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WHEN ROSS MEETS BELL: THE LINEX UTILITY FUNCTION

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Abstract

At first glance, there would appear to be no relationship between Bell's (1988) concept of one-switch utility function and that of a stronger measure of risk aversion due to Ross (1981). We show however that specific assumptions about the behavior of the stronger measure of risk aversion also gives rise to the linex utility function which belongs to the class of one-switch utility functions. In particular, this utility class is the only one that satisfies a stronger version of Kimball's (1993) standard risk aversion over all levels of wealth.

Key words and phrases: Ross risk aversion, decreasing prudence, one-switch utility function.

JEL code: D81.

1 Introduction

In a stimulating paper, Bell (1988) argued that decision-makers are likely to be characterized by a utility function satisfying the one-switch rule[1]. While four different families of utility functions satisfy such a rule, only one of them – the linear plus exponential, or linex – satisfies some desirable additional properties. Hence, this utility function seems to be relevant for many applications and its interest for the analysis of risky decisions was again illustrated in a recent paper by Sandvik and Thorlund-Petersen (2010).

Much earlier, Ross (1981) published an important paper devoted to an apparently different topic. The message of this paper can be summarized as follows. In Arrow (1965) and Pratt (1964), it is shown that a more risk averse decision-maker[2] is always willing to pay more for the elimination of a risk than a less risk averse one. Ross (1981) then shows that to obtain a similar result for risk reductions one needs a stronger definition of increased risk aversion, which has become known in the literature simply as "increased Ross risk aversion." Jindapon and Neilson (2007) consider an opportunity, at some monetary cost, to shift from a less preferred distribution to one that dominates via a mean-preserving contraction of the risk. They show that an individual with higher Ross risk aversion will always shift more of the risk.

In the present paper, we show that Ross' (1981) notion of stronger risk aversion coupled with specific assumptions about its behavior reinforces the arguments in favor of a linex utility function besides those already suggested by Bell (1988) on very different grounds.

Our paper is organized as follows. In Section 2, we are more specific about each of the two contributions briefly described in this introduction. Section 3 contains our main result and we indicate precisely how Ross' (1981) results and their extensions are related to the linex utility function. We briefly conclude in Section 4.

2 Bell's and Ross' contributions

Bell (1988) proposed "the one-switch rule as a reasonable property for utility functions to satisfy". The basic idea behind this rule can be summarized as follows. Consider two lotteries X and Y such that $\mathbb{E}[X] > \mathbb{E}[Y]$ while X has the worst payoff. At low wealth levels, because the worst outcome is avoided by holding Y, this lottery is likely to be preferred by a risk averter. However, as wealth increases, the relative attractiveness of Y decreases and there will exist a sufficiently high wealth level such that X becomes preferred because of its higher mean. When at most only one such preference reversal occurs[3], the utility function satisfies the one-switch rule.

In his paper, Bell (1988) shows that only four families of utility functions satisfy the one-switch rule. The linear plus exponential (linex, in short) defined by

$$u(x) = lx - c\exp(-\gamma x) \tag{2.1}$$

[1] In this case, there exists at most one unique critical wealth level at which the decision-maker switches from preferring one alternative to the other. See Section 2 for a definition and more information.

[2] See below for a formal definition.

[3] I.e. once X becomes preferred, further increases in wealth no longer induce a return to Y.

where x is wealth while l, c and γ are positive constants, is one of the four families. It turns out besides that it is the only relevant family if one adds to the one-switch rule some very reasonable requirements (e.g., the decision-maker is risk averse at all wealth levels and his risk aversion is decreasing in wealth). Such utility functions have been studied by Bell and Fishburn (2001), among others.

Ross' (1981) paper deals with a very different topic. From Arrow (1965) and Pratt (1964) we know that if a decision-maker becomes more risk averse, i.e. if his utility function is concavified[4], then he is willing to pay more for eliminating all risk in his wealth portfolio, i.e. the replacement of the risk by its expected value.

Since such a result seems so intuitive, one expects it should also hold for a mean preserving contraction of a risk. Using simple examples, Ross (1981) shows that unfortunately the extension to marginal changes in risk (instead of its elimination) does not always work. He suggests a stronger definition of "more risk aversion" that leads to the desired result.

Ross' (1981) definition can be compared to that of Arrow (1965) and Pratt (1964) as follows. According to Arrow (1965) and Pratt (1964), v is more risk averse than u if the coefficient of absolute risk aversion $-\frac{v''(x)}{v'(x)}$ of v exceeds the corresponding coefficient $-\frac{u''(x)}{u'(x)}$ for u at all wealth levels x. However, if we consider two different wealth levels x and y, say, we may have that $-\frac{u''(x)}{u'(x)}$ exceeds $-\frac{v''(y)}{v'(y)}$. Such a situation is prohibited under Ross' (1981) stronger notion of risk aversion, which requires that the minimum of $-\frac{v''}{v'}$ exceeds the maximum of $-\frac{u''}{u'}$, that is, there exists a positive constant λ such that the inequalities

$$\frac{v''(x)}{u''(x)} \geq \lambda \geq \frac{v'(x)}{u'(x)} \text{ are satisfied for all } x. \tag{2.2}$$

Ross (1981) shows next that the stronger measure of more risk aversion is obtained when the more risk averse v can be written as

$$v = \lambda u + \phi \tag{2.3}$$

where λ is a positive constant and where ϕ is a non-increasing and concave function of wealth (i.e., $\phi' \leq 0$, $\phi'' \leq 0$).

Ross' (1981) concept also applies when, instead of comparing two individuals, one compares a given individual at two wealth levels. One then obtains the notion of stronger decreasing absolute risk aversion (Ross DARA, in short). Whereas DARA implies that a decrease in wealth makes one behave in a more risk averse manner, Ross DARA implies that a stochastic decrease in wealth via first-order stochastic dominance makes one behave in a more risk averse manner, as shown by Eeckhoudt et al. (1996).

3 The link between these two contributions and the linex utility function

Bell (1988) had examined the conditions giving rise to the linex utility function without any reference to Ross' (1981) stronger measure. In this section, we show that the linex is a

[4]The utility function u is concavified when the resulting function is $v(x) = k(u(x))$ with $k' > 0$ and $k'' < 0$.

consequence of reasonable assumptions made about the behavior of Ross' stronger measure.

In the Arrow (1965) and Pratt (1964) environment as in Ross' (1981) one, it is straightforward to show that a utility function exhibits decreasing absolute risk aversion (DARA) whenever minus its marginal utility is more risk averse than itself. From (2.3), this implies that u is Ross DARA if there exists a function ϕ such that $\phi' \leq 0$, $\phi'' \leq 0$, and a constant $\lambda > 0$ such that

$$-u' = \lambda u + \phi \tag{3.1}$$

Solving this differential equation shows that u is of the form

$$u(x) \quad = \quad \left(-\int \exp(\lambda x)\phi(x)dx\right)\exp(-\lambda x). \tag{3.2}$$

Now, if we take $\phi(x) = -x$ then we get

$$u(x) = -\frac{1}{\lambda^2} + \frac{x}{\lambda} - c\exp(-\lambda x) \tag{3.3}$$

for some positive constant c, where we recognize the functional form of the linex utility function defined in (2.1). Hence, with potentially many other utility functions, the linex is one of the utility functions that satisfy the Ross DARA condition[5].

In order to further explore the properties of the linex utility function in Ross' (1981) framework, we now consider an additional restriction that is often made in the expected utility framework. Following Kimball (1993), it is often assumed that preferences satisfy the property of "standard risk aversion," which in addition to DARA requires the property of decreasing absolute prudence (DAP in short). While this assumption has been extensively analyzed in the Arrow (1965) and Pratt (1964) setting, recent papers by Keenan, Rudow and Snow (2009) and Keenan and Snow (2009) suggest its use also in the context of Ross' (1981) stronger measure. Moreover, Ross DARA and Ross DAP are sufficient for any second-order stochastic deterioration in a background risk to decrease risk-taking behavior, as shown by Eeckhoudt et al. (1996). We now show that the joint assumptions of Ross DARA and Ross DAP lead exclusively to a linex utility function.

Specifically, the utility function u is Ross DAP if the representation

$$u'' = -\theta u' + \psi$$

holds for some function ψ such that $\psi' \leq 0$, $\psi'' \leq 0$, and some constant $\theta > 0$. Combining this representation with (3.1) gives

$$u'' = -\theta u' + \psi = -\lambda u' - \phi'$$

so that

$$\lambda = \theta \text{ and } -\phi' = \psi.$$

But this implies

$$\psi' = -\phi'' \geq 0$$

[5]Interestingly, when Ross (1981) illustrates his concept of stronger risk aversion he implicitly uses the linex utility as an example.

which contradicts the assumption $\psi' \leq 0$ unless it is satisfied via an equality. This can only be true if ϕ is linear, that is, if u corresponds to a linear plus exponential utility function (3.3).

It is interesting to note that Ross prudence is not strictly decreasing here. Instead, it is constant everywhere. Our analysis shows that it is not possible to obtain both strict Ross DARA and strict Ross DAP at all wealth levels. Indeed, an example of both, as provided by Keenan and Snow (2009) does not hold for all wealth levels, but only over a restricted range of wealth.

4 Conclusion

In the setting of Arrow (1965) and Pratt (1964), many currently used utility functions (logarithmic, power, mixture of exponentials) satisfy Kimball's (1993) property of standard risk aversion. That is, they jointly satisfy the conditions of DARA and DAP. Besides, they are completely monotone.

When the Arrow (1965) and Pratt (1964) environment is replaced by Ross' (1981) stronger measure of risk aversion, the same well-accepted conditions of DARA and DAP, if assumed to hold at all wealth levels, yield a single class of utility functions: the linear plus exponential one. It thus appears that this (also completely monotone) utility function, justified a long time ago by Bell (1988) on very different grounds, has attractive features at the theoretical level, behind those already known for the assessment of a decision-maker's preferences.

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References

Arrow, K.J. (1965). Aspects of the Theory of Risk-Bearing (Yrjo Jahnsson Lectures). Yrjo Jahnssonin Saatio, Helsinki.

Bell, D. (1988). One-switch utility functions and a measure of risk. Management Science 34, 1416-1424.

Bell, D., & Fishburn, P.C. (2001). Strong one-switch utility. Management Science 47, 601-604.

Eeckhoudt, L., Gollier, C., & Schlesinger, H. (1996). Changes in background risk and risk taking behavior. Econometrica 64, 683-689.

Jindapon, P., & Neilson, W.S. (2007). Higher-order generalizations of Arrow-Pratt and Ross risk aversion: A comparative statics approach. Journal of Economic Theory 136, 719-728.

Keenan, D.C., Rudow, D.C., & Snow, A. (2009). Risk preferences and changes in background risk. Journal of Risk and Uncertainty 36, 139-152.

Keenan, D.C., & Snow, A. (2009). Ross risk vulnerability for introductions and changes in background risk. Paper presented at the EGRIE 2009 conference.

Kimball, M.S. (1993). Standard risk aversion. Econometrica 61, 589-611

Pratt, J.W. (1964). Risk aversion in the small and in the large. Econometrica 32, 122-136.

Ross, S.A. (1981). Some stronger measures of risk aversion in the small and in the large with applications. Econometrica 49, 621-663.

Sandvik, B., & Thorlund-Petersen, L. (2010). Sensitivity analysis of risk-tolerance. Decision Analysis 7, 313-321.