for a given functor of $(\infty, 1)$-categories we can construct a right fibration of cones and a limit is given by a terminal object in that $(\infty, 1)$-category of cones [4]. This notion is both easy to use with non-strict models, but for strict models coincides with the enriched notion of limits. This suggests a similar fibrational approach to limits for non-strict models of (∞, n)-categories.

It was observed, however, by clingman and Moser that the situation is far more complicated than one might expect. Indeed, in the very concrete case of 2-categories, they observed that any standard notion of 2-category of cones one would like to use is not able to capture the data of a limit [1]. In a follow up work they proposed a solution via double categories, which generalize 2-categories by allowing two kinds of 1-morphisms, horizontal and vertical 1-morphisms, which interact well with each other. Using this idea they define the *double category of cones* for a given functor of 2-categories and prove it can in fact be used to define limits of 2-categories via a terminal object in the double category [2]. Moreover, Grandis used a double categorical Grothendieck construction and double categorical fibrations in order to generalize their construction to weighted limits of 2-categories [3].

Independently, but also motivated by this insight, I developed a theory of fibrations of (∞, n)-categories [7] with the following key features:

1. It applies to a variety of non-strict models, such as n-fold complete Segal spaces.
2. There is a Grothendieck construction giving us a correspondence between fibrations over a given (∞, n)-category and functors valued in the (∞, n)-category of $(\infty, n-1)$-categories.
3. For a given functor of (∞, n)-categories, the domain of the corresponding fibration will be a double $(\infty, n-1)$-category.
4. In the case $n = 1$ the Grothendieck construction coincides with the one established in the literature, such as the unstraightening construction [4].

In particular, in the case $n = 2$, the domain of a fibration will be a double $(\infty, 1)$-categorical fibration, hence appropriately generalizing the double categorical Grothendieck construction.

In this talk I want to discuss ongoing work with Moser and Rovelli that combines the results by clingman and Moser for the case of 2-categories and my construction of fibrations of (∞, n)-categories to study

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limits in (∞, n)-categories. Given the previous explanations we now define the limit of a given functor of (∞, n)-categories as a terminal object in the corresponding double $(\infty, n-1)$-category of cones. This new notion satisfies the following conditions, hence making it a suitable theory of limits (∞, n)-categories:

1. In the case $n = 1$ our cone coincides with the $(\infty, 1)$-categorical cone that exists in the literature, such as the one studied by Lurie [4]. Hence our notion of limit for $(\infty, 1)$-categories coincides with the already established notion.
2. The embedding from double categories to double $(\infty, 1)$-categories defined by Moser [5] respects cones. Hence, our notion of limits for $(\infty, 2)$-categories recovers the work already done by clingman and Moser in the case where the $(\infty, 2)$-category is the nerve of a 2-category.
3. As part of our work we have constructed a homotopy coherent nerve of (∞, n)-categories [6]. This nerve takes (and reflects) enriched limits of strict (∞, n)-categories to fibrational limits of non-strict (∞, n)-categories. This means the existence and computation of a limit of a functor of strict (∞, n)-categories via enriched category theory coincides with the existence and computation of the limits of the non-strict functor given via the nerve using the double $(\infty, n-1)$-category of cones.

References

[1] clingman, t; Moser, L. 2-limits and 2-terminal objects are too different. arXiv preprint, 2020. arXiv:2004.01313.

[2] clingman, t; Moser, L. Bi-initial objects and bi-representations are not so different. arXiv preprint, 2020. arXiv:2009.05545.

[3] Grandis, M. Higher dimensional categories. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020. From double to multiple categories.

[4] Lurie, J. Higher topos theory, volume 170 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009.

[5] Moser, L. A double $(\infty, 1)$-categorical nerve for double categories. arXiv preprint, 2020. arXiv:2007.01848

[6] Moser, L; Rasekh, N.; Rovelli, M. A homotopy coherent nerve for (∞, n)-categories. arXiv preprint, 2022. arXiv:2208.02745

[7] Rasekh, N. Yoneda lemma for $\mathcal{D}$-simplicial spaces. arXiv preprint, 2021. arXiv:2108.06168.

Jónsson categories*

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In this work we generalise Jónsson's theorem, which characterises congruence distributive varieties of universal algebras [6], to the context of regular categories. The linear Mal'tsev condition extracted from the ternary Jónsson terms give rise to matrix conditions [5], denoted by J_n, $n \geqslant 1$. We call a regular category which satisfies the matrix condition J_n, for some $n \geqslant 1$, a *Jónsson category.* We characterise Jónsson categories $\mathbb{C}$ through properties involving equivalence and reflexive relations on a same object in $\mathbb{C}$. These properties on relations then allow us to show that, when $\mathbb{C}$ is an n-permutable category [1], $\mathbb{C}$ satisfies J_m, for some $m \geqslant 1$, if and only if $\mathbb{C}$ is equivalence distributive. It turns out that Jónsson categories $\mathbb{C}$ are such that the *Trapezoid Lemma* [2] holds in $\mathbb{C}$; consequently, every such $\mathbb{C}$ is factor permutable (see [4] and [3]).

References

[1] Carboni, A.; Kelly, G.M.; Pedicchio, M.C. Some remarks on Maltsev and Goursat categories. *Appl. Categ. Structures* 1 (1993), no.4, 385–421.

[2] I. Chajda, G. Czédli, E. Horváth, *Trapezoid Lemma and congruence distributivity*, Math. Slovaca 53 (2003), No. 3, 247–253.

[3] Gran, M. Applications of categorical Galois theory in universal algebra, Galois Theory, Hopf Algebras, and Semiabelian Categories (G. Janelidze, B. Pareigis, and W. Tholen, eds.). *Fields Inst. Commun., Amer. Math. Soc.* 43 (2004), 243–280.

[4] Gumm, H.P. Geometrical methods in congruence modular algebras.*Mem. Amer. Math. Soc.* 45 (1983), no. 286.

[5] Janelidze, Z. Closedness properties of internal relations I: A unified approach to Mal'tsev, unital and subtractive categories. *Theory Appl. Categ.* 16 (2006), 236–261.

[6] Jónsson, B. Algebras whose congruence lattices are distributive. *Mathematica Scandinavica* 21 (1967), 110–121.

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An $(\infty, 2)$-categorical pasting theorem*

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Power's 2-categorical pasting theorem [1], asserting that any pasting diagram in a 2-category has a unique composite, is at the basis of the 2-categorical graphical calculus, which is used extensively to develop the theory of 2-categories. In this talk we discuss an $(\infty, 2)$-categorical analog of the pasting theorem, asserting that the space of composites of any pasting diagram in an $(\infty, 2)$-category is contractible. This result [2], which is joint with Hackney-Ozornova-Riehl, rediscovers independent work by Columbus [3].

References

[1] Power, AJ. An $(\infty, 2)$-categorical pasting theorem. *J. Algebra* 129 (1990), no. 2, 439–445.

[2] Hackney, P; Ozornova, V; Riehl, E; Rovelli, M. An $(\infty, 2)$-categorical pasting theorem. *Trans. Amer. Math. Soc.* 376 (2023), no. 1, 555–597.

[3] Columbus, T. Pasting in simplicial categories. arXiv:2106.15861.

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Composing Dinatural Transformations: Towards a Calculus of Substitution*

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Dinatural transformations, which generalise the ubiquitous natural transformations to the case where the domain and codomain functors are of mixed variance, fail to compose in general; this has been known since they were discovered by Dubuc and Street in 1970 [1]. Many ad hoc solutions to this remarkable shortcoming have been found, but a general theory of compositionality was missing until Petrić, in 2003 [7], introduced the concept of g-dinatural transformations, that is, dinatural transformations together with an appropriate graph: he showed how acyclicity of the composite graph of two arbitrary dinatural transformations is a sufficient and essentially necessary condition for the composite transformation to be in turn dinatural. In this talk I would like to present the results of [6]: first I will give a brief overview of an alternative, semantic rather than syntactic, proof of Petrić's theorem, which we independently rediscovered with no knowledge of its prior existence; I will then show how to use it to define a generalised functor category, whose objects are functors of mixed variance in many variables, and whose morphisms are transformations that happen to be dinatural only in some of their variables.

I shall also define a notion of horizontal composition for dinatural transformations, extending the well-known version for natural transformations, and prove it is associative and unitary. Horizontal composition embodies substitution of functors into transformations and vice-versa, and is intuitively reflected from the string-diagram point of view by substitution of graphs into graphs.

This work represents the first, fundamental steps towards a substitution calculus for dinatural transformations as sought originally by Kelly, with the intention then to apply it to describe coherence problems abstractly; see more details below. There are still fundamental difficulties that are yet to be overcome in order to achieve such a calculus, which I will explain in the talk and will be the subject of future work.

In his seminal articles [4] and [5], Kelly argued that coherence problems are concerned with categories carrying an extra structure: a collection of functors and natural transformations subject to various equational axioms. For example, in a monoidal category $\mathbb{A}$ we have $\otimes\colon \mathbb{A}^2 \to \mathbb{A}$, $I\colon \mathbb{A}^0 \to \mathbb{A}$; if $\mathbb{A}$ is also closed then we would have a functor of mixed variance $(-) \implies (-)\colon \mathbb{A}^{\mathrm{op}} \times \mathbb{A} \to \mathbb{A}$. The natural transformations that are part of the data, like associativity in the monoidal case:

$$\alpha_{A,B,C}\colon (A \otimes B) \otimes C \to A \otimes (B \otimes C),$$

connect not the basic functors directly, but rather functors obtained from them by *iterated substitution*. By "substitution" we mean the process where, given functors

$$K\colon \mathbb{A} \times \mathbb{B}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{D}, \quad F\colon \mathbb{E} \times \mathbb{G} \to \mathbb{A}, \quad G\colon \mathbb{H} \times \mathbb{L}^{\mathrm{op}} \to \mathbb{B}, \quad H\colon \mathbb{M}^{\mathrm{op}} \to \mathbb{C}$$

we obtain the new functor

$$K(F, G^{\mathrm{op}}, H)\colon \mathbb{E} \times \mathbb{G} \times \mathbb{H}^{\mathrm{op}} \times \mathbb{L} \times \mathbb{M}^{\mathrm{op}} \to \mathbb{D}$$

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sending (A, B, C, D, E) to $K(F(A, B), G^{\mathrm{op}}(C, D), H(E))$. Hence substitution generalises composition of functors, to which it reduces if we only consider one-variable functors. In the same way, the equational axioms for the structure, like the pentagonal axiom for monoidal categories, involve natural transformations obtained from the basic ones by "substituting functors into them and them into functors", like $\alpha_{A\otimes B,C,D}$ and $\alpha_{A,B,C} \otimes D$.

By substitution of functors into transformations and transformations into functors we mean therefore a generalised *whiskering* operation or, more broadly, a generalised *horizontal composition* of transformations. For these reasons Kelly argued in [4] that an abstract theory of coherence requires "a tidy calculus of substitution" for functors of many variables and appropriately general kinds of natural transformations, generalising the usual Godement calculus [3, Appendice] for ordinary functors in one variable and ordinary natural transformations. (The "five rules of the functorial calculus" set down by Godement are in fact equivalent to saying that sequential composition of functors and vertical and horizontal composition of natural transformations are associative, unitary and satisfy the usual interchange law; see [8, Introduction] for more details.)

With the notion of "graph of a natural transformation", adapting his earlier work with Eilenberg on extranatural transformations [2], Kelly constructed a full Godement calculus for covariant functors only. When trying to deal with the mixed-variance case, however, he ran into problems. He considered the every-variable-twice extranatural transformations of [2] and, although he got "tantalizingly close", to use his words, to a sensible calculus, he could not find a way to define a category of graphs that can handle cycles in a proper way. This is the reason for the "I" in the title *Many-Variable Functorial Calculus, I* of [4]: he hoped to solve these issues in a future paper, which sadly has never seen the light of day.

What we do in this work is, in fact, consider transformations between mixed-variance functors whose type is even more general than Eilenberg and Kelly's, corresponding to $\underline{\underline{\mathrm{G}}}^*$ in [4], recognising that they are a straightforward generalisation of dinatural transformations in many variables. Our results on vertical and horizontal compositionality, mentioned in the beginning of this abstract, provide the basic ingredients for Kelly's substitution calculus extended to the mixed variance case.

References

[1] Dubuc, E., and Street, R. Dinatural transformations. In *Reports of the Midwest Category Seminar IV* (1970), vol. 137 of *Lecture Notes in Mathematics*, 126–137.

[2] Eilenberg, S., and Kelly, G. M. A Generalization of the Functorial Calculus. *Journal of Algebra 3*, 3 (May 1966), 366–375.

[3] Godement, R. *Topologie algébrique et théorie des faisceaux*. No. 13 in Publications de l'Institut de mathématique de l'Université de Strasbourg. Hermann, Paris, 1958. OCLC: 1216318.

[4] Kelly, G. M. Many-variable functorial calculus. I. In *Coherence in Categories* (1972), vol. 281 of *Lecture Notes in Mathematics*, 66–105.

[5] Kelly, G. M. An abstract approach to coherence. In *Coherence in Categories* (1972), vol. 281 of *Lecture Notes in Mathematics*, 106–147.

[6] McCusker, G., and Santamaria, A. Composing dinatural transformations: Towards a calculus of substitution. *Journal of Pure and Applied Algebra 225*, 10 (Oct. 2021), 106689.

[7] Petrić, Z. G-dinaturality. *Annals of Pure and Applied Logic 122*, 1 (Aug. 2003), 131–173.

[8] Santamaria, A. *Towards a Godement Calculus for Dinatural Transformations*. PhD thesis, University of Bath, Bath, July 2019.

Posetal closed Grothendieck construction*

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Let C be an arbitrary category and let Pos be the category of posets and order preserving functions. We study functors $Q : \mathsf{C} \longrightarrow \mathsf{Pos}$ and their total/Grothendieck categories $\int Q$. The aim is to understand when and how structure from C lifts to $\int Q$. For a functor $F : (\mathsf{C}^{op})^n \times \mathsf{C}^m \longrightarrow \mathsf{C}$, a *lifting* of F to $\int Q$ is a functor $\overline{F} : (\int Q^{op})^n \times \int Q^m \longrightarrow \int Q$ such that the following diagram commutes strictly:

$$\begin{array}{ccc} (\int Q^{op})^n \times \int Q^m & \xrightarrow{\overline{F}} & \int Q \\ \big\downarrow{\scriptstyle (\pi^{op})^n \times \pi^m} & & \big\downarrow{\scriptstyle \pi} \\ (\mathsf{C}^{op})^n \times \mathsf{C}^m & \xrightarrow{F} & \mathsf{C} \end{array}$$

Our staring point is the following observation:

Proposition 1. *Liftings of F from C to $\int Q$ bijectively correspond to lax extranatural transformations $\psi : \prod \circ Q^{n+m} \longrightarrow Q \circ F$.*

Here, for a *lax extranatural transformation* $\psi : \prod \circ Q^{n+m} \longrightarrow Q \circ F$, we mean a collection of order-preserving maps

$$\psi_{X,Y} : \prod_i Q(X_i)^{op} \times \prod_j Q(Y_j) \longrightarrow Q(F(X,Y))$$

indexed by objects (X, Y) of $\mathsf{C}^n \times \mathsf{C}^m$ such that, for each pair of maps $f : X \longrightarrow X'$ in C^n and $g : Y \longrightarrow Y'$ in C^m, the following diagram half-commutes:

$$\begin{array}{ccccc} & & \prod_i Q(X_i)^{op} \times \prod_j Q(Y_j) & & \\ & \swarrow{\scriptstyle \prod_i Q(f_i)^{op} \times \mathtt{id}} & & \searrow{\scriptstyle \mathtt{id} \times \prod_i Q(g_j)} & \\ \prod_i Q(X_i')^{op} \times \prod_j Q(Y_j) & & \leq & & \prod_i Q(X_i)^{op} \times \prod_j Q(Y_j') \\ \big\downarrow{\scriptstyle \psi_{X',Y}} & & & & \big\downarrow{\scriptstyle \psi_{X,Y'}} \\ Q(F(X',Y)) & & \xrightarrow{Q(F(f,g))} & & Q(F(X,Y')) \end{array} \tag{1}$$

It turns out that a lax extranatural transformation as above is natural (that is, the diagram (1) fully commutes) if and only if the lifting it gives rise preserves op-cartesian arrows.

When lifting a functor, we might also wish to lift some arrows. Lifting an arrows amounts to enforcing an inclusions, and lifting an isomorphism amounts to enforcing an identity. For example, in order to lift an associative tensor, we need to have a lax natural transformation $\mu_{X,Y} : Q(X) \times Q(Y) \longrightarrow Q(X \otimes Y)$

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such that the associator α lifts to $\int Q$. The latter requirement amounts to the inclusion

$$Q(\alpha_{X,Y,Z})(\mu_{X\otimes Y}(\mu_{X,Y}(x,y),z)) \leq \mu_{X,Y\otimes Z}(x,\mu_{Y,Z}(y,z))\,.$$

Considering that $\alpha_{X,Y,Z}$ needs to be invertible in $\int Q$, the above inclusion is further required to be an identity. We end up reconstructing one of the coherence conditions for Q to be monoidal, see e.g. [1].

These remarks are used to study when $\int Q$, more than merely lifting the symmetric monoidal structure by means $\mu_{X,Y}$, also lifts the closed structure by means of some lax extranatural transformation $\iota_{X,Y}$: $Q(X)^{op} \times Q(X) \longrightarrow Q(X \multimap Y)$. We prove the following statement:

Theorem 2. *If $\int Q$ lifts the symmetric monoidal closed structure of C, then, for each pair of objects X, Y if C and each $\alpha \in Q(X)$, the map*

$$Q(\mathsf{ev}_{X,Y})(\mu_{X,X\multimap Y}(\alpha,-)) : Q(X \multimap Y) \longrightarrow Q(X \otimes X \multimap Y) \longrightarrow Q(Y)$$

has a right adjoint. If the tensor in $\int Q$ preserves opcartesian arrows, then the converse holds.

Consequently we have:

Theorem 3. *If a monoidal functor Q factors (as a monoidal functor) through the forgetful functor U : $\mathsf{SLatt} \longrightarrow \mathsf{Pos}$ (where SLatt is the category of complete lattices ad sup-preserving maps), then the monoidal structure of $\int Q$ is closed and lifts the one of C.*

We use the above theorems to study concrete categories arising as $\int Q$ for some Q. These comprise the categories Q_F-$\mathsf{Set} = \int Q^{F(X)}$, see [2], arising from a quantale Q, a monoidal functors $\mathsf{Rel} \longrightarrow \mathsf{SLatt}$ sending X to Q^X, and a monoidal $F : \mathsf{Rel} \longrightarrow \mathsf{Rel}$, and $\mathsf{Nuts} = \int UP$, see [3], arising from the monoidal functor sending X to the set of upsets of the powerset of X.

The same methodology easily allows to abstractly characterise nuclear and dualizing objects of $\int Q$, and then to completely describe them in Q_F-Set and Nuts.

Using this approach, we shall further study liftings of other kind of structures, such as (co)monoids, (initial) algebras and (final) coalgebras of lifted functors.

References

[1] Moeller, J.; Vasilakopoulou, C. Monoidal Grothendieck construction. *Theory and Applications of Categories* 35 (2020), 1159–1207.

[2] Schalk, A.; De Paiva, V. Poset-valued sets or how to build models for linear logics. *Theoretical Computer Science* 315 (2004), 83-107.

[3] Ehrhard T.; Jafar-Rahmani. Categorical models of Linear Logic with fixed points of formulas. 36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021. *IEEE*, 2021. 13 pp.

A category of elements for enriched functors*

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The discrete Grothendieck construction

$$\int_{\mathcal{C}}\colon [\mathcal{C}^{\mathrm{op}}, \mathsf{Set}] \to \mathsf{Cat} \downarrow \mathcal{C},$$

most often called the *category of elements*, gives an equivalence between the categories of functors from $\mathcal{C}^{\mathrm{op}}$ to Set and of discrete fibrations over $\mathcal{C}$. It is intimately linked with the study of representable functors, as a well-known result shows that a functor F is representable if and only if its category of elements $\int_{\mathcal{C}} F$ has a terminal object. Hence, the category of elements gives us a way to characterize representable functors, and through them, universal properties, which are then used to understand key constructions such as adjunctions and (co)limits.

There exist several efforts to extend the (non-discrete) Grothendieck construction to other settings (e.g. [6, 8, 10, 11, 12, 13]), and in particular, to the setting of enriched categories. For $\mathcal{V} = \mathsf{Cat}$, [1, 3, 9] give an enriched Grothendieck construction and a correspondence with the appropriate notion of fibrations. For the general case of $\mathcal{V}$-categories, [2] defines a Grothendieck construction

$$\int_{\mathcal{C}}\colon [\mathcal{C}^{\mathrm{op}}, \mathcal{V}\text{-}\mathsf{Cat}] \to \mathcal{V}\text{-}\mathsf{Cat} \downarrow \mathcal{C}_{\mathcal{V}}$$

that takes a pseudofunctor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{V}\text{-}\mathsf{Cat}$, and produces a $\mathcal{V}$-category $\int_{\mathcal{C}} F$ with a projection to $\mathcal{C}_{\mathcal{V}}$, the free $\mathcal{V}$-category on $\mathcal{C}$. A similar construction in [14] deals with lax functors.

Unfortunately, neither of these enriched Grothendieck constructions have *enriched* functors as their input, since the category $\mathcal{V}\text{-}\mathsf{Cat}$ is generally not $\mathcal{V}$-enriched, even when $\mathcal{V}$ is cartesian closed. However, in this situation the category $\mathcal{V}$ is $\mathcal{V}$-enriched, and it makes sense to consider $\mathcal{V}$-functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$, which is the analogue of the discrete Grothendieck construction in the case $\mathcal{V} = \mathsf{Set}$.

In this talk, we will describe how the use of categories internal to $\mathcal{V}$ allows us to construct an enriched category of elements

$$\int_{\mathcal{C}}\colon \mathcal{V}\text{-}\mathsf{Cat}(\mathcal{C}^{\mathrm{op}}, \mathcal{V}) \to \mathsf{Cat}(\mathcal{V}) \downarrow \mathrm{Int}(\mathcal{C})$$

where $\mathrm{Int}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathsf{Cat}(\mathcal{V})$ is the internalization functor of [5]. This gives a characterization of the $\mathcal{V}$-representable functors as the ones whose category of elements has a terminal object, and provides a way to "flatten" the computation of weighted limits by instead considering internal limits from the category of elements of the weights. These applications respectively extend work of clingman–Moser [4] and of Grandis–Pare [7] for the case $\mathcal{V} = \mathsf{Cat}$.

References

[1] Bakovic, I. Grothendieck construction for bicategories. *http://www. irb. hr/korisnici/ibakovic/sgc. pdf* (2009).

*Joint work with Lyne Moser and Paula Verdugo. Abstract submitted to CT2023.

[2] Beardsley, J; Wong, L. Z. The enriched Grothendieck construction. *Advances in Mathematics* 344 (2019), 234–261.

[3] Buckley, M. Fibred 2-categories and bicategories. *Journal of Pure and Applied Algebra* 218 (2014), no. 6, 1034–1074.

[4] clingman, t; Moser, L. 2-limits and 2-terminal objects are too different. *Applied Categorical Structures* (2022).

[5] Cottrell, T; Fujii, S; Power, J. Enriched and internal categories: an extensive relationship. *Tbilisi Mathematical Journal* 10 (2017), no. 3, 239–254.

[6] Gepner, D; Haugseng, R; Nikolaus, T. Lax colimits and free fibrations in ∞-categories. *Documenta Mathematica* 22 (2017), 1225–1266.

[7] Grandis, M; Pare, R. Persistent double limits and flexible weighted limits. *https://www.mscs.dal.ca/ pare/DblPrs2.pdf* (2019).

[8] Harpaz, Y; Prasma, M. The Grothendieck construction for model categories. *Advances in Mathematics* 281 (2015), 1306–1363.

[9] Hermida, C. Some properties of **Fib** as a fibred 2-category. *Journal of Pure and Applied Algebra* 134 (1999), no. 1, 83–109.

[10] Lurie, J. Higher topos theory. *Princeton University Press, Princeton, NJ*, 2009. xviii+925pp.

[11] Moeller, J; Vasilakopoulou, C. Monoidal Grothendieck construction. *Theory and Applications of Categories* 35 (2020), No. 31, 1159–1207.

[12] Ramzi, M. A monoidal Grothendieck construction for ∞-categories. *arXiv:2209.12569* (2022).

[13] Rasekh, N. Yoneda Lemma for $\mathcal{D}$-Simplicial Spaces. *arXiv:2108.06168* (2021).

[14] Tamaki, D. The Grothendieck Construction and Gradings for Enriched Categories. *https://arxiv.org/abs/0907.0061* (2009).

Bicategorifying actions, strengths, and Freyd categories*

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Kock's notion of *strength* [1], and strong monads in particular, play a central role in modelling programming languages which interact with the world in the form of *effects* such as printing to screen, exceptions, or probability (*e.g.* [2, 3]). Recently, however, several semantic models have been proposed which are not categories but *bicategories* (*e.g.* [4, 5, 6]). Extending the classical theory of computational effects to these new models therefore requires a bicategorical treatment of strong monads and their associated structures.

The talk will be in two parts. In the first part of the talk I will introduce strong pseudofunctors and strong pseudomonads, and explain how one can be confident that these notions are "correct". In the second part, I will introduce bicategorical versions of closely-related structures arising in theoretical computer science, namely *premonoidal categories* and *Freyd categories*.

Part I: Strong pseudomonads and actions of monoidal bicategories. Working from an observation of Janelidze & Kelly [7], I will define actions of monoidal bicategories as certain degenerate tricategories, and show that the well-known 1-categorical correspondence between actions and monoidal functors lifts to a correspondence between $\mathcal{V}$-actions on a bicategory $\mathcal{B}$ and monoidal pseudofunctors $\mathcal{V} \to \mathrm{Hom}(\mathcal{B}, \mathcal{B})$.

I will then use actions to validate our definition of strong pseudomonads. Indeed, the starting point for our definition was that one should obtain a version of the classical correspondence between strengths on a monad T on a monoidal category $(\mathbb{C}, \otimes, I)$ and actions $\mathbb{C} \times \mathbb{C}_T \to \mathbb{C}_T$ on the Kleisli category which extend the canonical action of $\mathbb{C}$ on itself (see *e.g.* [8]). Perhaps remarkably, this starting point allows one to recover a significant amount of the expected theory; I will sketch some aspects, including connections to enrichment (à la Kock [1]) and a coherence result.

Part II: Premonoidal structure and Freyd bicategories. Premonoidal categories ([9, 10, 11]) axiomatise the structure of the Kleisli category of a strong monad. One has a tensor and unit similarly to a monoidal category, except the tensor is only assumed to be functorial in each argument separately. Every premonoidal category then comes with a *centre*, namely the wide sub-category of maps for which the tensor is functorial in two arguments. Central maps represent 'pure' programs, which do nothing but return values. A strict premonoidal category is exactly a monoid in **Cat** with the funny tensor product (equivalently, a one-object sesquicategory [12, 9]), but for a general premonoidal category one must ask for central isomorphisms playing the roles of associator and unitors.

Premonoidal categories are a useful abstraction, but when modelling programs it is often convenient to be able to specify a chosen category of central maps. This is achieved by a Freyd category ([13, 14]), which consists of a premonoidal category $\mathbb{C}$ (modelling programs that may interact with the world), a monoidal category $\mathbb{V}$ (modelling pure programs) and a functor $J : \mathbb{V} \to C$ strictly preserving premonoidal structure. The canonical example is the left adjoint $J : \mathbb{C} \to \mathbb{C}_T$ for a strong monad T.

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I will give bicategorical treatments of both premonoidal and Freyd structure. We will then see that the Kleisli bicategory of a strong pseudomonad has a canonical premonoidal structure, and that the canonical pseudofunctor $\mathcal{B} \to \mathcal{B}_T$ (for a strong pseudomonad T) defines a Freyd bicategory. There are some unexpected challenges: for instance, we believe that the centre of a premonoidal bicategory is not in general monoidal. If time permits, we will conjecture relationships to other forms of semi-strict higher-dimensional structure (*e.g.* [15, 16, 17]).

References

[1] Kock, A. Strong functors and monoidal monads. *Archiv der Mathematik* 23 (1972), no. 1, 113–120.

[2] Moggi, E. Computational lambda-calculus and monads. *Proceedings of the 4th Annual Symposium on Logic in Computer Science (LICS)* (1989).

[3] Moggi, E. Notions of computation and monads. *Information and Computation* 93 (1991), no. 1, 55–92.

[4] Melliès, P-A. Asynchronous Template Games and the Gray Tensor Product of 2-Categories. *Proceedings of the 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)* (2021).

[5] Fiore, M; Galal, Z; Paquet, H. A Combinatorial Approach to Higher-Order Structure for Polynomial Functors. *Proceedings of the 7th International Conference on Formal Structures for Computation and Deduction (FSCD)* (2022).

[6] Galal, Z. A Profunctorial Scott Semantics *Proceedings of the 5th International Conference on Formal Structures for Computation and Deduction (FSCD)* (2020).

[7] Janelidze, G; Kelly, G. M. A note on actions of a monoidal category. *Theory and Applications of Categories* 9 (2001), no. 4, 61–91.

[8] McDermott, D.; Uustalu, T. What makes a strong monad? *Electronic Proceedings in Theoretical Computer Science* 360 (2022), 113–133.

[9] Power, J; Robinson, E. Premonoidal categories and notions of computation. *Mathematical Structures in Computer Science* 7 (1997), no. 5, 453–468.

[10] Power, J. Premonoidal categories as categories with algebraic structure. *Theoretical Computer Science* 278 (2002), no. 1–2, 303–321.

[11] Levy, P B; Power, J; Thielecke, H. Modelling environments in call-by-value programming languages. *Information and Computation* 185 (2003), no. 2, 182–210.

[12] Street, R. Categorical Structures. Chapter 15 in Handbook of Algebra. *Elsevier*, 1995.

[13] Power, J; Thielecke, H. Environments, continuation semantics and indexed categories. *Lecture Notes in Computer Science* 1281 (1997), 391–414.

[14] Power, J; Thielecke, H. Closed Freyd- and κ-categories *Automata, Languages and Programming* (1999), 625–634.

[15] Weber, M. Free Products of Higher Operad Algebras. *arxiv* (2009).

[16] Forest S; Mimram, S. Free precategories as presheaf categories. *arxiv* (2022).

[17] Araújo, M. Computads and string diagrams for n-sesquicategories. *arxiv* (2022).

Relative Pseudomonads: Pseudocommutativity and Lax Idempotency Abstract submitted to CT2023.

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The presheaf pseudofunctor Psh sending a small category to its locally-small category of presheaves cannot be given the structure of an ordinary pseudomonad, due to size issues (Psh is not an endofunctor). This is in a particular sense the only obstacle; we can endow Psh instead with the structure of a *relative pseudomonad* along the inclusion Cat $\hookrightarrow$ CAT. Since Cat and CAT are particularly nice 2-categories (in particular, they are monoidal), we would like to extend notions like 'strong monad' and 'commutative monad' to this relative setting.

In this talk, I will extend the classical work of [1] Kock (1970) on strength and commutativity to define parameterised relative pseudomonads and pseudocommutative relative pseudomonads, of which Psh turns out to be both. To avoid a substantial amount of coherence tracking, I will work in the more general setting of 2-multicategories instead of monoidal 2-categories. I will prove analogous implications to the classical work: that a parameterised relative pseudomonad is a multilinear pseudofunctor, and that a pseudocommutative relative pseudomonad is a multilinear pseudomonad. I will close by extending the work of [2] López Franco (2011) with a proof that a strongly lax-idempotent relative pseudomonad (such as Psh itself) is pseudocommutative.

References

[1] Kock, A. Monads on Symmetric Monoidal Closed Categories. *Archiv der Mathematik* 21 (1970), 1–10

[2] López Franco, I. Pseudo-commutativity of KZ 2-monads. *Advances in Mathematics* 228 (2011), 2557–2605

Lax epimorphisms in CAT, $\mathcal{V}$-Cat and everywhere*

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Lax epimorphisms (also known as co-fully-faithful morphisms) in a 2-category A are the 1-cells e making $\mathsf{A}(e, c)$ fully faithful for all objects c. In this talk several features of lax epimorphisms will be presented.

We show that Cat (and CAT) has an orthogonal $(\mathcal{E}, \mathcal{M})$-factorization system where $\mathcal{E}$ is the class of lax epimorphisms, and we give two different descriptions of this factorization, one of them somehow imitates the construction of the comprehensive factorization system of Street and Walters [1].

Moreover, any 2-category has an orthogonal LaxEpi-factorization system under the existence of appropriate colimits.

We also give several characterizations of lax epimorphisms in the 2-category $\mathcal{V}$-Cat.

Part of this work is published in [2].

References

[1] Street, R.; Walters, R.F.C. A comprehensive factorization of a functor. *Bull. Amer. Math. Soc.* 79 (1973), 936-941.

[2] Lucatelli Nunes, F.; Sousa, L. On lax epimorphisms and the associated factorization. *Journal of Pure and Applied Algebra* 226 (2022), 27 pp.

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A logical analysis of Banach's fixpoint theorem*

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Banach's fixpoint theorem [1] from 1922 says that every contraction on a non-empty complete metric space admits a unique fixpoint. The gist of the proof is wonderfully simple: take any element x of the space (X,d) and, iterating the contraction $f\colon X \to X$, prove that the sequence $(f^n x)_{n\in\mathbb{N}}$ is Cauchy. In the complete space (X,d) this sequence converges, and one then shows that it does so to a (necessarily unique) fixpoint of f. Many generalizations and applications of Banach's theorem have been, and are still, studied.

In 1972, Lawvere [6] famously showed that metric spaces are a particular instance of enriched categories. More impressively still, Lawvere also showed how convergence of Cauchy sequences can adequately be understood via representability of left adjoint distributors, thus lifting the very concept of Cauchy completeness to the level of enriched categories. In his words, "specializing the constructions and theorems of general category theory we can deduce a large part of general metric space theory."

It is thus natural to investigate whether fixpoint theorems still make sense in the vast context of enriched categories. This is precisely the subject of this talk (and of our paper [2]).

More precisely, we shall take quantale-enriched categories as generalization of metric spaces. That is to say, we fix a quantale (= a posetal cocomplete monoidal closed category) Q, and work with categories, functors and distributors enriched in Q. Our contribution shows that fixpoint theorems for Q-categories depend on the interplay between three essential parameters. Indeed, a given contraction must be "strong enough" (we shall measure its strength by means of a control function); the space on which it acts must be "complete enough" for the Picard iteration to converge to a fixpoint (we shall take this to be Cauchy-completeness in the sense of Lawvere); but we also need sufficiently strong algebraic properties of the underlying quantale Q to allow for the formulation of precisely that convergence.

In concreto, we shall prove a fixpoint theorem for Cauchy-complete Q-categories that holds whenever the quantale Q has an underlying continuous lattice and the contraction is controlled by a sequentially lower-semicontinuous function on Q. Besides, we make plain when and why such a fixpoint is unique (up to isomorphism). As examples we find the classical Banach fixpoint theorem for metric spaces, and Boyd and Wong's [3] generalization thereof (taking the underlying quantale to be the positive real numbers); but we also formulate new results for fuzzy ordered sets (when working over a left-continuous t-norm [5]) and for probabilistic metric spaces (now the quantale is the tensor product of the positive reals with a left-continuous t-norm [4]).

References

[1] Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae* 3 (1922), 133–181.

*Joint work with Arij Benkhadra. Abstract submitted to CT2023.

[2] Benkhadra, A.; Stubbe, I. A logical analysis of fixpoint theorems. *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 64 (2023), 97–121.

[3] Boyd, D. W.; Wong, J. S. W. On nonlinear contractions. *Proceedings of the American Mathematical Society* 20 (1969), 458–464.

[4] Hofmann D.; Reis, C. Probabilistic metric spaces as enriched categories. *Fuzzy Sets and Systems* 210 (2013), 1–21.

[5] Klement, E.P.; Mesiar, R.; Pap, E. Triangular norms. *Springer Dordrecht*, 2000. xix+387.

[6] Lawvere, F. W. Metric spaces, generalized logic and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano* XLIII (1973), 135–166.

Factorization systems as double categories*

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We show that factorization systems, both strict and orthogonal, can equivalently be described as double categories satisfying certain properties. Specifically, every orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ gives rise to a certain double category $D_{\mathcal{E},\mathcal{M}}$ of commutative squares, every double category X with certain properties gives rise to an orthogonal factorization system $(\mathcal{E}_X, \mathcal{M}_X)$, and these processes are mutually inverse. This gives conceptual reasons for why categories like $\mathrm{Par}(\mathcal{C})$ (of objects and partial maps in $\mathcal{C}$) or $\mathrm{Cof}(\mathcal{E})$ (of categories and cofunctors internal to $\mathcal{E}$) admit orthogonal factorization systems.

We also demonstrate that a similar equivalence holds between strict factorization systems and certain double categories. As a consequence of the theory, we give explicit descriptions of a lax functor and a lax monoidal functor classifiers and explain why they admit strict factorization systems.

All of the above is done utilizing the *category of corners* construction associated to a *crossed double category*, concepts introduced by Mark Weber in [1] for the study of internal algebra classifiers.

References

[1] Weber, Mark. "Internal algebra classifiers as codescent objects of crossed internal categories." Theory & Applications of Categories 30.50 (2015).

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A novel fixed point theorem, towards a replacement for replacement

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Sadly often, we see "successor" and "limit" constructions recited as the construction of a fixed point, as if this were a proof. Missing are citations to John von Neumann (1928) for deriving recursion from induction and to Friedrich Hartogs (1917) for providing a suitable ordinal. Even then, we merely have two ordinals with the same value, so some other argument is needed to deduce a fixed point.

Casimir (Kasimierz) Kuratowski (1922) explained how many theorems in set theory, topology and measure theory that had been proved using transfinite recursion could be replaced with closure conditions.

Bourbaki (1949) and Ernst Witt (1951) showed how the problem provides its own ordinal, although Ernst Zermelo (1908) already had their proof and it was re-discovered repeatedly during the 20th century. It should have been fundamental to the curriculum, but only as an afterthought in Serge Lang's *Algebra* (1965) does it appear in a textbook. Wikipedia wrongly states that it is proved by transfinite recursion.

The huge clunking transfinite machine depends on excluded middle at every step. In the 1990s two intuitionistic approaches were developed, by André Joyal and Ieke Moerdijk (*Algebraic Set Theory*, 1994) and by me (1996). The key messages were that there are many kinds of ordinals, with distinct universal properties (forms of transfinite recursion) and the relations $\in$ and $\subset$ should be treated independently.

However, Hartogs' Lemma cannot be recovered, so there was no proof of the fixed point.

Then along came Dito Pataraia (1996), who threw off all of this set-theoretic baggage and used functions instead, to give an intuitionistic proof. It is breathtakingly simple and, unlike Bourbaki–Witt, could easily be reconstructed by a student in an exam. The key observation, to which domain theorists like me were somehow blind, is that composition makes the poset of inflationary monotone endofunctions *directed*, whilst also being directed *complete*.

In 2019, I returned to my 1990s work on well founded coalgebras, in order to meet the challenge of functors that preserve monos and not inverse images. I knew that I had to use Pataraia's theorem, but I tied myself in knots trying to deduce the result that I needed as a *corollary* of Pataraia: it was much more natural to do it the other way round.

So my version is this: Let $s : X \to X$ be an endofunction of a poset such that

- X has a least element $\bot$;
- X has joins ($\bigvee^{\uparrow}$) of directed subsets;
- s is monotone: $\forall xy.\ x \le y \Rightarrow sx \le sy$;
- s is inflationary: $\forall x.\ x \le sx$;
- $\forall xy.\ x = sx \le y = sy \Rightarrow x = y$ (the *Special Condition*).

Then

- X has a greatest element $\top$;
- $\top$ is the unique fixed point of s;

- if $\bot$ satisfies some predicate and it is preserved by s and directed joins then it holds for $\top$.

When I asked on MathOverflow whether anyone had seen my *Special Condition*, I was told that it was unnatural and given lectures on ordinal recursion.

How is the Special Condition to be achieved, just given $s : Y \to Y$ satisfying the other conditions? The Zermelo–Bourbaki–Witt theorem uses the subset $X \subset Y$ *generated* by $\bot$, s and $\bigvee^{\uparrow}$. But this already invokes second order logic or recursion as a preliminary to what should be the key tool for recursion.

Much more simply, if we take X to be

$$\{x : Y \mid x \le sx \ \wedge \ \forall a : Y.\ sa \le a \Rightarrow x \le a\}$$

or, if Y also has meets,

$$\{x : Y \mid x \le sx \ \wedge \ \forall u : Y.\ su \wedge x \le u \Rightarrow x \le u\},$$

then the Special Condition holds. These subsets are defined using the poset versions of the categorical notions of recursive and well founded coalgebras, so if x satisfies the second we call it a *well founded element* of Y with respect to the operation s.

As categorists we know that we have a small toolbox of very powerful tools: whenever we apply a general tool (unpack its definition) in some mathematical setting, this often turns out to be an important concept there.

The same seems to be true of well founded *elements*: in the natural structures, both well founded *relations* and well founded *coalgebras* are examples. Then most of the work of von Neumann's recursion theorem has been done in these settings.

In pure category theory, constructing coequalisers of algebras is perceived to be difficult and require transfinite recursion, but this is a simple direct application of the special condition itself.

This was meant to be a Lemma in a much larger programme. It warrants its own publicity because it is simple tool that any mathematician can take away and apply to their own subject, but also because of the regressive propaganda for transfinite methods that has been going on for over a century.

The larger programme, which I will not have time to discuss, generalises recursion and the "Mostowski" extensional quotient for well founded coalgebras well beyond their 1990s setting by replacing monos with factorisation systems. Applying this to posets and lower subsets instead of sets and subsets will give a much clearer explanation of the *plump* ordinals.

Based on that, we have a characterisation of transfinite iteration of functors that is another example of the generalised notion of extensionality.

It is a *characterisation* and not a *construction* because it cannot be done within the logic of an elementary topos. The set theorists will come in and tell us that we must use the Axiom-Scheme of Replacement in ZF to do it.

However, I am not prepared to allow them to dictate to us how to do our subject. I acknowledge that there are mathematical constructions that require transfinite iteration of functors, but this can be stated *within the native language of category theory.*

As Bill Lawvere taught us, it uses *Adjointness in Foundations.*

The numerous bibliographical details above, my past and present papers on this subject and slides for some recent seminars may be found on my website at

`www.paultaylor.eu/ordinals/`

Flatness, weakly-lex colimits, and free exact completions*

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Free regular and Barr-exact completions of lex categories (that is, categories with finite limits) have always played a key role in the theory of regularity and exactness. Given a lex category $\mathcal{C}$, its free *Barr-exact completion* is the data of a Barr-exact category $\mathcal{C}_{\text{ex}}$ together with an embedding $K\colon \mathcal{C} \hookrightarrow \mathcal{C}_{\text{ex}}$ for which precomposition with K induces an equivalence

$$\text{Reg}(\mathcal{C}_{\text{ex}}, \mathcal{E}) \simeq \text{Lex}(\mathcal{C}, \mathcal{E})$$

for any Barr-exact category $\mathcal{E}$. It was soon realized that such free completions could exist even when the category $\mathcal{C}$ is not finitely complete. Indeed, they where considered for any *weakly lex* category $\mathcal{C}$ (that is, a category with weak finite limits) separately by Carboni–Vitale and Hu.

In this context, lex functors in the universal property are replaced by *left covering functors*. When $\mathcal{C}$ is actually lex, $F\colon \mathcal{C} \to \mathcal{E}$ is left covering if and only if it is lex; while, if $\mathcal{C}$ is just weakly lex but $\mathcal{E} = \mathbf{Set}$, then F is left covering if and only if it is *flat* (equivalently, if its category of elements is filtered). For general $\mathcal{C}$ and $\mathcal{E}$ the explicit definition relies on some comparison map in $\mathcal{E}$ being a regular epimorphism, making left covering functors the suitable choice only for free regular and Barr-exact completions. It is in fact unclear what concept should be used instead when considering, for instance, free lextensive or pretopos completions of non-lex categories.

The main aim of the talk is to address this problem with the introduction of a notion of *flatness* for functors $F\colon \mathcal{C} \to \mathcal{E}$, from any category $\mathcal{C}$ into any lex category $\mathcal{E}$, which will specialize to left covering functors in the case where $\mathcal{C}$ is weakly lex. We do this in the framework of *lex-colimits* introduced by Garner and Lack; thus capturing not only regular and Barr-exact categories, but also (infinitary) lextensive, coherent, and adhesive categories as well as pretopoi. In particular, during the talk we shall also give necessary and sufficient conditions for the existence of free lextensive and free pretopos completions in the non-lex world, and in both cases describe flat functors in elementary terms.

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Sheafification as a geometric tripos-to-topos adjunction *

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The notion of *tripos*, acronym for "topos-representing indexed pre-orders sets", was originally introduced in [3] in order to explain from an abstract perspective in which sense the description of localic sheaf toposes in terms of A-*valued sets* provided by Higgs [1] and Hyland's realizability toposes [2] are instances of the same construction. The key insight is that Higgs's presentation of the category of sheaves $\mathsf{Sh}(\mathsf{A})$ via A-valued sets (for a given locale A) can be generalized to suitable Lawvere doctrines. This leads to a new construction called *tripos-to-topos* that produces a topos T_P of "P-valued sets" from a given tripos $P\colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{Hey}$. Localic and realizability toposes can both be shown to be instances of the tripos-to-topos construction, for suitable triposes.

Therefore, the "external" (or "explicit") perspective of the category of localic sheaves proposed by Higgs can be seen as the cornerstone for understanding the original motivations behind the tripos-to-topos construction. However, it is important to recall that sheaves on a locale A are typically described as presheaves on A satisfying a *glueing condition*. In particular, the category of sheaves $\mathsf{Sh}(\mathsf{A})$ and that of presheaves $\mathsf{PSh}(\mathsf{A})$ happen to be connected via the so-called *sheafification functor*, i.e. the left adjoint s to the inclusion i:

$$\mathsf{PSh}(\mathsf{A}) \underset{i}{\overset{s}{\underset{\bot}{\rightleftarrows}}} \mathsf{Sh}(\mathsf{A})$$

Therefore, the category of sheaves $\mathsf{Sh}(\mathsf{A})$ can be described as the sub-category of j-sheaves (in the "internal sense" according to [4]) of the category $\mathsf{PSh}(\mathsf{A})$, for the Lawvere-Tierny topology j corresponding to the sheafification.

The main purpose of this work is to show that the sheafification between a localic topos and its presheaf topos can be abstracted to a canonical adjunction between a tripos P and its full existential completion $P^{\exists}$ (as defined in [5]) so that *any tripos-to-topos construction* T_P of a Set-based tripos can be seen as the category of internal sheaves $\mathsf{Sh}_j(\mathsf{pT}_P)$ for the Lawvere-Tierney topology j induced by such an adjunction of triposes on the tripos-to-topos construction, called pT_P, of the full existential completion $P^{\exists}$ of P.

This result shows that the notion of tripos and the tripos-to-topos construction allows us to provide an abstract understanding of the notion of sheaf as presented traditionally.

In detail, we start by generalizing the construction of the category of localic presheaves to the level of triposes:

Definition. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Hey}$ be a tripos. We define the category of P-**presheaves** as the category $\mathsf{pT}_P := (\mathcal{G}_P)_{\mathsf{ex/lex}}$ given by the exact completion of the Grothendieck category of P.*

The ordinary definition of the category of (localic) presheaves can be obtained as particular case of the previous one since, when we consider the category of $\mathsf{A}^{(-)}$-presheaves given by the localic tripos $\mathsf{A}^{(-)}\colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{Hey}$ we obtain the category $\mathsf{pT}_\mathsf{A} := (\mathsf{A}_+)_{\mathsf{ex/lex}}$, that is known to be equivalent to the category $\mathsf{PSh}(\mathsf{A})$ of presheaves on A. In the following theorem we summarize our main results:

*Joint work with Maria Emilia Maietti. Abstract submitted to CT2023.

Theorem. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Hey}$ be a tripos. Then the following conditions are equivalent:*

1. *pT_P is a topos;*
2. *the full existential completion $P^{\exists}$ of P is a tripos;*
3. *$\mathcal{G}_P$ has weak dependent products and a generic proof.*

Furthermore, when one of the previous conditions holds then $\mathsf{pT}_P \cong \mathsf{T}_{P^{\exists}}$ and there exists an adjuction of triposes

$$P^{\exists} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} P$$

whose counit is an iso and which induces a geometric embedding of toposes

$$\mathsf{pT}_P \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathsf{T}_P.$$

Corollary. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Hey}$ be a tripos. If pT_P is a topos then the previous adjunction of triposes induces a Lawvere-Tierny topology j on pT_P such $\mathsf{T}_P \equiv \mathsf{Sh}_j(\mathsf{T}_P)$.*

We conclude by presenting some sufficient conditions for triposes such that pT_P is guaranteed to be topos, which allow us to recognize a wide variety of examples, including all Set-based triposes:

Theorem. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Hey}$ be a tripos such that*

1. *$\mathcal{C}$ has weak dependent products;*
2. *the predicate classifier Ω of P has a power object $\mathrm{P}\Omega$ in $\mathcal{C}$;*
3. *$\mathcal{C}$ admits a proper, stable factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$, and every epi of $\mathcal{E}$ splits.*

Then we have that pT_P is a topos.

Corollary. *For every Set-based tripos P, the category pT_P is a topos and $\mathsf{T}_P \equiv \mathsf{Sh}_j(\mathsf{pT}_P)$.*

Finally, we will show that our approach combining the tripos-to-topos with the full existential completion allows us to abstract –as triposes adjunctions – other known sheafification-like adjunctions, including the adjunction between $(\mathcal{C})_{\mathsf{ex/reg}}$ and $(\mathcal{C})_{\mathsf{ex/lex}}$ presented in [6], and that between Set and a realizability topos.

References

[1] Higgs, D. Injectivity in the topos of complete heyting algebra valued sets. *Canad. J. Math.* 36 (1984), no. 3, 550–568.

[2] Hyland, J.M.E. The effective topos. In A.S. Troelstra and D. van Dalen, editors, *The L.E.J. Brouwer Centenary Symposium*, volume 110 of *Stud. Logic Foundations Math.*, Elsevier, 1982. 165–216 pp.

[3] Hyland, J.M.E; Johnstone, P.T; Pitts, A.M. Tripos theory. *Math. Proc. Camb. Phil. Soc.* 88 (1980), 205–23280.

[4] S. MacLane and I. Moerdijk. Sheaves in Geometry and Logic. A First Introduction to Topos Theory. *Spinger*, 1992.

[5] Maietti, M.E; Trotta, D. A characterization of generalized existential completions. *Ann. Pure Appl. Log.* 174 (2023), no. 4, 103234.

[6] Menni, M. Closure operators in exact completions. *Theory Appl. Categories* 8 (2001), no. 21, 522–540.

Radon-Nikodym derivatives and martingales

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The Radon-Nikodym theorem is a result in measure theory that describes a correspondence between random variables and measures, using Radon-Nikodym derivatives. The result has many applications in probability theory and stochastic calculus. In particular, the result implies the existence of conditional expectation, an important concept in martingale theory.

In this talk we will give **a categorical proof of the Radon-Nikodym theorem**. We will do this by describing the trivial version of the result on finite probability spaces as a natural isomorphism. We then proceed to Kan extend this isomorphism to obtain the result for general probability spaces. Moreover, we observe that conditional expectation naturally appears in the construction of the right Kan extensions. Using this we can represent martingales, a special type of stochastic processes, categorically.

We then repeat the same construction for the case where everything is enriched over $\mathbf{CMet}_1$, the category of complete metric spaces and 1-Lipschitz maps. In the enriched context, we can give **a categorical proof of a martingale convergence theorem**, by showing that a certain functor preserves certain cofiltered limits.

The dependent Gödel fibration*

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Summary We present ongoing joint work towards providing an internal characterization of when a given Grothendieck fibration is a generalized *dependent Dialectica construction*, *i.e.*, a completion of some fibration by sums and then products, along a given class of *display maps*. This complements the work of Hofstra [1] by an internal viewpoint, and generalizes both this and the work by Trotta–Spadetto–de Paiva [3] to the *dependent case*, replacing the class of cartesian projections in a fixed category by arbitrary display maps. We discuss how, besides being of intrinsic interest, this recovers a range of relevant examples in categorical logic.

1. From the Dialectica interpretation to Dialectica categories. In the 1950s, Gödel published the *Dialectica interpretation*, a proof interpretation used to prove relative consistency of intuitionistic arithmetic. In the late 80s, Valeria de Paiva introduced [2] as a categorification of this syntactic process, by assigning to a finitely complete category C its *Dialectica category* $\mathsf{Dial}(\mathsf{C})$. By invoking additional structure these Dialectica categories give rise to models of linear logic.

2. From Dialectica categories to Dialectica fibrations. Work of Hyland, Biering, Hofstra, von Glehn, and Moss, generalized the Dialectica construction even further, assigning to a Grothendieck fibration $\mathsf{p} : \mathsf{E} \to \mathsf{B}$ (over a finitely complete category B) its *Dialectica fibration* $\mathfrak{Dial}(\mathsf{p}) : \mathfrak{Dial}(\mathsf{E}) \to \mathsf{B}$. The Dialectica category $\mathsf{Dial}(\mathsf{C})$ in the sense outlined above is recovered as the total category of the Dialectica construction $\mathfrak{Dial}(\mathrm{sub}_{\mathsf{C}})$ of the subobject fibration $\mathrm{sub}_{\mathsf{C}} : \mathrm{Sub}_{\mathsf{C}} \to \mathsf{C}$ of C. In fact, $\mathfrak{Dial}(\mathsf{p})$ turns out to be fibered equivalent to the iterated completion of the fibration p by first adding fibered sums and then fibered products, which suggests a close connection to von Glehn's *polynomials*.

3. Intrinsic characterization of simple Dialectica fibrations. Trotta–Spadetto–de Paiva [3] proved an *internal* characterization of the Dialectica by introducing the notion of a *Gödel fibration*. Inspired by Gödel's original interpretation and by the notion of *existential-free element* (presented in the proof-irrelevant setting of Lawvere doctrines in [4]) they introduced the notion of a *quantifier-free element* of a fibration. Intuitively, given a logical formula in a fibration p one says that α is $\exists$-quantifier-free if and only if there exists a proof π of a formula $\exists i \beta(i)$ assuming α, then there exists a witness $t = t(\pi)$ together with a proof of $\beta(t)$. Furthermore, since quantifier-freeness should be stable under substitution, this must hold for all reindexings $\alpha(f)$ of α, in the fibrational sense. There exists also a dual notion, called $\forall$-quantifier-freeness. A *Skolem fibration* is a Grothendieck fibration (over a finitely complete base) that has simple, *i.e.*, non-dependent sums and products, *enough* $\exists$-quantifier free elements in a suitable sense, and in which $\exists$-quantifier free elements are closed under simple products. One can show that every Skolem

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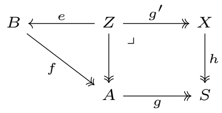
fibration validates the *Skolem principle*, internally expressed as $\forall x \exists y \alpha(i,x,y) \cong \exists f \forall x \alpha(i,x,f(x))$. A Skolem fibration is called a *Gödel fibration* if its full subfibration of $\exists$-quantifier-free elements has enough $\forall$-quantifier-free elements. It is shown in [3] that (up to fibered equivalence) a fibration is a Dialectica fibration if and only if it is a Gödel fibration. Since the latter is a notion completely intrinsic to the structure of the given fibration this yields an internal characterization of when a fibration arises as the Dialectica construction. Moreover, it provides a means to construct the latter.

4. Current work: Intrinsic characterization of generalized dependent Dialectica fibrations. The characterization of [3] considers the case of Dialectica fibrations in the sense of [1], where the Dialectica objects are given as tuples (I,U,X,α) where I,U,X are objects in the base and α is an element in the fiber over $I \times U \times X$, playing the role of a predicate of the form $\exists u \forall x \alpha(i,u,x)$ as justified by invoking the result of viewing this as a completion. From an internal point of view, the transition to a dependent version of the Dialectica construction replaces the tuples (I,U,X,α) by tuples $\big(I,U,\sum_u X_u, \alpha(i,u,x)\big)$ where α is an object over the fiber of X. Then the object α is to be identified with $\exists u \forall x \alpha(i,u(i),x(i,u))$. In the simple, non-dependent case the completion process starts by adding sums with respect to the class of cartesian projections $\{I \times X \to I\}_{I,X \in \mathsf{C}}$. The results of [3] make crucial use of the explicit description of this class. We abstract the previously introduced notions to *dependent Skolem and dependent Gödel fibrations*, replacing cartesian projections by maps of a fixed class of display maps $\mathcal{F}$ and, resp., (cartesian) exponents by *$\mathcal{F}$-dependent products*. Correspondingly, the (simple) Dialectica fibration of a fibration p gets replaced by its generalized variant $\mathfrak{Dial}_{\mathcal{F}}(\mathsf{p})$, which arises by freely adding fibered products and then sums, both along maps in $\mathcal{F}$. Foundationally, we are considering (a) the Dialectica construction on the level of *fibrations*, and (b) relative to a class of *display maps* just as Moss and von Glehn [5] do. Our main quest is, however, different. Namely, we give novel results characterizing intrinsically when a fibration is an $\mathcal{F}$-dependent Dialectica fibration. This work in progress envisions the following two main results. In both cases, let $(\mathsf{B},\mathcal{F})$ be a display map category with $\mathcal{F}$-dependent products, and in the first theorem, for an arrow u in the base $\coprod_u \dashv u^* \dashv \prod_u$ denotes the adjoints of the cartesian reindexing u^*.

Theorem 1 (Dependent Skolemization)**.** *Let $\mathsf{p}: \mathsf{E} \to \mathsf{B}$ a dependent Skolem fibration, and $g: A \twoheadrightarrow S$ and $f: B \twoheadrightarrow A$ be maps in $\mathcal{F}$. Consider the $\mathcal{F}$-dependent product of f along g, as given by the diagram:*



Then there is a natural isomorphism $\prod_g \coprod_f \cong \coprod_h \prod_{g'} e^*$ between functors from E_B to E_S.

Theorem 2 (Dependent Dialectica is equivalent to dependent Gödel)**.** *Up to fibered equivalence over* B, *a fibration* $\mathsf{p}: \mathsf{E} \to \mathsf{B}$ *is of the form* $\mathfrak{Dial}_{\mathcal{F}}(\mathsf{q})$ *if and only if it is a dependent Gödel fibration. Moreover,* q *is (equivalent to) the sub-fibration of $\mathcal{F}$-quantifier free elements of* p.

References

[1] P Hofstra. The Dialectica monad and its cousins. Models, logics, and higher-dimensional categories: A tribute to the work of Mihály Makkai. *AMS & CRM*, 2011. CRMP/53. 107-139.

[2] V de Paiva. The Dialectica categories. *Categories in Computer Science and Logic* 92 (1989). 47-62.

[3] D Trotta; M Spadetto; V de Paiva. The Gödel fibration. *46th International Symposium on Mathematical Foundations of Computer Science (MFCS 2021)*, 202 (2021). 1-16.

[4] D Trotta; M E Maietti. A characterization of generalized existential completions. *Annals of Pure and Applied Logic*, 174(4), 2023.

[5] S Moss; T von Glehn. Dialectica Models of Type Theory. *Logic in Computer Science*, 2018.

When are there enough model isomorphisms? Representing groupoids for classifying toposes.*

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Topological Galois theory. Some models of logical theories have a rich enough automorphism structure that we would expect the theory to be in some way recoverable from the group of model automorphisms. For example, the rationals with the usual ordering is a conservative, homogeneous model for the theory of dense linear orders $\mathbb{T}_{\mathrm{DLO}}$.

This problem of recoverability can be formalised via *classifying topos theory* since, in addition to a logical description, toposes can be *represented* by topological/algebraic data, e.g. as the topos of continuous actions by a topological group. That $\mathbb{T}_{\mathrm{DLO}}$ is recoverable from the automorphisms of $(\mathbb{Q}, <)$ can be expressed by the fact that the topos of continuous actions by the automorphism group $\mathrm{Aut}(\mathbb{Q})$, equipped with the Krull topology, classifies the theory $\mathbb{T}_{\mathrm{DLO}}$.

This is an application of the 'topological Galois theory' of [3], where it is shown that an atomic theory is represented by the topological group of automorphisms of a (set-based) model if and only if that model is both conservative and homogeneous (this is in contrast to the 'localic Galois theory' theory of [4], where the *localic* automorphism group of any model suffices).

Representing groupoids. The same story can be repeated for arbitrary theories by generalising from topological groups to *topological groupoids*. While Joyal and Tierney famously showed in [6] that every topos is the topos of sheaves on some open localic groupoid, Butz and Moerdijk give in [2] a parallel result that every topos with enough points is represented by an open topological groupoid.

When a topos with enough points is known to classify a theory $\mathbb{T}$, the papers of Awodey and Forssell [1], [5] give an explicitly logical description of a representing topological groupoid. In essence, the construction of [1] and [5] expresses that $\mathbb{T}$ is recoverable from the groupoid of all *κ-indexed models*, for a sufficiently large cardinal κ.

Our Contribution. However, we may wonder what is the minimum information required of a groupoid of $\mathbb{T}$-models in order to recover a theory $\mathbb{T}$. In this talk we exposit a holistic treatment of which groupoids of $\mathbb{T}$-models yield representing open topological groupoids. We will observe that, just as with the 'topological Galois theory' of [3], it does not suffice to have only a *conservative* set of models for the theory, but instead model-theoretic conditions must be imposed on the groupoid as well. Specifically, we prove the following result.

Definitions 1. Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a (small) groupoid of (set-based) $\mathbb{T}$-models.

(i) Given a formula in context $\{ \vec{x}, \vec{y} : \varphi \}$ and a tuple of parameters $\vec{m}$ from our models (note that we allow our models to share parameters), the corresponding *definable subset with parameters* $[\![\vec{x}, \vec{m} : \varphi]\!]_{\mathbb{X}}$

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is the set of pairs $\langle \vec{n}, M \rangle$ where $M \in X_0$ and $M \vDash \varphi(\vec{n}, \vec{m})$, considered as a subset of the disjoint coproduct of the models. If $\vec{m} = \emptyset$, the subset $[\![\vec{x} : \varphi]\!]_{\mathbb{X}}$ is said to be definable *without parameters*.

(ii) We say that groupoid $\mathbb{X} = (X_1 \rightrightarrows X_0)$ of $\mathbb{T}$-models *eliminates parameters* if the *isomorphism closure* of a definable subset with parameters is definable without parameters.[1]

Theorem 2. *Let $\mathbb{T}$ be a geometric theory, and let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a (small) groupoid of $\mathbb{T}$-models.*

(i) If X_0 is conservative and $\mathbb{X}$ eliminates parameters, then there exist topologies on X_0 and X_1 making $\mathbb{X}$ an open topological groupoid such that $\mathbf{Sh}(\mathbb{X})$ *classifies $\mathbb{T}$.*

(ii) Conversely, if there exist topologies on X_0 and X_1 making $\mathbb{X}$ an open topological groupoid such that $\mathbf{Sh}(\mathbb{X})$ *classifies $\mathbb{T}$, then there is an isomorphism of groupoids $\mathbb{X} \cong \mathbb{X}'$ such that X'_0 is conservative and $\mathbb{X}'$ eliminates parameters.*

Both the groupoid of κ-indexed models constructed in [5] and the automorphism groups considered in [3] can be recovered as examples of conservative model-groupoids that eliminate parameters.

References

[1] Awodey, S.; Forssell, H. First-order logical duality. *Annals of Pure and Applied Logic* 164 (2013), no. 3, 319–348.

[2] Butz, C.; Moerdijk, I. Representing topoi by topological groupoids. *Journal of Pure and Applied Algebra* 130 (1998), no. 3, 223–235.

[3] Caramello, O. Topological Galois theory. *Advances in Mathematics* 291 (2016), 646–695.

[4] Dubuc, E.J. Localic Galois theory. *Advances in Mathematics* 175 (2003), no. 1, 144–167.

[5] Forssell, H. Topological representation of geometric theories. *Mathematical Logic Quarterly* 58 (2012), no. 6, 380–393.

[6] Joyal, A.; Tierney, M. An extension of the Galois theory of Grothendieck. *Memoirs of the American Mathematical Society* 51 (1984), no. 309, v–71.

[7] Poizat, B. Une théorie de Galois imaginaire. *Journal of Symbolic Logic* 48 (1983), no. 4, 1151–1170.

[1] One can compare our definition of 'elimination of parameters' with Poizat's 'elimination of imaginaries' [7].

On monadicity of strict ω-categories*

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It is well known that the category of strict ω-categories ωCat is monadic over globular sets, i.e., over presheaves on globular category. It is also known that the category ωCat is monadic over globular polygraphs, i.e., the category of those polygraphs whose generators in positive dimensions have always generators as their domains and codomains. The category of globular polygraphs is again a presheaf category. In this talk I will show that the category ωCat is monadic over the category of positive-to-one polygraphs **pPoly**, with the free ω-category functor being (again) an embedding. Thus the category of opetopic set, equivalent to **pPoly**, is yet another category equivalent to a category of presheaves so that ωCat is monadic over it. As in the previously mentioned cases the monad T_ω on category **pPoly** for strict ω-categories is strongly cartesian and it decomposes into two strongly cartesian monads related by a distributive law, the first monad responsible for compositions and the second monad responsible for identities.

The category **Poly** of all polygraphs is not a presheaf category and it seem that ωCat might fail to be monadic over **Poly**. In [1] Simon Henry characterised the subcategories of the category **Poly** that are the so-called good presheaf categories. They are identified by shapes of generators in the polygraphs, or equivalently by some subobjects of the terminal object in **Poly**. It seem interesting to characterise those subcategories of the category **Poly** that are good presheaf categories over which ωCat is monadic in a canonical way, i.e., with the free ω-category functor being an embedding.

References

[1] Henry, S. Non-unital polygraphs form a presheaf category. Higher Structures 3(1):248–291, 2019.

[2] Zawadowski, M. On positive face structures and positive-to-one computads. arXiv:0708.2658 [math.GT], 1-77.

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Duality in Monoidal Categories[*]

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There are at least three categorical gadgets that capture various notions of monoidal duality: closed monoidal; *-autonomous [4], also called Grothendieck–Verdier [2]; as well as rigid monoidal categories. These concepts are all interlinked: rigid monoidal categories are always Grothendieck–Verdier, and these, in turn, are necessarily closed monoidal.

While the internal-hom[1] functor for closed monoidal categories need only be right adjoint to the tensor product, it has even more structure in the other two cases. For example, given a Grothendieck–Verdier category $(\mathcal{C}, \otimes, 1)$ with contravariant duality functor $D\colon \mathcal{C} \longrightarrow \mathcal{C}$, the internal-hom for $x, y \in \mathcal{C}$ is implemented by $[x, y] \coloneqq D(x \otimes D^{-1}y)$. If $\mathcal{C}$ is rigid, then D is monoidal—the tensor–hom adjunction simplifies to $- \otimes x \dashv - \otimes Dx$.

This raises the naive question: is a monoidal category with tensor-representable internal-hom automatically rigid? For a closed monoidal category $(\mathcal{C}, \otimes, 1, [-, -])$, rigidity boils down to the existence of a natural isomorphism $\phi_x\colon [x, 1] \otimes x \longrightarrow [x, x]$ for every $x \in \mathcal{C}$, such that the following diagram commutes, where ε^x and η^x are the unit and counit of the tensor–hom adjunction:

$$\begin{array}{ccc} x \otimes [x,1] \otimes x & & \\ {\scriptstyle x\otimes\phi_x}\downarrow & \searrow^{x\otimes\varepsilon^x_1} & \\ x \otimes [x,x] & \xrightarrow[\eta^x_x]{} & x \end{array}$$

The (rigid) duality can then be defined as $D \coloneqq [-, 1]$. Thus, by applying ε^x and η^x to the monoidal unit, one obtains natural *evaluation* and *coevaluation* maps

$$\mathrm{coev}_x \coloneqq \eta^x_1\colon 1 \longrightarrow [x, x] \qquad \text{and} \qquad \mathrm{ev}_x \coloneqq \varepsilon^x_1\colon [x, 1] \otimes x \longrightarrow 1.$$

There is no immediate reason why the above diagram should commute. At the BCQT2022 summer school, C. Heunen suggested that it is not true and asked for counterexamples. We will present one by way of an explicit construction of a "free" monoidal category in the spirit of [3]. While not rigid, it turns out that the left and right adjoints to $- \otimes x$ being tensor-representable implies that $\mathcal{C}$ is Grothendieck–Verdier.

References

[1] Halbig, S.; Zorman, T. Duality in Monoidal Categories. *arXiv e-prints*, 2023.

[2] Boyarchenko, M.; Drinfeld, V. A duality formalism in the spirit of Grothendieck and Verdier. Quantum Topol., 4(4):447–489, 2013.

[3] Kassel, C. Quantum groups. In Algebra and operator theory (Tashkent, 1997), pages 213–236. Kluwer Acad. Publ., Dordrecht, 1998.

[4] Barr, M. *-autonomous categories. With an appendix by P.-H. Chu, vol. 752 of Lect. N. Math. Springer, Cham, 1979.

[*] Joint work with Sebastian Halbig [1].

[1] By this we shall always mean *left* internal-hom. Likewise, whenever we use closed we really mean *left* closed.

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