Off-line model reduction for on-line linear MPC of nonlinear large-scale distributed systems

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ABSTRACT

Model predictive control (MPC) is an efficient method for the controller design of a large number of processes. However, linear MPC is often inappropriate for controlling nonlinear large-scale systems, while non-linear MPC can be computationally costly. The resulting optimization-based procedure can lead to local minima due to the, non-convexities that non-linear systems can exhibit. To overcome the excessive computational cost of MPC application for large-scale nonlinear systems, model reduction methodology in conjunction with efficient system linearizations have been exploited to enable the efficient application of linear MPC for nonlinear distributed parameter systems (DPS). An off-line model reduction technique, the proper orthogonal decomposition (POD) method, combined with a finite element Galerkin projection is first used to extract accurate non-linear low-order models from the large-scale ones. Trajectory Piecewise-Linear (TPWL) methodologies are subsequently developed to construct a piecewise linear representation of the reduced nonlinear model, both in a static and in a dynamic fashion. Linear MPC, based on quadratic programming, can then be efficiently performed on the resulting low-order, piecewise affine system. Our combined methodology is readily applicable in combination with advanced MPC methodologies such as multi-parametric MPC (MP-MPC) (Pistikopoulos, 2009). The stabilisation of the oscillatory behaviour of a tubular reactor with recycle is used as an illustrative example to demonstrate our methodology.

1. Introduction

Model predictive control (MPC) is widely used in the process industries. It links low-level fast PID controllers with high-level slow scheduling in appropriate control hierarchies and achieves the set goals by a sequence of on-line optimisations. Optimisation and scheduling are crucial to any large scale processing plant. Over the last two decades, linear MPC has become a popular and effective advanced control strategy. However, it often leads to poor performance for non-linear systems except near the point at which the model was identified (Tenny, Rawlings, & Wright, 2004). Nonlinear MPC is a more natural choice for non-linear systems. In general, however, non-linear models lead to considerably more highly demanding and expensive computations in particular for performing the numerous on-line optimization runs required in the MPC framework, which restricts the applications of MPC. Therefore, nonlinear MPC is mostly used for (well-mixed) batch systems which are typically characterised by a relatively small number of state variables, while linear MPC is more often used in continuous (distributed parameter) systems, which are typically large-scale (Heath, Li, Wills, & Lennox, 2006).

Model reduction plays a very important role for the applicability of nonlinear MPC on complex systems. An efficient model reduction technique can lead to a successful implementation of nonlinear MPC. In general, model reduction techniques can drastically reduce the complexity of internally complex dynamical systems and at the same time preserve the accuracy of their input–output behaviour so as to significantly reduce simulation times. There are some basic requirements for the efficiency of model reduction techniques for control applications: to perform the reduction process in an automated fashion; to provide a good approximation of the original (full) system; to preserve the properties specified by the user and; to maintain explicit dependence of important parameters and/or system inputs. Model reduction technology has been used in conjunction with both linear and nonlinear MPC for distributed (linear and nonlinear) systems. A brief overview of model reduction techniques that have been used in conjunction with MPC technology is given below:

Balanced truncation has been extensively used for reduction purposes in MPC applications for linear time invariant systems. See (Stykel, 2006) for an overview of the method and for an application on the control of semi-discretised Stokes equations. Another

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important method is the multi-parametric MPC (MP-MPC) framework (see Pistikopoulos, 2009), where explicit solutions of the linear MPC problem are solved off-line and are tabulated, thus minimising on-line computations, significantly reducing the computational expense. Furthermore, model reduction for large-scale linear robust control problems includes a class of techniques such as H_1, H_2 (Huang, Yan, & Teo, 2001) and Hankel (Glover, 1984) norm methods. The Proper Orthogonal Decomposition method (Holmes, Lumley, & Berkooz, 1996) has been extensively used for control applications mainly in a non-linear MPC framework (e.g. Garcia, Vilas, Banga, & Alonso, 2008; Li & Christofides, 2008) and also for MPC in conjunction with mesoscopic simulators (Oguz & Gallivan, 2008). Moreover, *equation-free* approaches (e.g. Kevrekidis et al., 2003; Theodoropoulos, Qian, & Kevrekidis, 2000) based on the on-line matrix-free computation of low-dimensional system subspaces have been used for optimization (Luna-Ortiz & Theodoropoulos, 2005; Theodoropoulos & Luna-Ortiz, 2006) and MPC applications see e.g. (Armaou, Theodoropoulos, & Kevrekidis, 2005; Bonis & Theodoropoulos, 2010). For a more comprehensive review of recent developments in model reduction-based optimization/MPC applications see (Theodoropoulos, 2010) and references within.

The main purpose of this work is to combine an off-line model reduction technique, namely the POD method, with linear MPC for nonlinear large-scale systems. POD combined with a finite element (FEM) Galerkin projection is first used to extract non-linear reduced models following an "adequate" sampling procedure in the *allowable* range of parameters. The nonlinear low-order models are then linearised with a Trajectory Piecewise-Linear (TPWL) method. The resulting reduced piece-wise linear system is then efficiently handled by linear MPC. It is worthwhile to note here that in our POD-FEM/TPWL/MPC methodology, despite the possible high nonlinearity of the original system, a quadratic objective function is always extracted for the MPC formulation. Furthermore, the POD-FEM-based reduced model is nonlinear only in the time dimension. Therefore, the TPWL-based linearisation becomes essentially a set of 1-D linearisations in time. Moreover, since the POD-FEM-TPWL procedure is performed off-line, all the on-line MPC computations are computationally inexpensive. This technique, briefly presented in (Xie & Theodoropoulos, 2010), is especially promising for multi-parametric MPC (Pistikopoulos, 2009) in order to enhance its capabilities to handle large-scale nonlinear problems. We believe that the POD-FEM/TPWL/MPC technique can have significant impact in the applicability of linear MPC to nonlinear large-scale systems. The rest of the paper is organised as follows: In Sections 2.1 and 2.2 a brief introduction of the standard linear and nonlinear MPC methods is given. In Sections 2.3 and 2.4 the basic components of our methodology are discussed starting with linear MPC, and continuing with the POD and TPWL methods. In Section 3 a case study, namely the control of a tubular reactor is used to illustrate the features of our technique. Finally, the conclusions of this work are discussed in Section 4.

2. Model reduction/linear MPC methodology

2.1. Linear MPC

Linear MPC is often formulated as a state-space model with linear discrete time (Morari & Lee, 1999).

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0 \tag{1}$$

where $x(k) \in \Re^n$ and $u(k) \in \Re^m$ denote state and control inputs, respectively. Receding horizon methods are performed by open-loop optimization with objective function (Morari & Lee, 1999):

$$J_{(p,m)}(x_0) = \min_{u(\cdot)}[x^T(p)P_0x(p) + \sum_{i=0}^{p-1} x^T(i)Qx(i) + \sum_{i=0}^{m-1} u^T(i)Ru(i)] \tag{2}$$

where $(p \geq m)$ and p denotes the length of the prediction horizon or output horizon, and m the length of the control horizon or input horizon.

As both the control horizon and the prediction horizon approach infinity and without constraints, the above problem reduces to the standard linear quadratic regulator (LQR) problem. The optimal control sequence is generated by a static state feedback law where the feedback law gain matrix is found from the solution of an algebraic Riccati equation (ARE). This feedback control law can guarantee closed-loop stability for any positive semi-definite weighting matrix Q and any positive definite R. For the case with constraints, and by choosing both the control and the output horizons to be finite, the quadratic program is finite-dimensional and can be solved relatively easily on-line at every time step.

2.2. Nonlinear MPC

Nonlinear model predictive control is recently gaining popularity in the industrial community. However, the number of reported nonlinear MPC applications is far fewer than those of linear MPC, mainly because the linear MPC problem is simpler and easier to implement. The formulations for these controllers vary widely, and almost the only common principle is to retain nonlinearities in the process model (Tenny et al., 2004).

Nonlinear MPC is often formulated in a nonlinear set of differential equations.

$$\dot{x} = f(x(t), u(t)), \quad x(0) = x_0 \tag{3}$$

subject to input and state constraints of the form:

$$x(t) \in \Re^n \text{ and } u(t) \in \Re^m, \quad \forall t \geq 0 \tag{4}$$

where $x(t)$ and $u(t)$ denote the state and control input, respectively. The same receding horizon idea is also the principle of nonlinear MPC but the model itself is nonlinear.

An extensive review has been made by Morari and Lee (1999). Closed-loop stability of many algorithms has been studied extensively and addressed satisfactorily from a theoretical point of view. Nevertheless feasibility and the possible difference between the open-loop performance objective and the actual closed-loop performance are very difficult to solve.

In addition, convergence to a global optimum by using quadratic programming for linear MPC can be guaranteed, while it can not be ensured for nonlinear MPC due to the possible non-convexities of the underlying nonlinear models. Local minima of nonlinear problems can result in poor performance for the nonlinear MPC. The cost for nonlinear MPC computation is often very high compared to that of linear MPC. Local minima imply that even if a feasible solution exists it sometimes cannot be reached. In order to benefit from the advantages of linear MPC and to reduce the cost of computation, some type of linearisation on the nonlinear MPC formulation can be exploited.

2.2.1. Linearisation on nonlinear MPC

Many efforts have been made on the linearisation of the nonlinear MPC formulation. Nevistić and Morari (1995) applied first feedback linearisation and then used MPC in a cascade arrangement for the resulting linear system, so that quadratic programming can be used and global stability can also be achieved. This method is limited to low-order systems, which fulfill the conditions required for feedback linearisation.

García (1984) first used at each step a different linear model derived from a local Jacobian linearisation, and employed standard dynamic matrix control (DMC) on the actual nonlinear system. Gattu and Zafiriou (1992) and Lee and Ricker (1994) proposed to add the extended Kalman filter to deal with unstable dynamics and to improve disturbance estimation. De Oliveira (1996) made further progress by applying contraction constraints and deriving explicit stability conditions which show the dependence of the solution on the quality of the linear approximation and on the various tuning parameters.

Nevistić (1997) reported excellent simulation results when a linear time varying (LTV) system approximation is computed at each time step over the predicted system trajectory. The time-invariant MPC algorithm can be easily modified to accommodate the LTV system.

Zheng (1998) developed a closed-loop control strategy into the MPC formulation to reduce the online computational demand. The nonlinear MPC control law is approximated with a linear controller (by linearising the nonlinear model and assuming no constraints). This linear controller is used to compute all the future control moves. The online computation effort is significantly reduced in this manner since only the first control move is needed to solve the optimisation problem.

2.3. Proper orthogonal decomposition (POD)

POD essentially employs the spectral theory of compact, self-adjoint operators expressed in the Karhunen-Loeve decomposition theorem (Wong, 1971). POD is probably the most efficient linear decomposition in terms of data compression, and can contain the most "energy" in an average sense (Holmes et al., 1996). The energy of a given mode is related to the magnitude of the eigenvalue corresponding to that mode. The method of snapshots (Sirovich, 1987) is often used to determine semi-*empirically* the reduced set of global eigenfunctions (basis functions), which efficiently span the system. The procedure for POD involves (i) an empirical collection of data points in time from the dynamic model or from experiments for an appropriate range of parameters; (ii) Construction of a two-point correlation matrix of the gathered dynamic responses; (iii) Calculation of the low-order set of $l \ll n$ (n being the dimension of the full model) global basis functions, which capture most of the system's energy, through eigenvalue analysis of this two-point correlation matrix; (iv) Expression of the state variables $x(y,t)$ of the system (where y are spatial coordinates) as linear combinations of the eigenfunctions $\varpi(y)$ (which are functions only of space) and of some coefficients $a(t)$, which are functions only of time:

$$x(y,t) = \sum_{j=1}^{l} a_j(t)\varpi_j(y) + \bar{x}(y) \tag{5}$$

$\bar{x}(y)$ being the average snapshot; and (v) Calculation of these time coefficients using a Galerkin projection of the original model onto the global eigenfunctions, resulting in a reduced order model consisting of l equations. It should be noted here that if the original full model is non-linear, then the reduced model is also non-linear since the Galerkin projection preserves this nonlinearity. It should, however, be also mentioned that this nonlinearity is fully "absorbed" in the l time coefficients $\alpha_j(t)$, $j = 1,\ldots l$, since the basis functions $\varpi_j(y)$ are constant. Hence, and as mentioned above, the resulting non-linear problem is 1-dimensional irrespective of the high-dimensionality of the original problem. POD has been used for the model reduction of a variety of systems, e.g. to produce low-order models for the nonlinear MPC of parabolic PDEs systems (Baker & Christofides, 2000), and for the optimization (Bendersky & Christofides, 2000) and control (Shvartsman et al., 2000) of reduced order models of transport-reaction processes.

2.4. Trajectory piecewise-linear method

A trajectory piecewise linear technique is used in this work to derive a piecewise affine representation of the obtained (nonlinear) reduced model. It should be mentioned here again that only a 1-dimensional piecewise linearization of the time coefficients $\alpha_i(t)$ is needed. Trajectory piecewise-linear methods are presented in Rewieński and White (2003). In particular the weighted method is of interest to us. It is obviously computationally expensive to generate TPWL models of nonlinear large-scale dynamic systems. However, it is computationally efficient to use the POD reduced models for this purpose. Furthermore, an automated TPWL-based procedure to obtain the optimal linearization horizons has been developed. Piecewise linear interpolations are constructed for a number of intervals $i \in [1, n-1]$ in time $\tau \in [t_i, t_{i+1}]$ where n is determined by a preset tolerance (see equations 8–10 below). Also the linearization, L_i, within one time interval, τ, is given by

$$L_i(\tau) = \gamma_i + \sigma_i(\tau - t_i) \tag{6}$$

where the linearization coefficients are defined by

$$\gamma_i = \alpha_j(t_i) \tag{6a}$$

and

$$\sigma_i = (\alpha_j(t_{i+1}) - \alpha_j(t_i))/(t_{i+1} - t_i) \tag{6b}$$

Here $\alpha_j(t_i)$ is the j-th POD time coefficient at time t_i.

To obtain the optimal number of TPWL horizons, the mean value theorem (Van Loan, 1997) is used:

$$\alpha_j(\tau) = L_i(\tau) + \frac{\alpha_j^{(2)}(\eta)}{2}(\tau - t_i)(\tau - t_{i+1}) \tag{7}$$

where $\eta \in [t_i, t_{i+1}]$. If the second derivative of the j-th time coefficient $\alpha_j^{(2)}(\eta)$ is bounded by M_2 and $\bar{h}$ is the length of the longest time interval, then

$$\left|a_j(\tau) - L_i(\tau)\right| \le \frac{M_2\bar{h}^2}{8} \tag{8}$$

This error bound indicates the smallest integer satisfying the inequality in Eq. (8) if the linearizations $L_i(\tau)$ are based on a uniform partition of the time domain $t_{init} = t_1 < t_2 < \cdots < t_n = t_{fin}$, where $t_i = t_{init} + (i-1)(t_{fin} - t_{init})/(n-1)$. The inequality in Eq (9) below guarantees that the difference (error) between the linearisation L_i and the actual function a_j is less than or equal to a positive tolerance δ,

$$\left|a_j(\tau) - L_i(\tau)\right| \le \frac{M_2\bar{h}^2}{8} = \frac{M_2}{8}\left(\frac{t_{fin} - t_{init}}{n-1}\right)^2 \le \delta \tag{9}$$

Hence n, the number of TPWL horizons must satisfy

$$n \ge 1 + (t_{fin} - t_{init})\sqrt{\frac{M_2}{8\delta}} \tag{10}$$

We term the piece-wise linearisation obtained with the use of Eqs. (6)–(10) "static" TPWL.

For highly nonlinear problems, static TPWL produces a large number of horizons eventually leading to an equally large number of linear MPC problems, which can be computationally expensive. Therefore, we have developed an adaptive version of TPWL, which is based on an iterative procedure. Starting from the number of intervals computed from the static TPWL method, we iteratively increase the interval sizes by joining (two or more) adjacent intervals. The interval $[t_L, t_R]$ is acceptable if $\left|a_j((t_L + t_R)/2) - ((a_j(t_L) + a_j(t_R))/2)\right| \le \delta$ or if $t_R - t_L \le h_{\min}$, where

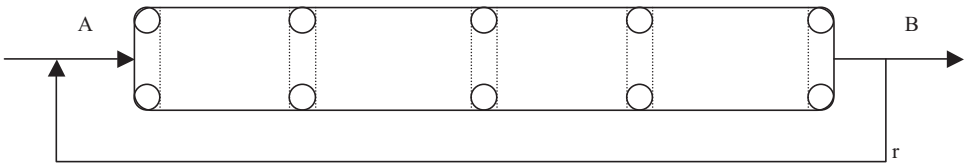

Fig. 1. Tubular reactor with recycle.

$h_{\min} > 0$ are heuristically defined parameters (Van Loan, 1997). In addition, a partition $t_1 < \cdots < t_n$ is acceptable if each subinterval is acceptable.

It is obvious that the combination of a POD reduced model, presented in the previous section with the TPWL method (static or dynamic) will efficiently produce a piece-wise linear representation of the, generally, nonlinear time coefficients of the reduced model in an automated way (Fig. 1).

3. Case study

3.1. Tubular reactor with recycle

The tubular reactor, where an exothermic reaction A → B takes place, considered here is described by two dimensionless partial differential equations for concentration and temperature evolution/distribution (Jensen & Ray, 1982) defined on the spatial domain $z \in [0, 1]$:

$$C_t = -\frac{\partial C}{\partial z} + \frac{1}{Pe_c}\frac{\partial^2 C}{\partial z^2} - f(C, T)$$

$$T_f = -\frac{\partial T}{\partial z} + \frac{1}{Pe_T}\frac{\partial^2 T}{\partial z^2} + B_T f(C, T) + \beta_T(T_c - T) \tag{11}$$

where C and T are the (dimensionless) reactant concentration and the temperature, respectively. T_C corresponds to the temperature of the cooling medium and $f(C, T) = B_c C \exp(\gamma T/(1+T))$ is the reaction term. The parameters used are: $Pe_C = 7.0$, $Pe_T = 7.0$, $B_C = 0.1$, $B_T = 2.5$, $g = 10.0$ and $b_T = 2.0$. A recycle is used here to return part of the reactant at the output stream to the feed stream at a ratio r. Hence, the boundary conditions for concentration and temperature at $z = 0$ become (Antoniades & Christofides, 2001):

$$\frac{\partial C}{\partial z} = -Pe_c[(1-r)(1+C_0) + rC(t, 1) - C(t, 0)] \text{ and}$$
$$\frac{\partial T}{\partial z} = -Pe_T[(1-r)(1+T_0) + rT(t, 1) - T(t, 0)] \tag{12}$$

The boundary conditions at $z = 1$ are $dC/dz = 0$ and $dT/dz = 0$. We have constructed a FEM-based simulator for this reactor by discretising the model in 16 quadratic elements. This resulted in $16 \times 2 = 32$ ordinary differential equations, which were solved using the backward Euler method. The tubular reactor has a very rich parametric behaviour including a wealth of bifurcations such as turning points, Hopf bifurcations, etc. Results from the FEM simulator are shown in Fig. 2. The initialisation point (time, $t = 0$) corresponds to zero dimensionless temperature (see Fig. 2a and b) and correspondingly maximum reactant concentration (Fig. 3a and b). In Figs. 2a and 3a it can be seen that the tubular reactor shows stable behaviour for $r = 0$. The exothermic reaction raises the temperature rather fast shortly after $t = 0$ to approximately 1.8 along the whole reactor (Fig. 2a). The subsequent reactant depletion (Fig. 3a) reduces the temperature to the equilibrium values at approximately $t = 2$. As depicted in Fig. 2a the equilibrium temperature is expectedly higher near the entrance of the reactor where we have higher reactant concentrations and lower near the outlet where higher reactant concentrations are consumed (see Fig. 3a). For $r = 0.5$ a Hopf bifurcation exists and the system undergoes sustained oscillations as it can be seen in Figs. 2b and 3b.

In order to demonstrate the POD-FEM-TPWL-MPC methodology we chose as our control objective to stabilize the reactor with $r = 0.5$ in order to behave like the system with $r = 0$ by introducing a number of jacket temperature zones (actuators). The objective function is:

$$J = \min_{du}\{(T(t) - T_{ref}(t))^{\mathrm{T}} Q(T(t) - T_{ref}(t)) + DU^{\mathrm{T}} R_c DU\} \tag{13}$$

where $T_{ref}(t)$ is the reference state ($r = 0$) and DU is the control on the actuators.

3.2. Sampling of dynamic responses and POD-based reduction

We have applied a method using Heaviside step functions to automate the (largely empirical) sampling in order to cover efficiently the range of the parameter space. Taking in account the step functions (which have 2 states, 0 and 1) we have for 8 actuators $2^8 = 256$ states. Taking (ad hoc) 11 samples, using the full-scale model, over the range of the (dimensionless) cooling temperature [−1,1] we obtain $256 \times 11 = 2816$ samples. The sampling time was 15 s. Obviously, the choice of sampling intervals and sampling times is heuristic, justifying the semi-empirical nature of PODs.

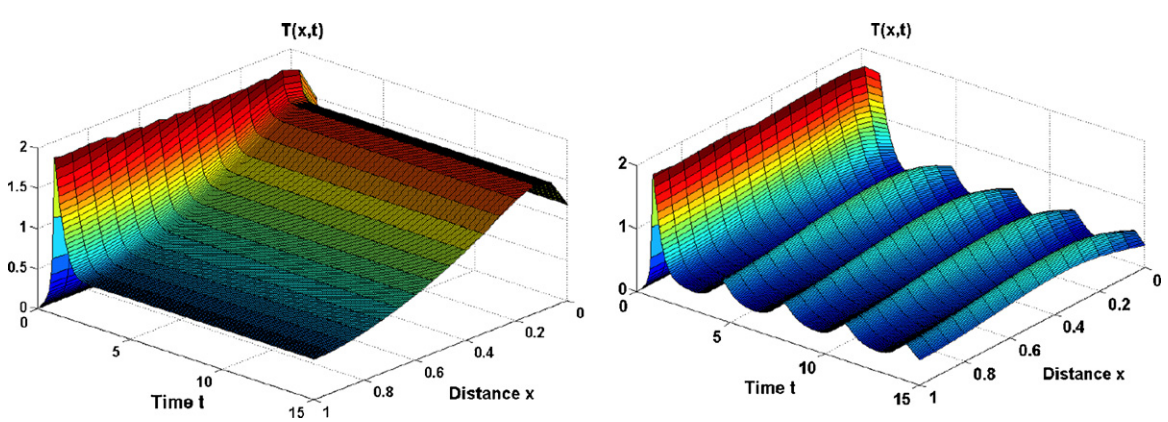

Fig. 2. Temperature profiles for the tubular reactor (a) $r = 0$; (b) $r = 0.5$

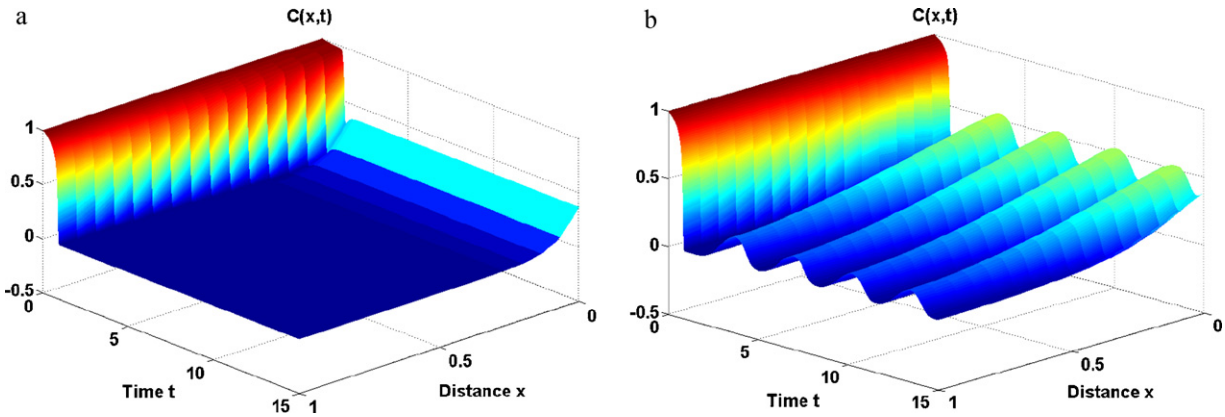

Fig. 3. Concentration profiles for the tubular reactor (a) $r = 0$; (b) $r = 0.5$

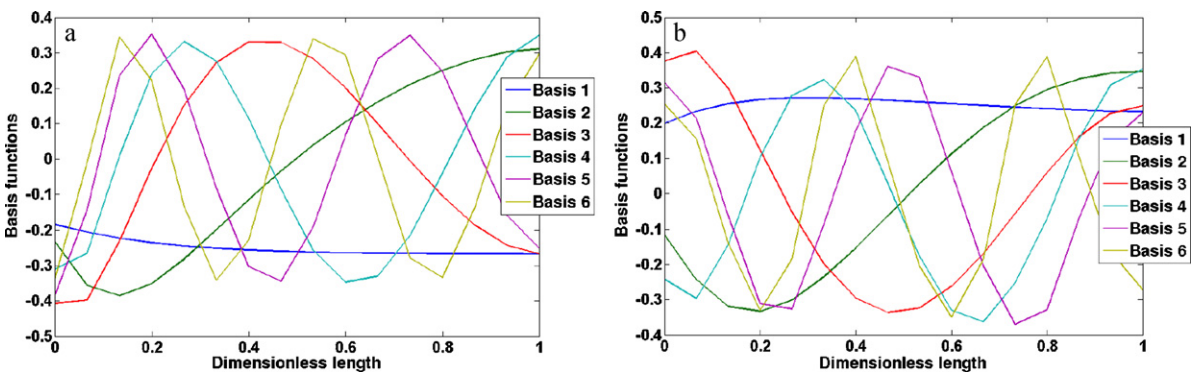

Fig. 4. Global basis functions for (a) concentration; (b) temperature from the sampling data of the tubular reactor with $r = 0.5$

Based on these 2816 samples we constructed the corresponding two-point correlation matrix. Singular value decomposition of this matrix revealed that $l = 8$, eigenfunctions for concentration and $l = 8$ eigenfunctions for temperature capture 99.5% of the system's energy. This is a 50% reduction of the original system.

In Fig. 4a and b, the six most important eigenfunctions for concentration and temperature for the case of $r = 0.5$ are shown. As we can see they can effectively capture the system's (long and short term) dynamics.

To compute the (nonlinear) time coefficients, the FEM model was then projected onto the POD basis functions. The reduced model then results from Eq. (5). As it can be seen from Fig. 5 there is an excellent agreement between the full and the reduced model, which can accurately capture the oscillatory behaviour of the system.

3.3. TPWL method for the time coefficients

We used both static and adaptive TPWL to produce piece-wise affine representations of the time coefficients for concentration and for temperature. Fig. 6 shows the adaptive TPWL segments for the temperature time coefficients for $\delta = 0.001$ here for the case of $r = 0$ (since this corresponds to the reference trajectory). Obviously the corresponding piece-wise linear representation of the time coef-

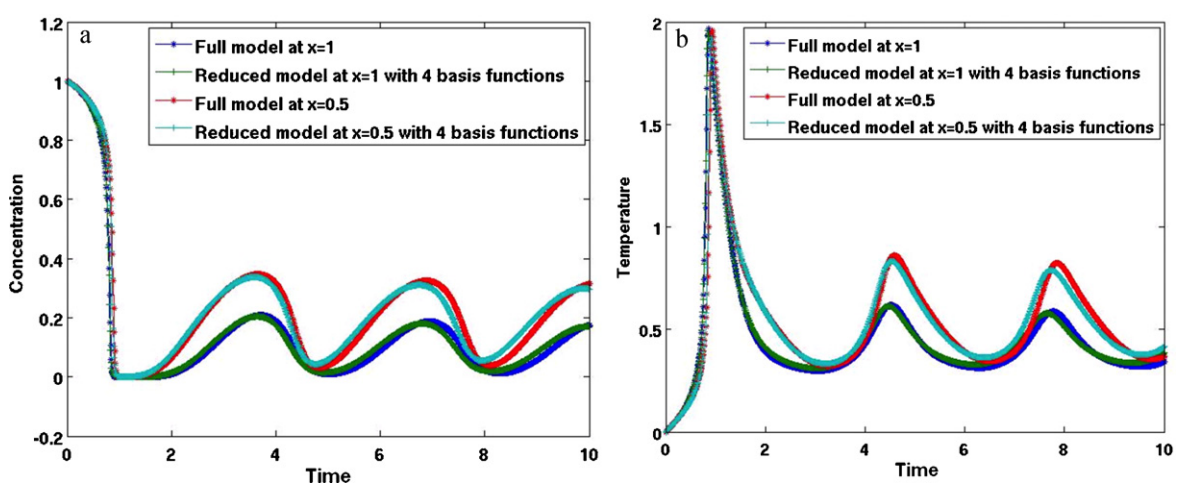

Fig. 5. Comparison between (a) concentration; (b) temperature predictions of full model and reduced model at the middle and output points for the tubular reactor with $r = 0.5$

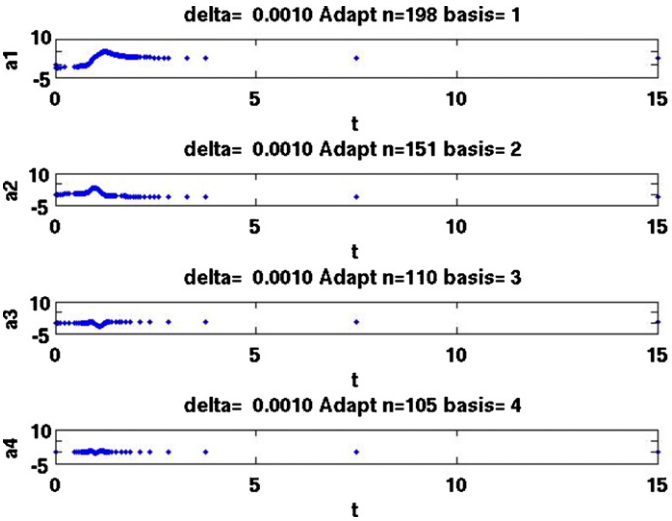

Fig. 6. TPWL for time coefficients on temperature of the tubular reactor using the POD method.

ficients for $r = 0.5$ can also be easily computed. It should be noted that adaptive TPWL produces 198 time segments, which are much less compared to the 3000 static TPWL intervals.

3.4. Control law for linear MPC

The POD method was applied on the control objective:

$$J = \min_{du} \left(\left(\sum_{k=1}^{l} \alpha_{k\_T}(t)\varpi_{k\_t}(x) + \overline{T_{16}} \right) - T_{ref}(t) \right)^{\mathrm{T}} \times Q \left(\left(\sum_{k=1}^{l} \alpha_{k\_T}(t)\varpi_{k\_t}(x) + \overline{T_{16}} \right) - T_{ref}(t) \right) + DU^{\mathrm{T}} R_c DU \tag{14}$$

which is a quadratic function due to the linear POD representation of the state variables as seen in Eq. (5). Q and R_c are non-negative definite matrices. Applying POD on the nonlinear Eqs. (11) and (12) resulted in a reduced set of nonlinear characteristic equations of the system (equality constraints) which are functions of the time coefficients $a_j(t)$. TPWL was then applied on these time coefficients, a one-dimensional linearization only, to obtain piece-wise linear equality constraints:

$$\begin{aligned} \alpha_j(t + 1/t_n) &= L_1\alpha_j(t) + G_i U(t) + C_1 \\ \alpha_j(t + i/t_n) &= L_i\alpha_j(t + (i-1)/t_n) + G_i U(t + (i-1)/t_n) + C_i \\ w(t) &= H\alpha_j(t) + \overline{T}_{exit} \end{aligned} \tag{15}$$

where $i = 1,\ldots p$, $\alpha_j(t)$ includes time coefficients both for concentration and temperature and t_n is the control horizon.

$$H = [\underbrace{0, 0 \ldots, 0}_{l}, \varpi_{1\_T}(z_{exit}), \varpi_{2\_T}(z_{exit}), \ldots, \varpi_{l\_T}(z_{exit})]^{\mathrm{T}} \tag{16}$$

where $\varpi_{i\_T}(z_{exit})$ denotes the i-th eigenfunction for temperature at the reactor's exit. In order to eliminate the time-independent matrices C_i, an operation inspired from (Wang, 2009) was performed on

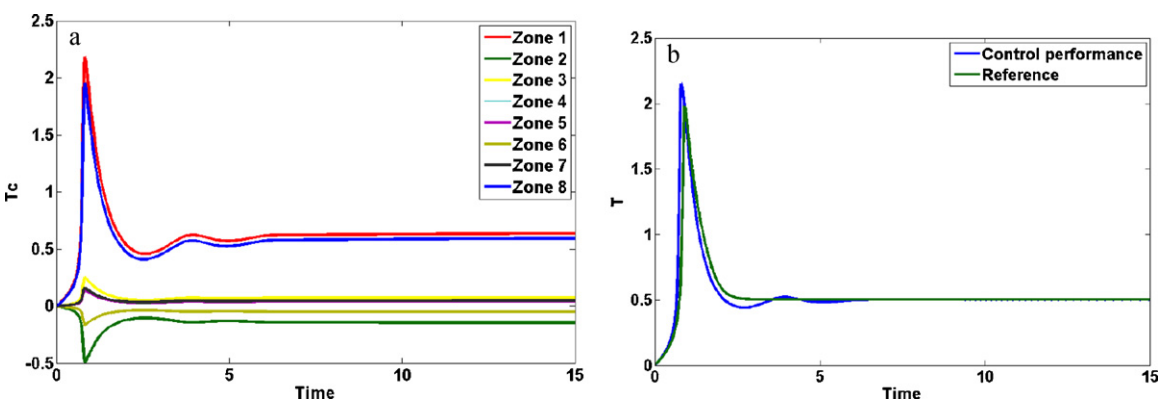

Fig. 7. Linear MPC with static TPWL results (a) time evolution profile for the 8 actuators; (b) control and reference profile.

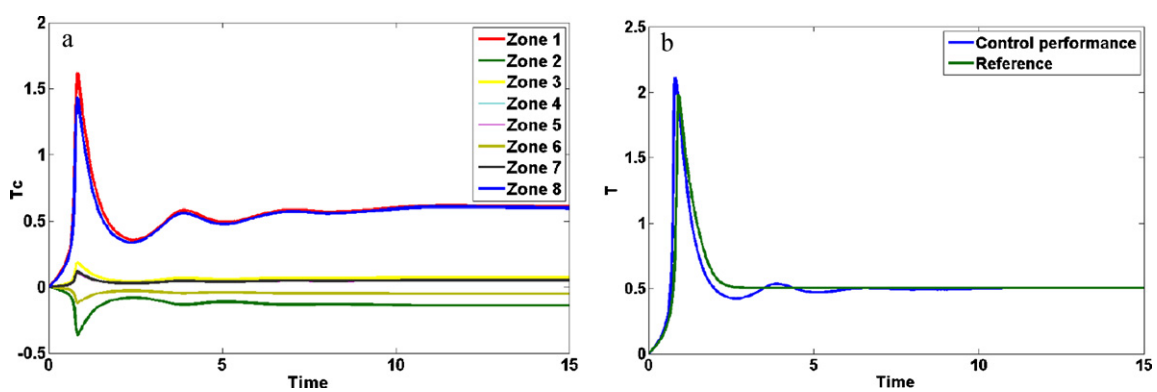

Fig. 8. Linear MPC with adaptive TPWL results (a) time evolution profile for the 8 actuators; (b) temperature control and reference profile.

both sides of Eq. (14) as follows:

$$\begin{aligned}\alpha_j(t+1/t_n)-\alpha_j(t)&=L_i(\alpha_j(t)-\alpha_j(t-1/t_n))\\&+G_1(U(t)-U(t-1/t_n))\\\alpha_j(t+p/t_n)-\alpha_j(t+(i-1)/t_n)&=L_i(\alpha_j(t+(i-1)/t_n)\\&-\alpha_j(t+(i-2)/t_n))+G_i(U(t+(i-1)/t_n)-U(t+(i-2/t_n))\end{aligned}\tag{17}$$

Let the vector of state variables for this reduced piece-wise affine representation be the difference $\Delta\alpha_j(k)=\alpha_j(k)-\alpha_j(k-1)$ of time coefficients at two consecutive time points $t=k$ and $t=k-1$ and the output $w(k)$, i.e. $x(k)=[\Delta\alpha_j(k)^{\mathrm{T}}\ w(k)]^{\mathrm{T}}$.

This then leads to the following piecewise linear state-space model:

$$\begin{aligned}x(k+1)&=A_ix(k)+B_i\Delta U(k)\\w(k)&=Cx(k)\end{aligned}\tag{18}$$

where $A_i=\begin{bmatrix}L_i & 0_i^{\mathrm{T}}\\ HL_i & 1\end{bmatrix}$, $B_i=\begin{bmatrix}G_i\\ HG_i\end{bmatrix}$, $C=[0_i \quad 1]$,

$0_i=\overbrace{\begin{bmatrix}0 & 0 & \dots & 0\end{bmatrix}}^{l}$, $i=1,\dots N_c$.

Therefore, we obtain a quadratic objective function (Eq. (14)) subject to the piece-wise linear equality constraints (Eq. (17)). The control law can then be obtained explicitly from

$$\Delta U=(\Phi^{\mathrm{T}}Q\Phi+R_c)^{-1}\Phi^{\mathrm{T}}Q[R_sY_{ref}(k)-Fx(k)]\tag{19}$$

where, $R_s=\overbrace{[1\,1\dots1]^{\mathrm{T}}}^{N_P}$,

$$\Phi=\begin{bmatrix}HB_1 & 0 & \dots & 0\\ HA_2B_1 & HB_2 & \dots & 0\\ \vdots & \vdots & \dots & \vdots\\ HA_{N_p}A_{N_p-1}\cdots A_2B_1 & HA_{N_p}A_{N_p-1}\cdots A_3B_2 & \dots & A_{N_p-1}\cdots A_{N_c+1}B_{N_c}\end{bmatrix},$$

$$F=\begin{bmatrix}HB_1\\ HA_2B_1\\ \vdots\\ HA_{N_p}A_{N_p-1}\dots A_2B_1\end{bmatrix},$$

N_p is the number of predictive horizons, and N_c is the number of control horizons.

The control output variables can then be calculated from

$$w(k)=Fx(k)+\Phi\Delta U\tag{20}$$

In Fig. 6, results of the piece-wise linear MPC with the static TPWL method for 8 actuators are shown. Fig. 7a shows the control law for the 8 actuators (zones). The control output and the reference profile are shown in Fig. 7b. As it can be seen the reactor is efficiently stabilized, while the on-line computation time for an implementation in MATLAB (R2007b) single-threaded code and execution on a workstation based on 2 Intel Xeon 5160 processors (3.00 GHz) and 4GB of RA is about 15 min for 3000 equal time intervals.

In Fig. 8, results of the linear MPC with the adaptive TPWL method for 8 actuators are shown. Fig. 8a shows the control law for the 8 actuators (zones). The control output and the reference profile are shown in Fig. 8b. As it can be seen the reactor is effectively stabilized, while the on-line computation time is less than 1 min for 198 unequal time intervals.

Hence, our POD-TPWL-MPC methodology can be effectively used to control distributed parameter systems with computational efficiency.

4. Conclusions

We have developed an efficient model reduction-based technique which combines the proper orthogonal decomposition coupled with the finite element method for reduction of the large-scale constraints (system model), a trajectory piece-wise linear method to provide a piece-wise affine representation of the reduced constraints and a (piece-wise) linear MPC implementation. This POD-FEM-TPWL methodology enables the use of linear MPC for highly non-linear systems. The computation of an appropriate number of linear segments is performed in an automated way via the TPWL method, while dynamic TPWL produces the optimal number of linear segments for a given tolerance. Despite the potential high dimensionality of the original nonlinear model, the POD reduced model is nonlinear in one dimension only, time. The representation of the reduced state variables as a linear combination of time coefficients and basis functions, yields directly a quadratic objective function for the MPC problem, hence the corresponding control laws can be obtained analytically due to the piece-wise affine representation of the constraints. This method can efficiently facilitate the use of linear MPC for non-linear systems, as it was demonstrated in our case study, where stabilization of a tubular reactor undergoing sustained oscillations was performed. We are currently working on the implementation of the POD-FEM-TPWL in the multi-parametric control framework.

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