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# On the estimation of dynamic conditional correlation models

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**Abstract**

It is now well recognized that the maximum likelihood estimator applied to the dynamic conditional correlation model is severely biased in high dimensions and, in particular, in cases where the time series dimension is close to the sample size. In this paper, we argue that one of the reasons for the bias lies in an ill-conditioned sample covariance matrix, which is used in the so-called variance targeting technique to match sample and theoretical unconditional covariances. We propose to reduce the bias by using shrinkage to target methods for the sample covariance matrix. As targets we use, alternatively, the identity matrix, a single factor model, and equicorrelation. Since the shrinkage intensity decreases towards zero with increasing sample size, the estimator is asymptotically equivalent to the efficient maximum likelihood estimator. The finite sample performance of the proposed estimator over alternative estimators is demonstrated through a Monte Carlo study. Finally, we provide an illustrative application to financial time series.

**Keywords:** DCC, shrinkage, sample covariance matrix, financial time series

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# 1 Introduction

Over the last 20 years, modelling of time-varying covariances of financial asset returns has become an integral part of financial econometrics. Estimation of time-varying volatilities and co-movements between financial series has proved to be a useful tool in financial management. Correlations between returns are relevant for problems such as time-varying beta coefficients in CAPM-type models, estimation of hedge ratios, and Value-at-Risk (VaR) of a portfolio.

Alternative multivariate GARCH (generalized autoregressive conditional heteroscedasticity) models, which are aiming to give simple solutions to the problems described above, have become the subject of wide discussions. The most well known of them are the VEC model of Bollerslev et al. (1988), the BEKK model of Engle and Kroner (1995), and the DCC model of Engle (2002). Recent proposals include the Flexible Multivariate GARCH model of Ledoit et al. (2003), the asymmetric DCC model of Cappiello et al. (2006), Generalized Autoregressive Conditional Correlation (GARCC) model of McAleer et al. (2008), and the Dynamic Equicorrelation (DE) model of Engle and Kelly (2008). For recent reviews of multivariate GARCH models, see Bauwens et al. (2006) and Silvennoinen and Teräsvirta (2008).

The Dynamic Conditional Correlation (DCC) model of Engle (2002) is one of the most cited works related to the parametric modelling of time-varying correlations for multivariate portfolios. It is a generalization of the Constant Conditional Correlation (CCC) model of Bollerslev (1990), where volatilities are time-varying but conditional correlations are assumed to be constant. The CCC model is, however, too restrictive as it does not take into account the time variation in co-movements of the assets during the periods of economic stability, growth or crises. Engle (2002) and Tse and Tsui (2002) extended the CCC model by allowing the correlation to change over time. Both models are quite similar but, while Tse and Tsui (2002) model the correlation process directly, Engle (2002) specifies the model using a nonlinear transformation of a GARCH-type process to ensure that the resulting process is a sequence of correlation matrices. Engle and Sheppard (2001) provide some theoretical properties for the DCC model, but due to the complexity of the model, a rigorous treatment of the theory is not yet available.

Although the DCC model is easy to implement and is widely used, it is now well accepted that it does not perform well for the case of large dimensions. To a minor extent, the reason is the assumption of the same parameters driving all correlations. Hafner and Franses (2009) extend the DCC model by allowing the parameters to vary across the assets. More importantly, it turned out that in high dimensions the parameter estimates encounter severe negative biases, resulting in increasingly smooth correlation trajectories, which eventually become virtually flat and constant. Engle and Sheppard (2001) recognize the presence of downward bias in the estimated parameters but do not propose solutions.

Aielli (2008) reveals a weak point in the DCC model of Engle (2002). His theoretical computations and results of the simulation study show that the DCC model possesses a significant asymptotic bias in the estimator of the sample covariance matrix which is a constituent of the correlation evolution process. Aielli (2008) proposes a consistent DCC (cDCC) model. He modifies the form of the correlation driving process of the DCC model in such a way that it has martingale difference innovations.

Engle et al. (2008) suggest a composite likelihood estimator based on summing up the quasi-likelihood functions of subsets of assets and thus avoids working with high dimensional matrices. They work with the specification of Aielli (2008) and suppose that the bias problem of the standard DCC model stems from the bias in the covariance targeting parameter. However, we show in this paper that the problem of biased parameter estimates prevails in the specification proposed by Aielli (2008) in high dimensions. This suggests that the problem is genuine to the dimensionality issue.

We claim that the main problem in estimating either the standard DCC model or the modified version of Aielli (2008) in high dimensions is an ill-conditioned estimator of the sample covariance matrix, which is used for covariance targeting. We suggest using the shrinkage technique of Ledoit and Wolf (2004a), which is a solution to obtain a well-conditioned and asymptotically accurate estimator of the covariance matrix. We show that this approach considerably improves the downward bias of the DCC model as well as the cDCC model. Moreover, our estimator is asymptotically efficient, because the shrinkage intensity goes to zero when keeping the dimension fixed and letting the sample size go to infinity. This contrasts the composite likelihood estimator of Engle et al. (2008), which will be less efficient due to the information loss by ignoring the joint likelihood.

This paper is organized as follows. In Section 2 we review the benchmark model, namely the DCC model of Engle (2002) and the related model of Aielli (2008). We discuss the main problems of the parametric estimation of time-invariant correlation and suggest alternative solution, which considerably improves the results of the DCC model for vast dimensions. Section 3 provides the results of a Monte Carlo simulation study. The last section presents an empirical example. Finally, Section 5 gives the conclusions.

## 2 Dynamic conditional correlation model

Let us consider the log-returns $r_t = (r_{1t}, \dots, r_{Kt})'$ for $t = 1, \dots, T$ with $E(r_t|\mathcal{F}_{t-1}) = 0$ and $\text{Var}(r_t|\mathcal{F}_{t-1}) = H_t$, where $\mathcal{F}_{t-1}$ is the information available up to time $t-1$. The objective is to model the time varying behavior of the conditional covariance matrix $H_t$. The Constant Conditional Correlation (CCC) model of Bollerslev (1990) decomposes the conditional covariance as

$$H_t = D_t R D_t, \tag{1}$$

where $D_t = D_t(\theta)$ is the diagonal matrix of time varying volatilities $\sqrt{h_{ii,t}}, i = 1, \dots, K$, parametrized by a vector $\theta$, and $R$ is a $(K \times K)$ constant conditional correlation matrix. The correlation matrix $R$ is usually estimated by the sample correlation matrix of standardized returns $\varepsilon_t = D_t^{-1} r_t$. However, the assumption of time invariant correlation is too restrictive. Engle (2002) extended this model to the more general case of Dynamic Conditional Correlation (DCC) where the conditional correlations are driven by lagged standardized residuals and an autoregressive term. The DCC model is defined as

$$\begin{aligned} H_t &= D_t R_t D_t, && (2)\\ R_t &= \text{diag}\{Q_t\}^{-1/2} Q_t \text{diag}\{Q_t\}^{-1/2}, && (3)\\ Q_t &= (1-\alpha-\beta)S + \alpha\varepsilon_{t-1}\varepsilon_{t-1}' + \beta Q_{t-1}, && (4) \end{aligned}$$

where $R_t = R_t(\phi, \theta)$ is the matrix of time varying conditional correlations with the elements on the diagonal equal to one, $\phi = (\alpha, \beta)$ is the parameter vector and $S$ is the sample covariance matrix of $\varepsilon_t$. It is also easy to show that $R_t$ is the conditional covariance matrix of $\varepsilon_t$.

The estimation of the parameters is done in two steps by quasi-maximum likelihood (QML), assuming that the innovations are Gaussian. The joint log-likelihood of the model can be split into two parts and maximized sequentially. First, the univariate volatilities are modeled for each series of returns. Then, the parameters of the correlation process are estimated. The joint likelihood of the model is

$$\begin{aligned} \mathcal{L}(\theta, \phi) &= -0.5 \sum_{t=1}^{T} \left( K \log(2\pi) + \log|H_t| + r_t' H_t^{-1} r_t \right) && (5)\\ &= \mathcal{L}_v(\theta) + \mathcal{L}_c(\theta, \phi), && (6) \end{aligned}$$

with

$$\mathcal{L}_v(\theta) = -0.5\sum_{t=1}^{T}\left(K\log(2\pi)+\log|D_t|^2+r_t'D_t^{-2}r_t\right), \tag{7}$$

$$\mathcal{L}_c(\theta,\phi) = -0.5\sum_{t=1}^{T}\left(\log|R_t|+\varepsilon_t'R_t^{-1}\varepsilon_t-\varepsilon_t'\varepsilon_t\right). \tag{8}$$

Firstly, we estimate the parameters of the volatility model. The estimate of $\theta$ is

$$\widehat{\theta} = \arg\max_{\theta}\mathcal{L}_v(\theta). \tag{9}$$

Once $\widehat{\theta}$ is obtained, $S$ is estimated as $\frac{1}{T}\sum_{t=1}^{T}\widehat{\varepsilon}_t\widehat{\varepsilon}_t'$, where $\widehat{\varepsilon}_t = D(\widehat{\theta})_t^{-1}r_t$. Thus, we proceed to the second step, where the parameters of the correlation model are estimated:

$$\widehat{\phi} = \arg\max_{\phi}\mathcal{L}_c(\widehat{\theta},\phi). \tag{10}$$

This estimation technique is called "correlation targeting" (Engle, 2009, Ch.11), since $S$ in equation (4) is not estimated as a MLE, but is a moment estimator $\widehat{S}$ and obtained prior to step two. This technique is based on the assumption that $\bar{Q} = \frac{1}{T}\sum_{t=1}^{T}Q_t$ is equal to $\widehat{S}$. However, this assumption cannot be justified as the unconditional expectation of $Q_t$ is not equal to the unconditional expectation of $\varepsilon_t\varepsilon_t'$. Aielli (2008) performed a simulation study where he demonstrated that such model specification produces quite a high bias of $\widehat{S}$. His results show that the estimator of the $S$ matrix is inconsistent.

Aielli (2008) proposes a similar model called consistent DCC (cDCC). He suggests that the $Q_t$ process should take a slightly different form than in (4), namely

$$Q_t = (1-\alpha-\beta)\Psi + \alpha\varepsilon_{t-1}^{*}\varepsilon_{t-1}^{*'} + \beta Q_{t-1}, \tag{11}$$

where $\varepsilon_t^{*} = \text{diag}\{Q_t\}^{1/2}\varepsilon_t$. Here, it is apparent that the corrected standardized residuals have conditional variance $\text{Var}(\varepsilon_t^{*}|\mathcal{F}_{t-1})$ equal to $Q_t$ and the unconditional covariance matrix $E[\varepsilon_t^{*}\varepsilon_t^{*'}]$ equal to $\Psi$. This expectation can be estimated consistently by the sample covariance matrix of $\varepsilon_t^{*}$.

The estimation of the cDCC model is based on the maximization by QML of the same likelihood function as the DCC model. In the first step, the parameter vector $\theta$ of the univariate volatility models is estimated. The parameter vector $\phi$ of the correlation process of the cDCC model is estimated as

$$\widehat{\phi} = \arg\max_{\phi} \mathcal{L}_c(\widehat{\theta}, \widehat{\Psi}(\widehat{\theta}, \phi), \phi). \tag{12}$$

The estimator of $\Psi$ is then $\widehat{\Psi}(\widehat{\theta}, \widehat{\phi}) = \frac{1}{T}\sum_{t=1}^{T} \varepsilon_t^* \varepsilon_t^{*\prime}$.

The cDCC model of Aielli (2008) is elegant and easy to implement. His simulation studies show that there is a considerable bias in the estimator of $S$ for the DCC even when the cross-section size is small. However, it does not solve the problem of the downward bias in $\alpha$ and $\beta$, which is a recognized problem of large systems (Engle and Sheppard, 2001). Engle et al. (2008) suggest a composite likelihood approach to overcome the issue of high dimension. Such an approach allows to estimate models even when the size of the cross-section is larger than the number of observations. The DCC composite likelihood (DCC CL) model is based on subsets of observations, say for example subset of 2 assets $Y_{jt} = \{r_{1j,t} r_{2j,t}\}$ $j = 1, \ldots, N$. There are different ways to select subsets: it could be all possible combinations of pairs of assets ($N = K(K-1)/2$), contiguous pairs ($N = K$) or subsets of pairs selected randomly. The simulation study in Engle et al. (2008) shows that the results for different types of subsets are approximately the same. The conditional variance of $Y_{jt}$ is

$$\mathrm{Var}(Y_{j,t}|\mathcal{F}_{t-1}) = H_{j,t}. \tag{13}$$

The innovations obtained upon the devolatilizing are defined as

$$\varepsilon_{j,t} = \begin{pmatrix} h_{1j,t}^{-1/2} & 0 \\ 0 & h_{2j,t}^{-1/2} \end{pmatrix} \begin{pmatrix} r_{1j,t} \\ r_{2j,t} \end{pmatrix}, \tag{14}$$

so the conditional covariance of $\varepsilon_{j,t}$ is the same as the conditional correlation of $Y_{j,t}$. The dynamics of the correlation process is defined as

$$Q_{j,t} = S_j(1 - \alpha - \beta) + \alpha \varepsilon_{j,t-1} \varepsilon_{j,t-1}^{\prime} + \beta Q_{j,t-1}, \tag{15}$$

where $S_j$ is the sample covariance matrix of $\varepsilon_{j,t}$. The parameter vector $\phi = (\alpha, \beta)$ is estimated as

$$\widehat{\phi} = \arg\max_{\phi} \mathcal{L}_{CL}(\phi), \tag{16}$$

where

$$\mathcal{L}_{CL}(\phi) = \sum_{t=1}^{T} \frac{1}{N} \sum_{j=1}^{N} \log \mathcal{L}_{j,t}(\phi), \tag{17}$$

$$\log \mathcal{L}_{j,t}(\phi) = -0.5 \left( \log |R_{j,t}| + \varepsilon'_{j,t} R_{j,t}^{-1} \varepsilon_{j,t} - \varepsilon'_{j,t} \varepsilon_{j,t} \right), \tag{18}$$

$$R_{j,t} = \text{diag}\{Q_{j,t}\}^{-1/2} Q_{j,t} \text{diag}\{Q_{j,t}\}^{-1/2}. \tag{19}$$

Engle et al. (2008) also extend the estimator to the cDCC case. Their simulation study shows that for these estimates the problem of the downward bias does not exist. The method also has an advantage that it is computationally easy and does not require to invert large dimensional covariance matrices.

Thus, we can see that estimation of the DCC model for large cross-sections leads to two problems, the problem of downward bias of $\alpha$ and $\beta$ and the problem of inconsistent estimator of the sample covariance matrix. The model of Aielli (2008) improves the latter, whereas the method of Engle et al. (2008) combined with the cDCC model, improves both. However, it is important to notice that the estimates of the composite likelihood method are not efficient. We propose a method, which can be applied to both, DCC and cDCC models and improves the estimates for high dimensions.

We propose to use a shrinkage technique to reduce the bias of parameters of the correlation process. We suppose that the root of the problem is the fact that when the number of series is large and approaches the number of observations, which is quite common in financial econometrics, the sample covariance matrix is ill-conditioned. We suggest applying the shrinkage methods of Ledoit and Wolf (2003, 2004a, 2004b) to the DCC model. The main idea of their approach is that the positive (negative) error comes from the extra high (low) coefficients and by pulling them downwards (upwards) one can obtain a well-conditioned sample covariance matrix. By doing so they impose a particular structure, which depends on the *shrinkage target* and *shrinkage intensity* $\delta$. The optimal shrinkage intensity is estimated consistently from the minimization of a loss function. The loss function is based on the Frobenius norm $||\cdot||^2$, basically the quadratic distance between the shrinkage estimator $\Sigma^*$ and the true covariance matrix $\Sigma$ and does not require to invert the covariance matrix.

Ledoit and Wolf (2004a) suggest shrinkage to identity and use $\mu I_K$ as the shrinkage target, where $I_K$ is the identity matrix. Thus, the optimal linear combination is

$$\Sigma_I^* = \delta_I \mu I_K + (1 - \delta_I) S, \tag{20}$$

where $\delta_I$ is the shrinkage intensity and $\mu = \langle \Sigma, I_K \rangle = \text{tr}(\Sigma)/K$ is the Frobenius inner product, which in this case is the mean of the diagonal elements of $\Sigma$. At the estimation stage the true covariance matrix $\Sigma$ is substituted by its sample counterpart $S$.

Ledoit and Wolf (2003) use the covariance matrix given by the single-index model of Sharpe (1963) as shrinkage target, thus they impose a factor structure on the estimator of the sample covariance matrix. This technique is called shrinkage to market. The model of Sharpe (1963) assumes that stock returns $\varepsilon_{it}$ depend on market returns $\varepsilon_{0t}$ as

$$\varepsilon_{it} = a_i + b_i \varepsilon_{0t} + \nu_{it}, \tag{21}$$

where $a_i$ and $b_i$ are parameters. The covariance matrix will be

$$M = \sigma_{00}^2 bb' + \Delta, \tag{22}$$

where $\sigma_{00}^2$ is the variance of the market returns, $b = (b_1, \ldots, b_K)'$, $\nu_{it}$ are the idiosyncratic errors of the model and $\Delta = \text{diag}\{\text{Var}(\nu_{1t}), \ldots, \text{Var}(\nu_{Kt})\}$. Then, the optimal linear combination is

$$\Sigma_M^* = \delta_M M + (1 - \delta_M) S. \tag{23}$$

Finally, Ledoit and Wolf (2004b) propose shrinkage to equicorrelation. By using equicorrelation, they shrink the extreme values of the sample covariance matrix towards the *center*. Let $s_{ij}$ be the elements of the sample covariance matrix $S$. Then a typical element of the correlation matrix is $\rho_{ij} = s_{ij}/\sqrt{s_{ii}s_{jj}}$ and the average correlation is calculated as

$$\bar{\rho} = \frac{2}{(K-1)K} \sum_{i=1}^{K-1} \sum_{j=i+1}^{K} \rho_{ij}. \tag{24}$$

Thus, the elements of the equicorrelation matrix $E$ are $e_{ij} = \bar{\rho}\sqrt{s_{ii}s_{jj}}$. In this case, the optimal linear combination is

$$\Sigma_E^* = \delta_E E + (1 - \delta_E) S. \tag{25}$$

The estimates of the optimal shrinkage intensities $\delta_x,\ x = I, M, E$ are given by

$$\widehat{\delta}_x = \max\left(0, \min\left(1, \frac{\widehat{\kappa}_x}{T}\right)\right). \tag{26}$$

This form keeps the shrinkage intensity in the interval $[0, 1]$. It is important to notice that $\widehat{\delta}_x = O(\frac{1}{T})$, so asymptotically the effect of the shrinkage vanishes.

Hence, for fixed $K$ and $T \to \infty$, our estimator becomes equivalent to the efficient maximum likelihood estimator. The estimated constants $\widehat{\kappa}_x$ are given by

$$\widehat{\kappa}_I = \frac{T^{-1}\sum_{t=1}^{T}\|\varepsilon_t\varepsilon_t' - S\|^2}{\|S - \widehat{\mu} I_K\|^2}, \quad (27)$$

$$\widehat{\kappa}_M = \frac{T^{-1}\sum_{t=1}^{T}\|\varepsilon_t\varepsilon_t' - S\|^2 - \widehat{\Omega}_M}{\|S - M\|^2}, \quad (28)$$

where $\widehat{\Omega}_M = \sum_{i=1}^{K}\sum_{j=1}^{K}\widehat{\text{AsyCov}}\left(\sqrt{T}m_{ij}, \sqrt{T}s_{ij}\right)$ with $m_{ij}$ and $s_{ij}$ being the elements of $M$ and $S$ respectively, and finally

$$\widehat{\kappa}_E = \frac{T^{-1}\sum_{t=1}^{T}\|\varepsilon_t\varepsilon_t' - S\|^2 - \widehat{\Omega}_E}{\|S - E\|^2}, \quad (29)$$

where $\widehat{\Omega}_E = \sum_{i=1}^{K}\sum_{j=1}^{K}\widehat{\text{AsyCov}}\left(\sqrt{T}e_{ij}, \sqrt{T}s_{ij}\right)$.

Implementation of the shrinkage technique to the DCC model is straitforward. In equation (4), the matrix $S$ is replaced by the required shrinkage estimator $\Sigma^*$. The same approach can be used in the cDCC model, where the shrinkage method is applied to the $\widehat{\Psi}$ matrix and the new shrinkage estimator replaces $\Psi$ in equation (11). In the next sections, we name the DCC and cDCC models with shrinkage by adding a letter associated with the shrinkage target: I for Identity (DCCI, cDCCI), M for Market (DCCM,cDCCM) and E for Equicorrelation (DCCE, cDCCE). Obviously, the DCC model with shrinkage is a special case of the standard DCC model, to which it *reduces* when the shrinkage intensity is equal to zero. Thus, the models are nested and can be compared by likelihood-ratio tests.

# 3 Monte Carlo study

In this section we examine the performance of the models described previously. We follow the approach of Engle and Sheppard (2001) and Engle et al. (2008) and estimate the models for different samples, varying the number of observations $T$ and number of assets $K$. We examine the statistical behavior of the estimators, given that the true parameters are known.

For simplicity, let us assume that the data series $r_{jt}, j = 1, \dots, K$ have no trend and that each series comes from a univariate GARCH process

$$
\begin{aligned}
r_{jt} &= \varepsilon_{jt}\sqrt{h_{jt}}, && (30)\\
h_{jt} &= 0.01 + 0.05\varepsilon^2_{j,t-1} + 0.9h_{j,t-1}, && (31)
\end{aligned}
$$

where $\varepsilon_{j,t}|\mathcal{F}_{t-1} \sim N(0,1)$ and $E[\varepsilon_{it}\varepsilon_{jt}|\mathcal{F}_{t-1}] = \rho_{ij,t}$.

Engle et al. (2008) report the results of their simulation study for the parameters of the $Q$ process $(\alpha, \beta) = (0.05, 0.93)$. In order to compare the results, we run the simulation study for the same setup. To simulate from the DCC and cDCC models, we use sample correlation matrices estimated from a real data series of daily S&P500 for the period from 2005 until 2009 (each set includes the index itself and the series are ordered alphabetically).

The performance of the models is examined for dimension $K = 5, 10, 50, 100$ and 200, allowing the time span to be $T = 100, 250, 500, 1000$ and 2000. The simulation study is based on $M = 100$ replications. We also estimate the parameters of the models using the true correlation matrix $S_0$ instead of $\widehat{S}$ for the DCC model (DCCT) and $\Psi_0$ instead of $\widehat{\Psi}$ for cDCC model (cDCCT).

For each data series we simulate $1.5 \cdot T$ observations and then we discard the first $0.5 \cdot T$ observations. For the estimation of the DCC and cDCC parameters, the constraints $\alpha \in [0, 1)$, $\beta \in [0, 1)$ and $\alpha + \beta < 1$ are used. By using these constraints, we guarantee that the matrix $Q_t$ is positive semi-definite.

We present the results for 8 parametric models described in Section 2, namely: DCC, cDCC, DCC and cDCC with shrinkage to the identity matrix (DCCI, cDCCI), to the single-index model (DCCM,cDCCM) and to equicorrelation (DCCE,cDCCE), using the shrinkage methods of Ledoit and Wolf (2003, 2004a, 2004b).

For each DGP and each model we report the root mean squared errors (RMSE) and the bias of the estimated parameters of the correlation process $\widehat{\phi} = (\widehat{\alpha}, \widehat{\beta})$ calculated across the number of replications M

$$RMSE(\widehat{\phi}) = \sqrt{\frac{1}{M}\sum_{m=1}^{M}\left(\widehat{\phi}_m - \phi^0\right)^2}, \tag{32}$$

$$Bias(\widehat{\phi}) = \frac{1}{M}\sum_{m=1}^{M}\left(\widehat{\phi}_m - \phi^0\right), \tag{33}$$

where $\phi^0$ is the true vector of parameters.

The results of the simulation study are presented in Tables 1-4. We also report the mean of the shrinkage intensity for the models with shrinkage in Table 5.

(1) From Table 1 it is seen that the DCC estimator of $\alpha$ possesses high negative bias, which considerably grows with the increase in dimension $K$. This issue is discussed in Engle and Sheppard (2001) and is the main motivation for this research.

(2) The results of the DCC CL model show that for the case of small dimensions and small number of observations, the bias is quite substantial. However, in contrast to the DCC model, the bias of the DCC CL model is similar the same for any size of the cross-section.

(3) The bias of $\alpha_{DCC}$ is small when the number of observations $T$ is relatively high, but when the ratio $K/T$ is high the bias is considerable. This is vividly seen, when comparing with the results of the DCCT, for which the bias of the estimated $\alpha$ stays on about the same level, not exceeding 0.005 in absolute value.

(4) The models DCCI, DCCM, DCCE show substantial improvement of the DCC estimates. The bias for all three models is 2 or 3 times smaller than the bias of DCC. The difference is considerable for small number of observations and vanishes when T increases. The shrinkage techniques improve the estimators even for small cross-sections.

(5) The RMSE of the shrinkage estimators (Table 2) shows the same behavior as the bias and is much smaller for small $T$ than the RMSE of the DCC model.

(6) Table 5 reports the mean of the shrinkage intensities $\delta$ for the DCC and cDCC models with shrinkage. The behavior of the estimators of the shrinkage intensities is quite similar. For fixed number of dimensions and increasing number of observations, the shrinkage intensity decreases. At the same time, for fixed T and increasing dimension, the shrinkage intensities do not fluctuate much. This is also clearly seen in Figure 1 and Figure 2.

Table 3 and Table 4 report the results for cDCC and relevant models. As in the previous case, the shrinkage estimators show considerable improvement to the standard estimator both in terms of bias and RMSE. However, it seems that the more natural specification of the Aielli model does not improve the estimates of $\alpha$ and $\beta$, so that the bias of both the DCC and cDCC models is of comparable size.

Table 1: Results for the simulation study for the DCC model with the true parameter vector $(\alpha, \beta) = (0.05, 0.93)$. The estimated models are: DCC, DCCT, DCC CL, DCCI, DCCM and DCCE. Number of replications M = 100.

**Bias**

| | **DCC** | | **DCCT** | | **DCC CL** | | **DCCI** | | **DCCM** | | **DCCE** | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **K=5** | | | | | | | | | | | |
| **T** | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ |
| **100** | -0.016 | -0.084 | 0.008 | -0.067 | -0.014 | -0.094 | -0.006 | -0.068 | -0.015 | -0.082 | -0.005 | -0.060 |
| **250** | -0.002 | -0.025 | 0.000 | -0.006 | -0.003 | -0.027 | 0.000 | -0.024 | -0.002 | -0.025 | 0.000 | -0.022 |
| **500** | -0.001 | -0.009 | -0.001 | 0.001 | -0.001 | -0.009 | 0.000 | -0.008 | -0.001 | -0.009 | 0.000 | -0.008 |
| **1000** | 0.000 | -0.002 | 0.000 | -0.001 | 0.000 | -0.005 | 0.000 | -0.002 | 0.000 | -0.002 | 0.000 | -0.002 |
| **2000** | 0.000 | -0.001 | 0.001 | -0.001 | 0.001 | -0.002 | 0.000 | -0.001 | 0.000 | -0.001 | 0.000 | -0.001 |
| **T** | **K=10** | | | | | | | | | | | |
| **100** | -0.026 | -0.110 | -0.002 | 0.002 | -0.009 | -0.095 | -0.014 | -0.087 | -0.019 | -0.098 | -0.010 | -0.070 |
| **250** | -0.007 | -0.016 | -0.001 | 0.002 | -0.001 | -0.027 | -0.005 | -0.015 | -0.006 | -0.016 | -0.004 | -0.013 |
| **500** | -0.002 | -0.004 | -0.002 | 0.003 | 0.000 | -0.008 | -0.002 | -0.004 | -0.002 | -0.005 | -0.001 | -0.004 |
| **1000** | -0.001 | -0.001 | -0.001 | 0.001 | 0.000 | -0.003 | -0.001 | -0.001 | -0.001 | -0.001 | -0.001 | -0.001 |
| **2000** | 0.000 | 0.000 | 0.000 | 0.001 | 0.000 | -0.001 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| **T** | **K=50** | | | | | | | | | | | |
| **100** | -0.050 | -0.073 | -0.005 | 0.006 | -0.016 | -0.088 | -0.020 | -0.124 | -0.015 | -0.115 | -0.004 | -0.081 |
| **250** | -0.024 | -0.031 | -0.003 | 0.004 | -0.002 | -0.020 | -0.016 | -0.025 | -0.012 | -0.022 | -0.010 | -0.019 |
| **500** | -0.013 | -0.002 | -0.002 | 0.002 | -0.001 | -0.006 | -0.011 | -0.003 | -0.008 | -0.004 | -0.008 | -0.003 |
| **1000** | -0.008 | 0.003 | -0.001 | 0.001 | 0.000 | -0.003 | -0.007 | 0.003 | -0.006 | 0.002 | -0.006 | 0.002 |
| **2000** | -0.005 | 0.003 | -0.001 | 0.001 | 0.000 | -0.002 | -0.004 | 0.003 | -0.004 | 0.002 | -0.004 | 0.003 |
| **T** | **K=100** | | | | | | | | | | | |
| **100** | | | | | | | | | | | | |
| **250** | -0.040 | -0.114 | -0.004 | 0.005 | -0.004 | -0.018 | -0.021 | -0.057 | -0.013 | -0.038 | -0.010 | -0.032 |
| **500** | -0.022 | -0.010 | -0.002 | 0.003 | -0.001 | -0.006 | -0.017 | -0.010 | -0.012 | -0.009 | -0.011 | -0.008 |
| **1000** | -0.016 | 0.004 | -0.001 | 0.002 | 0.000 | -0.002 | -0.012 | 0.002 | -0.009 | 0.001 | -0.009 | 0.001 |
| **2000** | -0.009 | 0.004 | -0.001 | 0.001 | 0.000 | 0.002 | -0.008 | 0.005 | -0.007 | 0.004 | -0.007 | 0.004 |
| **T** | **K=200** | | | | | | | | | | | |
| **100** | | | | | | | | | | | | |
| **250** | -0.050 | -0.254 | -0.005 | 0.006 | -0.003 | -0.020 | -0.013 | -0.111 | -0.004 | -0.056 | -0.002 | -0.045 |
| **500** | -0.035 | -0.047 | -0.004 | 0.004 | 0.005 | -0.080 | -0.022 | -0.034 | -0.012 | -0.022 | -0.010 | -0.020 |
| **1000** | -0.023 | -0.003 | -0.002 | 0.002 | 0.000 | -0.002 | -0.018 | -0.004 | -0.012 | -0.004 | -0.012 | -0.004 |
| **2000** | -0.015 | 0.003 | -0.001 | 0.001 | 0.001 | -0.001 | -0.013 | 0.003 | -0.003 | -0.023 | -0.010 | 0.002 |

Table 2: Results for the simulation study for the DCC model with the true parameter vector $(\alpha, \beta) = (0.05, 0.93)$. The estimated models are: DCC, DCCT, DCC CL, DCCI, DCCM and DCCE. Number of replications M = 100.

**RMSE**

| | **DCC** | | **DCCT** | | **DCC CL** | | **DCCI** | | **DCCM** | | **DCCE** | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **K=5** | | | | | | | | | | | |
| **T** | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ |
| **100** | 0.026 | 0.095 | 0.047 | 0.224 | 0.028 | 0.101 | 0.022 | 0.085 | 0.026 | 0.094 | 0.021 | 0.079 |
| **250** | 0.010 | 0.036 | 0.017 | 0.041 | 0.012 | 0.041 | 0.010 | 0.034 | 0.010 | 0.036 | 0.010 | 0.033 |
| **500** | 0.006 | 0.016 | 0.012 | 0.020 | 0.010 | 0.021 | 0.006 | 0.015 | 0.006 | 0.016 | 0.006 | 0.015 |
| **1000** | 0.005 | 0.008 | 0.008 | 0.014 | 0.006 | 0.011 | 0.005 | 0.008 | 0.005 | 0.008 | 0.005 | 0.007 |
| **2000** | 0.003 | 0.005 | 0.005 | 0.008 | 0.004 | 0.007 | 0.003 | 0.005 | 0.003 | 0.005 | 0.003 | 0.005 |
| **T** | **K=10** | | | | | | | | | | | |
| **100** | 0.028 | 0.113 | 0.009 | 0.014 | 0.021 | 0.102 | 0.018 | 0.094 | 0.022 | 0.102 | 0.015 | 0.078 |
| **250** | 0.009 | 0.019 | 0.005 | 0.008 | 0.011 | 0.039 | 0.008 | 0.018 | 0.009 | 0.019 | 0.007 | 0.017 |
| **500** | 0.005 | 0.008 | 0.004 | 0.005 | 0.007 | 0.014 | 0.004 | 0.008 | 0.004 | 0.008 | 0.004 | 0.008 |
| **1000** | 0.003 | 0.004 | 0.003 | 0.004 | 0.005 | 0.009 | 0.003 | 0.004 | 0.003 | 0.004 | 0.003 | 0.004 |
| **2000** | 0.002 | 0.003 | 0.002 | 0.003 | 0.003 | 0.005 | 0.002 | 0.003 | 0.002 | 0.003 | 0.002 | 0.003 |
| **T** | **K=50** | | | | | | | | | | | |
| **100** | 0.050 | 0.074 | 0.007 | 0.008 | 0.020 | 0.094 | 0.022 | 0.125 | 0.017 | 0.117 | 0.008 | 0.084 |
| **250** | 0.024 | 0.031 | 0.004 | 0.005 | 0.006 | 0.023 | 0.016 | 0.026 | 0.012 | 0.022 | 0.010 | 0.019 |
| **500** | 0.013 | 0.003 | 0.003 | 0.003 | 0.004 | 0.009 | 0.011 | 0.004 | 0.009 | 0.004 | 0.008 | 0.004 |
| **1000** | 0.008 | 0.004 | 0.002 | 0.002 | 0.003 | 0.006 | 0.007 | 0.003 | 0.006 | 0.002 | 0.006 | 0.002 |
| **2000** | 0.005 | 0.003 | 0.001 | 0.001 | 0.002 | 0.003 | 0.004 | 0.003 | 0.004 | 0.003 | 0.004 | 0.003 |
| **T** | **K=100** | | | | | | | | | | | |
| **100** | | | | | | | | | | | | |
| **250** | 0.040 | 0.115 | 0.005 | 0.006 | 0.007 | 0.020 | 0.021 | 0.058 | 0.013 | 0.038 | 0.010 | 0.033 |
| **500** | 0.022 | 0.010 | 0.003 | 0.003 | 0.003 | 0.008 | 0.017 | 0.010 | 0.012 | 0.009 | 0.011 | 0.009 |
| **1000** | 0.016 | 0.004 | 0.002 | 0.002 | 0.002 | 0.004 | 0.012 | 0.002 | 0.009 | 0.001 | 0.009 | 0.001 |
| **2000** | 0.009 | 0.004 | 0.001 | 0.001 | 0.000 | 0.002 | 0.008 | 0.005 | 0.007 | 0.004 | 0.007 | 0.004 |
| **T** | **K=200** | | | | | | | | | | | |
| **100** | | | | | | | | | | | | |
| **250** | 0.050 | 0.274 | 0.006 | 0.006 | 0.007 | 0.023 | 0.014 | 0.113 | 0.006 | 0.057 | 0.004 | 0.046 |
| **500** | 0.035 | 0.047 | 0.004 | 0.005 | 0.005 | 0.090 | 0.022 | 0.035 | 0.012 | 0.022 | 0.011 | 0.020 |
| **1000** | 0.023 | 0.003 | 0.002 | 0.002 | 0.000 | 0.003 | 0.018 | 0.004 | 0.012 | 0.004 | 0.012 | 0.004 |
| **2000** | 0.015 | 0.003 | 0.001 | 0.002 | 0.001 | 0.001 | 0.013 | 0.003 | 0.021 | 0.078 | 0.010 | 0.003 |

Table 3: Results for the simulation study for the cDCC model with the true parameter vector $(\alpha, \beta) = (0.05, 0.93)$. The estimated models are: cDCC, cDCCT, cDCC CL, cDCCI, cDCCM and cDCCE. Number of replications M = 100.

**Bias**

| | **cDCC** | | **cDCCT** | | **cDCC CL** | | **cDCCI** | | **cDCCM** | | **cDCCE** | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **K=5** | | | | | | | | | | | |
| **T** | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ |
| **100** | -0.021 | -0.088 | 0.008 | -0.067 | -0.020 | -0.079 | -0.012 | -0.068 | -0.020 | -0.089 | -0.010 | -0.057 |
| **250** | -0.002 | -0.026 | 0.000 | -0.006 | -0.001 | -0.025 | -0.001 | -0.023 | -0.002 | -0.026 | -0.001 | -0.020 |
| **500** | -0.001 | -0.006 | -0.001 | 0.001 | -0.002 | -0.008 | 0.000 | -0.006 | -0.001 | -0.006 | 0.000 | -0.006 |
| **1000** | -0.001 | -0.001 | 0.000 | -0.001 | 0.000 | -0.004 | -0.001 | -0.001 | -0.001 | -0.001 | -0.001 | -0.001 |
| **2000** | -0.001 | 0.000 | 0.001 | -0.001 | 0.000 | -0.002 | -0.001 | 0.000 | -0.001 | 0.000 | -0.001 | 0.000 |
| **T** | **K=10** | | | | | | | | | | | |
| **100** | -0.029 | -0.104 | 0.000 | 0.001 | -0.018 | -0.080 | -0.018 | -0.086 | -0.023 | -0.096 | -0.013 | -0.066 |
| **250** | -0.008 | -0.017 | -0.002 | 0.004 | -0.004 | -0.025 | -0.006 | -0.016 | -0.007 | -0.017 | -0.005 | -0.013 |
| **500** | -0.005 | -0.002 | -0.002 | 0.003 | -0.003 | -0.005 | -0.004 | -0.002 | -0.004 | -0.002 | -0.004 | -0.002 |
| **1000** | -0.002 | -0.001 | -0.001 | 0.001 | 0.000 | -0.003 | -0.002 | -0.001 | -0.002 | -0.001 | -0.002 | -0.001 |
| **2000** | -0.001 | 0.000 | 0.000 | 0.000 | 0.000 | -0.002 | -0.001 | 0.000 | -0.001 | 0.000 | -0.001 | 0.000 |
| **T** | **K=50** | | | | | | | | | | | |
| **100** | -0.050 | -0.105 | -0.007 | 0.009 | -0.021 | -0.094 | -0.024 | -0.124 | -0.015 | -0.107 | -0.007 | -0.073 |
| **250** | -0.025 | -0.032 | -0.006 | 0.008 | -0.004 | -0.019 | -0.018 | -0.025 | -0.014 | -0.020 | -0.012 | -0.016 |
| **500** | -0.014 | -0.002 | -0.005 | 0.006 | -0.002 | -0.007 | -0.012 | -0.002 | -0.010 | -0.003 | -0.009 | -0.002 |
| **1000** | -0.010 | 0.004 | -0.003 | 0.004 | -0.001 | -0.002 | -0.009 | 0.003 | -0.008 | 0.003 | -0.007 | 0.003 |
| **2000** | -0.006 | 0.003 | -0.003 | 0.003 | 0.000 | -0.001 | -0.007 | 0.005 | -0.006 | 0.005 | -0.006 | 0.005 |
| **T** | **K=100** | | | | | | | | | | | |
| **100** | | | | | | | | | | | | |
| **250** | -0.039 | -0.112 | -0.012 | -0.074 | -0.005 | -0.017 | -0.019 | -0.059 | -0.012 | -0.037 | -0.010 | -0.030 |
| **500** | -0.022 | -0.010 | -0.007 | 0.008 | -0.002 | -0.006 | -0.017 | -0.010 | -0.013 | -0.009 | -0.011 | -0.008 |
| **1000** | -0.014 | 0.001 | -0.005 | 0.006 | -0.001 | -0.002 | -0.014 | 0.003 | -0.012 | 0.002 | -0.011 | 0.003 |
| **2000** | -0.010 | 0.001 | -0.004 | 0.005 | -0.001 | -0.002 | -0.014 | 0.003 | -0.012 | 0.002 | -0.011 | 0.003 |
| **T** | **K=200** | | | | | | | | | | | |
| **100** | | | | | | | | | | | | |
| **250** | -0.050 | -0.121 | -0.011 | 0.014 | -0.004 | -0.015 | 0.000 | -0.126 | 0.003 | -0.062 | 0.003 | -0.049 |
| **500** | -0.032 | -0.063 | -0.009 | 0.011 | -0.002 | -0.005 | -0.019 | -0.039 | -0.009 | -0.027 | -0.009 | -0.022 |
| **1000** | -0.021 | -0.006 | -0.007 | 0.009 | -0.001 | -0.003 | -0.019 | -0.005 | -0.013 | -0.005 | -0.013 | -0.005 |
| **2000** | -0.015 | 0.003 | -0.003 | 0.005 | 0.000 | -0.001 | -0.014 | 0.004 | -0.012 | 0.003 | -0.012 | 0.003 |

Table 4: Results for the simulation study for the cDCC model with the true parameter vector $(\alpha, \beta) = (0.05, 0.93)$. The estimated models are: cDCC, cDCCT, cDCC CL, cDCCI, cDCCM and cDCCE. Number of replications M = 100.

**RMSE**

| | **cDCC** | | **cDCCT** | | **cDCC CL** | | **cDCCI** | | **cDCCM** | | **cDCCE** | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **K=5** | | | | | | | | | | | |
| **T** | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ |
| **100** | 0.030 | 0.102 | 0.047 | 0.224 | 0.032 | 0.095 | 0.026 | 0.086 | 0.029 | 0.102 | 0.025 | 0.078 |
| **250** | 0.012 | 0.038 | 0.017 | 0.041 | 0.014 | 0.040 | 0.011 | 0.035 | 0.012 | 0.038 | 0.011 | 0.031 |
| **500** | 0.008 | 0.013 | 0.012 | 0.020 | 0.010 | 0.018 | 0.008 | 0.013 | 0.008 | 0.013 | 0.008 | 0.013 |
| **1000** | 0.005 | 0.008 | 0.008 | 0.014 | 0.006 | 0.010 | 0.005 | 0.008 | 0.005 | 0.008 | 0.005 | 0.008 |
| **2000** | 0.004 | 0.006 | 0.005 | 0.008 | 0.004 | 0.075 | 0.004 | 0.006 | 0.004 | 0.006 | 0.004 | 0.006 |
| **T** | **K=10** | | | | | | | | | | | |
| **100** | 0.031 | 0.109 | 0.009 | 0.017 | 0.029 | 0.091 | 0.022 | 0.094 | 0.026 | 0.102 | 0.018 | 0.078 |
| **250** | 0.010 | 0.021 | 0.006 | 0.009 | 0.011 | 0.042 | 0.009 | 0.020 | 0.009 | 0.021 | 0.008 | 0.018 |
| **500** | 0.006 | 0.006 | 0.004 | 0.007 | 0.008 | 0.014 | 0.006 | 0.007 | 0.006 | 0.007 | 0.005 | 0.006 |
| **1000** | 0.003 | 0.004 | 0.003 | 0.004 | 0.006 | 0.009 | 0.003 | 0.004 | 0.003 | 0.004 | 0.003 | 0.004 |
| **2000** | 0.002 | 0.003 | 0.002 | 0.003 | 0.004 | 0.006 | 0.002 | 0.003 | 0.002 | 0.003 | 0.002 | 0.003 |
| **T** | **K=50** | | | | | | | | | | | |
| **100** | 0.050 | 0.118 | 0.008 | 0.011 | 0.025 | 0.100 | 0.026 | 0.125 | 0.017 | 0.110 | 0.012 | 0.079 |
| **250** | 0.025 | 0.032 | 0.007 | 0.008 | 0.009 | 0.024 | 0.018 | 0.026 | 0.014 | 0.021 | 0.012 | 0.017 |
| **500** | 0.015 | 0.003 | 0.005 | 0.006 | 0.005 | 0.010 | 0.012 | 0.004 | 0.010 | 0.004 | 0.009 | 0.004 |
| **1000** | 0.010 | 0.004 | 0.004 | 0.004 | 0.004 | 0.005 | 0.009 | 0.004 | 0.008 | 0.003 | 0.007 | 0.003 |
| **2000** | 0.006 | 0.003 | 0.003 | 0.003 | 0.002 | 0.004 | 0.007 | 0.005 | 0.006 | 0.005 | 0.006 | 0.005 |
| **T** | **K=100** | | | | | | | | | | | |
| **100** | | | | | | | | | | | | |
| **250** | 0.039 | 0.113 | 0.017 | 0.279 | 0.008 | 0.020 | 0.020 | 0.061 | 0.013 | 0.038 | 0.011 | 0.031 |
| **500** | 0.022 | 0.011 | 0.007 | 0.008 | 0.005 | 0.008 | 0.017 | 0.011 | 0.013 | 0.010 | 0.012 | 0.009 |
| **1000** | 0.014 | 0.001 | 0.005 | 0.006 | 0.003 | 0.004 | 0.014 | 0.004 | 0.012 | 0.003 | 0.011 | 0.003 |
| **2000** | 0.010 | 0.001 | 0.004 | 0.005 | 0.003 | 0.004 | 0.014 | 0.004 | 0.012 | 0.003 | 0.011 | 0.003 |
| **T** | **K=200** | | | | | | | | | | | |
| **100** | | | | | | | | | | | | |
| **250** | 0.050 | 0.121 | 0.011 | 0.014 | 0.007 | 0.018 | 0.005 | 0.126 | 0.007 | 0.064 | 0.006 | 0.050 |
| **500** | 0.032 | 0.090 | 0.009 | 0.011 | 0.005 | 0.007 | 0.019 | 0.040 | 0.012 | 0.032 | 0.009 | 0.022 |
| **1000** | 0.021 | 0.006 | 0.007 | 0.009 | 0.001 | 0.004 | 0.019 | 0.005 | 0.014 | 0.005 | 0.013 | 0.005 |
| **2000** | 0.015 | 0.003 | 0.003 | 0.005 | 0.001 | 0.002 | 0.015 | 0.004 | 0.012 | 0.003 | 0.012 | 0.003 |

Table 5: The average shrinkage intensity, estimated for the models: DCCI, DCCM, DCCE, cDCCI, cDCCM and cDCCE.

Shrinkage intensity

| | **DCCI** | **DCCM** | **DCCE** | **cDCCI** | **cDCCM** | **cDCCE** |
|---|---|---|---|---|---|---|
| **T** | **K=5** | | | | | |
| **100** | 0.093 | 0.086 | 0.229 | 0.090 | 0.089 | 0.264 |
| **250** | 0.038 | 0.045 | 0.143 | 0.038 | 0.044 | 0.151 |
| **500** | 0.021 | 0.025 | 0.083 | 0.019 | 0.024 | 0.081 |
| **1000** | 0.010 | 0.014 | 0.050 | 0.010 | 0.013 | 0.047 |
| **2000** | 0.005 | 0.008 | 0.027 | 0.005 | 0.007 | 0.027 |
| **T** | **K=10** | | | | | |
| **100** | 0.090 | 0.165 | 0.226 | 0.088 | 0.180 | 0.246 |
| **250** | 0.043 | 0.099 | 0.131 | 0.040 | 0.094 | 0.129 |
| **500** | 0.023 | 0.066 | 0.088 | 0.022 | 0.065 | 0.087 |
| **1000** | 0.012 | 0.045 | 0.060 | 0.012 | 0.043 | 0.057 |
| **2000** | 0.006 | 0.028 | 0.036 | 0.006 | 0.027 | 0.035 |
| **T** | **K=50** | | | | | |
| **100** | 0.081 | 0.170 | 0.205 | 0.079 | 0.187 | 0.224 |
| **250** | 0.037 | 0.114 | 0.128 | 0.035 | 0.120 | 0.137 |
| **500** | 0.020 | 0.089 | 0.097 | 0.018 | 0.088 | 0.093 |
| **1000** | 0.011 | 0.067 | 0.068 | 0.010 | 0.061 | 0.064 |
| **2000** | 0.005 | 0.046 | 0.044 | 0.006 | 0.045 | 0.042 |
| **T** | **K=100** | | | | | |
| **100** | | | | | | |
| **250** | 0.036 | 0.111 | 0.126 | 0.035 | 0.115 | 0.130 |
| **500** | 0.020 | 0.087 | 0.094 | 0.018 | 0.084 | 0.093 |
| **1000** | 0.010 | 0.065 | 0.066 | 0.009 | 0.067 | 0.065 |
| **2000** | 0.005 | 0.043 | 0.041 | 0.005 | 0.040 | 0.041 |
| **T** | **K=200** | | | | | |
| **100** | | | | | | |
| **250** | 0.036 | 0.108 | 0.123 | 0.035 | 0.116 | 0.131 |
| **500** | 0.019 | 0.085 | 0.093 | 0.019 | 0.083 | 0.094 |
| **1000** | 0.010 | 0.068 | 0.070 | 0.010 | 0.060 | 0.063 |
| **2000** | 0.005 | 0.042 | 0.044 | 0.005 | 0.044 | 0.043 |

Figure 1: Results of the simulation study for DCC models. Bias of estimated parameters $\widehat{\alpha}$ and $\widehat{\beta}$ for the setups with different number of observations $T$ equal to $100, 250, 500, 1000$ and 2000.

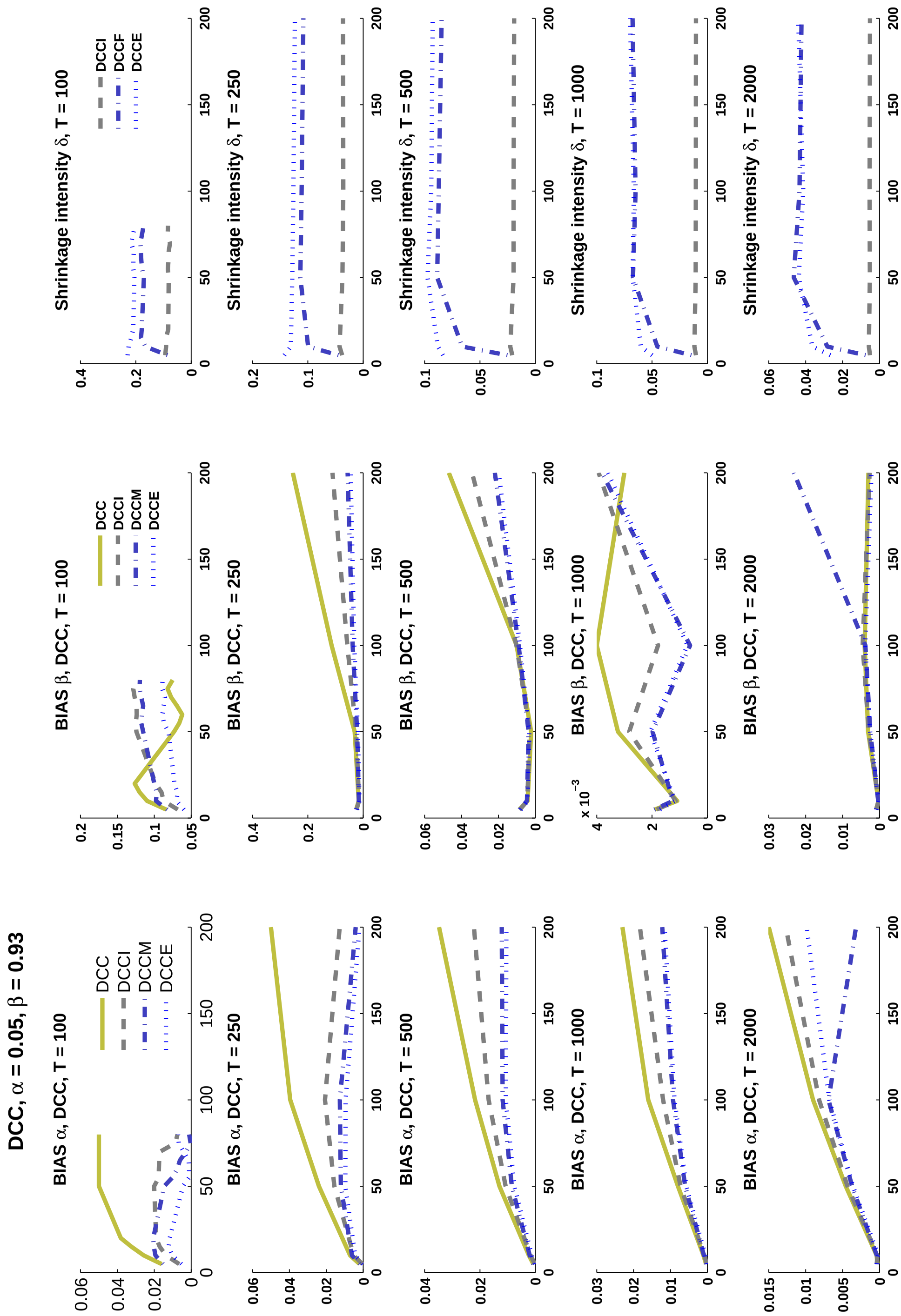
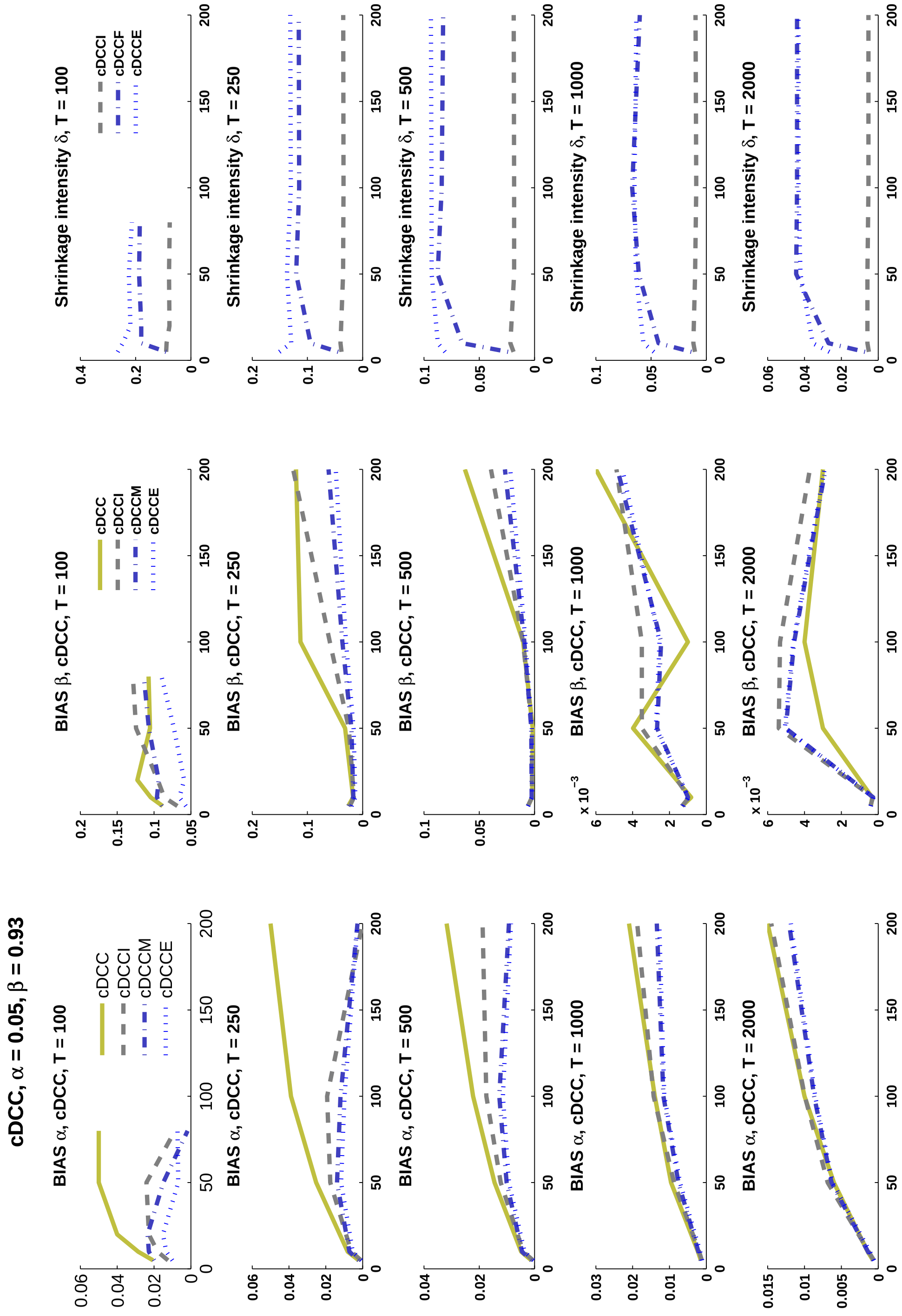


Figure 2: Results of the simulation study for cDCC models. Bias of estimated parameters $\widehat{\alpha}$ and $\widehat{\beta}$ for the setups with different number of observations $T$ equal to $100, 250, 500, 1000$ and 2000.



# 4 Empirical example

The S&P 500 is an index based on the 500 largest stocks in terms of market capitalization, traded in the NYSE. In this section the methods discussed above are applied to the equities which were components of the S&P500 between 8 December 1999 and 7 December 2009. Here we consider weekly log-returns. Thus, the number of observations is $T = 521$. Only those series which are available over the entire period of time are included in the data set. The series are arranged according to the market capitalisation on the last day of the observation period. The data set also includes the index itself. The models were estimated for different sizes of cross-section: $K = \{5, 10, 25, 50, 75, 100\}$. Before fitting the DCC and related models, the data series are tested for the presence of autocorrelation. We used the BIC criterion to select the order $m$ of an $AR(m)$ process. The residuals of the estimated $AR(m)$ are then used to model the volatilities and correlations.

At the first step of the QML, we estimate the univariate volatilities. We fit GARCH$(p, q)$ models and select orders $p$ and $q$ by Engle's LM test. Then the standardized residuals $\widehat{\varepsilon}_t$ are used to estimate the time-varying correlation.

The estimates of the DCC and cDCC type of models are presented in Table 6 and Table 7 respectively. The estimates of $\alpha$ and $\beta$ of the DCC and cDCC model is decreasing considerably with the increase of the size of the cross-sections $K$, which corresponds to the findings of Engle and Sheppard (2001) and Engle et al. (2008). On the contrary, the estimates obtained from the composite likelihood method (DCC CL and cDCC CL) of Engle et al. (2008) do not suffer from the typical problem of negative bias and stay on the relatively stable level. The estimates of $\alpha$ and $\beta$ for the DCC and cDCC models with shrinkage (DCCI, DCCM, DCCE, cDCCI, cDCCM, cDCCE) also decrease with increasing dimension $K$, but less than the standard estimates. In general, the estimated parameters of the cDCC type models are slightly larger than those of the DCC.

As was shown in the simulation study, the estimates of the DCC model with shrinkage also possess a negative bias. However, it is not as large as the bias of the DCC estimates. The estimation results confirm the results of the simulation study, namely that the parameter estimates of the DCC and cDCC models are of comparable magnitude.

Table 6: Estimated parameters for the DCC, DCC CL and DCC models with shrinkage to Identity (I), Market (M) and Equicorrelation (E). Data: S&P500 weekly returns from 8 December 1999 until 2 December 2009. Number of observations $T = 521$. Data are arranged according to the capitalization on the last day of the observation period.

| | **DCC** | | **DCC CL** | | **DCCI** | | **DCCM** | | **DCCE** | |
|---|---|---|---|---|---|---|---|---|---|---|
| **T** | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ |
| **5** | 0.015 | 0.948 | 0.037 | 0.939 | 0.017 | 0.975 | 0.015 | 0.951 | 0.016 | 0.963 |
| **10** | 0.007 | 0.954 | 0.025 | 0.951 | 0.008 | 0.989 | 0.008 | 0.959 | 0.009 | 0.978 |
| **25** | 0.005 | 0.919 | 0.024 | 0.936 | 0.007 | 0.990 | 0.007 | 0.949 | 0.007 | 0.973 |
| **50** | 0.001 | 0.833 | 0.018 | 0.930 | 0.006 | 0.979 | 0.004 | 0.929 | 0.004 | 0.946 |
| **75** | 0.000 | 0.757 | 0.018 | 0.935 | 0.006 | 0.965 | 0.003 | 0.867 | 0.003 | 0.887 |
| **100** | 0.000 | 0.835 | 0.018 | 0.943 | 0.008 | 0.935 | 0.003 | 0.813 | 0.003 | 0.830 |

Table 7: Estimated parameters for the cDCC, cDCC CL and cDCC models with shrinkage to Identity (I), Market (M) and Equicorrelation (E). Data: S&P500 weekly returns from 8 December 1999 until 2 December 2009. Number of observations $T = 521$. Data are arranged according to the capitalization on the last day of the observation period.

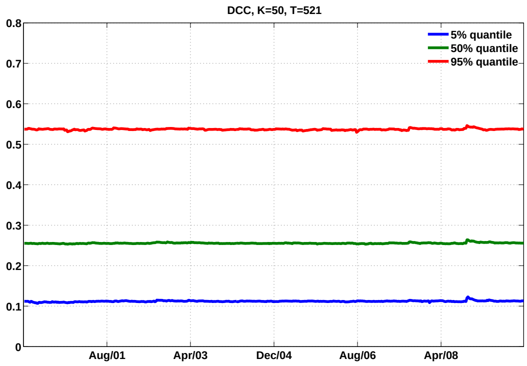
| | **cDCC** | | **DCC CL** | | **cDCCI** | | **cDCCM** | | **cDCCE** | |
|---|---|---|---|---|---|---|---|---|---|---|
| **T** | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ |
| **5** | 0.017 | 0.942 | 0.040 | 0.938 | 0.018 | 0.975 | 0.018 | 0.946 | 0.019 | 0.956 |
| **10** | 0.007 | 0.953 | 0.025 | 0.952 | 0.009 | 0.988 | 0.008 | 0.958 | 0.010 | 0.973 |
| **25** | 0.005 | 0.919 | 0.022 | 0.940 | 0.007 | 0.988 | 0.008 | 0.948 | 0.008 | 0.970 |
| **50** | 0.001 | 0.832 | 0.017 | 0.933 | 0.007 | 0.975 | 0.004 | 0.926 | 0.004 | 0.944 |
| **75** | 0.000 | 0.838 | 0.018 | 0.937 | 0.007 | 0.959 | 0.004 | 0.862 | 0.004 | 0.884 |
| **100** | 0.000 | 0.836 | 0.018 | 0.945 | 0.006 | 0.917 | 0.004 | 0.798 | 0.004 | 0.818 |

Table 8: Shrinkage intensity $\delta_x, x = I, M, E$ for DCC and cDCC models with shrinkage to Identity (I), Market (M) and Equicorrelation (E).

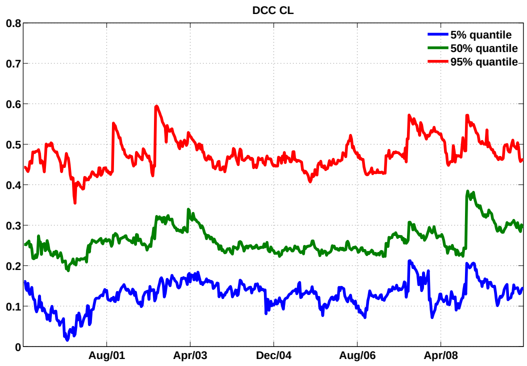
| **K** | **DCCI** | **DCCM** | **DCCE** | **cDCCI** | **cDCCM** | **cDCCE** |
|---|---|---|---|---|---|---|
| **5** | 0.070 | 0.028 | 0.121 | 0.073 | 0.028 | 0.111 |
| **10** | 0.044 | 0.100 | 0.136 | 0.044 | 0.100 | 0.131 |
| **25** | 0.031 | 0.100 | 0.115 | 0.031 | 0.100 | 0.115 |
| **50** | 0.035 | 0.142 | 0.153 | 0.035 | 0.142 | 0.153 |
| **75** | 0.034 | 0.145 | 0.155 | 0.034 | 0.145 | 0.155 |
| **100** | 0.035 | 0.148 | 0.159 | 0.035 | 0.148 | 0.159 |

To compare the results discussed above we plot the pointwise median, 5% and 95% quantiles of the estimated correlation paths for the case of 50 cross-sections (1225 paths). As it was expected, the DCC and cDCC models in Figures 3(a) and 4(a) do not show any variation in the correlation. On the other hand, the DCC CL and cDCC CL models in Figures 3(b) and 4(b) are quite volatile. The correlation paths for DCCI, DCCE, cDCCI and cDCCE models in Figures 3(c,d) and 4(c,d) are much smoother than those for DCC CL and cDCC CL.

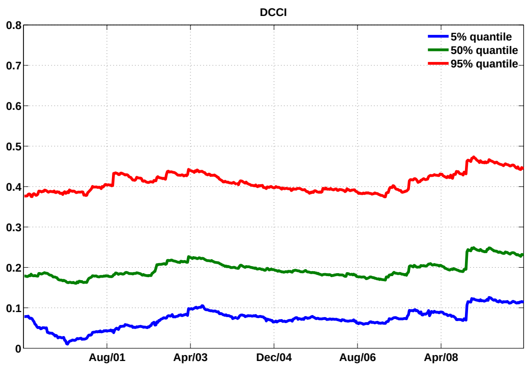
Figure 3: The pointwise median, 5% and 95% quantiles of estimated correlation of the DCC, DCC CL, DCCI and DCCE models. Data: S&P500 weekly returns from 8 December 1999 until 2 December 2009. Number of observations $T = 521$. Data are arranged according to the capitalization on the last day of the observation period.



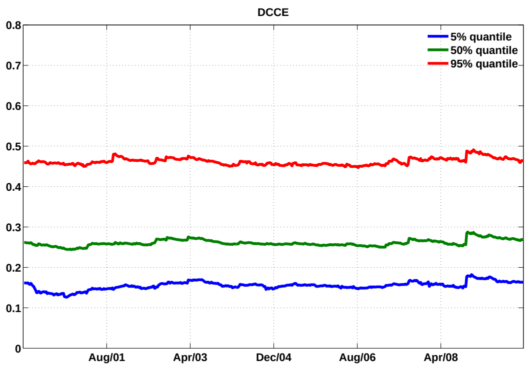

(a) DCC




(b) DCC CL

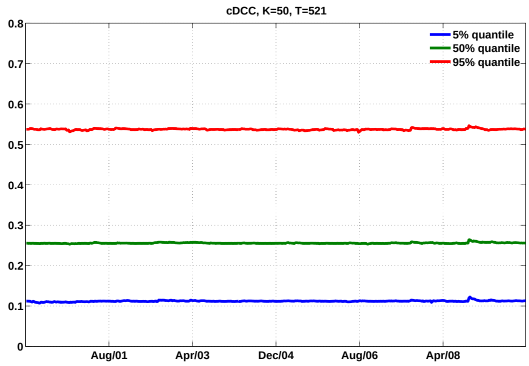



(c) DCCI

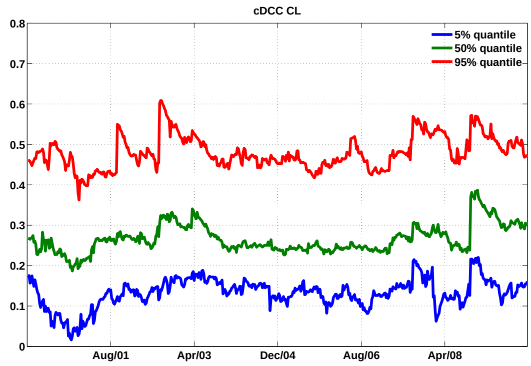



(d) DCCE

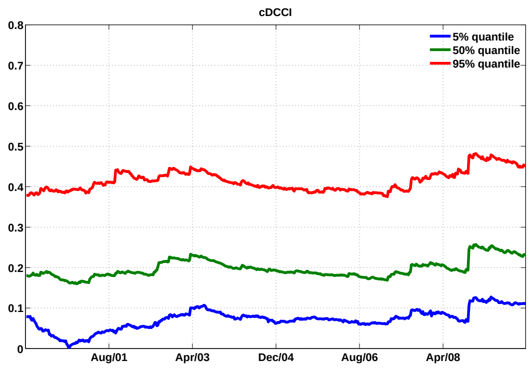
Figure 4: The pointwise median, 5% and 95% quantiles of estimated correlation of the cDCC, cDCC CL, cDCCI and cDCCE models. Data: S&P500 weekly returns from 8 December 1999 until 2 December 2009. Number of observations $T = 521$. Data are arranged according to the capitalization on the last day of the observation period.



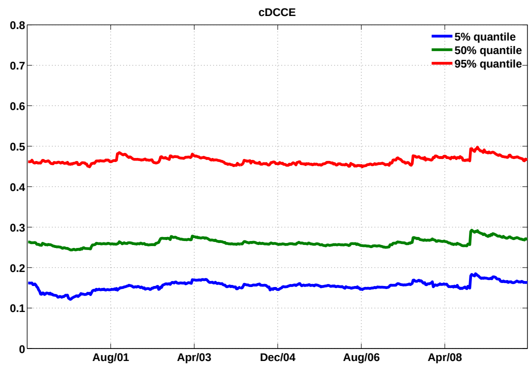

(a) DCC




(b) DCC CL




(c) DCCI




(d) DCCE

Following Hafner and Franses (2009), we apply the Portmanteau test (Lütkepohl, 1993) to check for residual autocorrelation of the standardized returns $\widehat{R}_t^{-1/2}\widehat{\varepsilon}_t$. The modified statistics for the multivariate Portmanteau test is given by

$$P_h = T^2 \sum_{i=1}^{h} (T-i)^{-1} \mathrm{tr}\left(\widehat{C}_i' \widehat{C}_0^{-1} \widehat{C}_i \widehat{C}_0^{-1}\right), \tag{34}$$

where

$$\widehat{C}_i = \frac{1}{T} \sum_{t=i+1}^{T} \widehat{R}_t^{-1/2} \widehat{\varepsilon}_t \left(\widehat{R}_{t-i}^{-1/2} \widehat{\varepsilon}_{t-i}\right)'. \tag{35}$$

The asymptotic distribution of $P_h$ is $\chi^2$ with $hK^2$ degrees of freedom, where $h$ is the number of lags. In Table 9 and Table 10 the results of the Portmanteau test with $h = 10$

for the data with different dimensions $K$ are reported. Small values of the statistics $P_h$ indicate a good fit of the models to the data. For every $K$, the value of the statistics $P_{10}$ is in general higher for DCC than for DCC with shrinkage. $P_{10}$ for DCC CL is the smallest. Among the models with shrinkage, the best fit has the DCCI model. The cDCC type models show similar results. In general, the values of $P_{10}$ for cDCC models are smaller. Eventually, all of the statistics have corresponding p-values smaller than 0.05.

Table 9: Results of the Portmanteau test for for the DCC, DCC CL and DCC models with shrinkage to Identity (I), Market (M) and Equicorrelation (E). Data: S&P500 weekly returns from 8 December 1999 until 2 December 2009. Number of observations $T = 521$. Data are arranged according to the capitalization on the last day of the observation period. The table reports the value of $P_h$ for number of lags $h = 10$ and the corresponding p-value.

| **K** | $h * K^2$ | **DCC** | | **DCC CL** | | **DCCI** | | **DCCM** | | **DCCE** | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $P_{10}$ | Pvalue | $P_{10}$ | Pvalue | $P_{10}$ | Pvalue | $P_{10}$ | Pvalue | $P_{10}$ | Pvalue |
| 5 | 250 | 294 | 0.028 | 290 | 0.041 | 292 | 0.036 | 294 | 0.028 | 293 | 0.031 |
| 10 | 1000 | 1129 | 0.003 | 1071 | 0.058 | 1097 | 0.017 | 1126 | 0.003 | 1113 | 0.007 |
| 25 | 6250 | 6765 | 0.000 | 6691 | 0.000 | 6709 | 0.000 | 6752 | 0.000 | 6742 | 0.000 |
| 50 | 25000 | 26409 | 0.000 | 26132 | 0.000 | 26267 | 0.000 | 26380 | 0.000 | 26373 | 0.000 |
| 75 | 56250 | 58217 | 0.000 | 57605 | 0.000 | 58096 | 0.000 | 58180 | 0.000 | 58175 | 0.000 |
| 100 | 100000 | 103121 | 0.000 | 102218 | 0.000 | 102982 | 0.000 | 103067 | 0.000 | 103059 | 0.000 |

Table 10: Results of the Portmanteau test for for the cDCC, cDCC CL and cDCC models with shrinkage to Identity (I), Market (M) and Equicorrelation (E). Data: S&P500 weekly returns from 8 December 1999 until 2 December 2009. Number of observations $T = 521$. Data are arranged according to the capitalization on the last day of the observation period. The table reports the value of $P_h$ for number of lags $h = 10$ and the corresponding p-value.

| **K** | $h * K^2$ | **cDCC** | | **DCC CL** | | **cDCCI** | | **cDCCM** | | **cDCCE** | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $P_{10}$ | Pvalue | $P_{10}$ | Pvalue | $P_{10}$ | Pvalue | $P_{10}$ | Pvalue | $P_{10}$ | Pvalue |
| 5 | 250 | 294 | 0.029 | 294 | 0.030 | 305 | 0.010 | 295 | 0.028 | 295 | 0.027 |
| 10 | 1000 | 1128 | 0.003 | 1138 | 0.002 | 1110 | 0.008 | 1125 | 0.004 | 1114 | 0.007 |
| 25 | 6250 | 6765 | 0.000 | 6714 | 0.000 | 6728 | 0.000 | 6751 | 0.000 | 6742 | 0.000 |
| 50 | 25000 | 26409 | 0.000 | 26126 | 0.000 | 26281 | 0.000 | 26375 | 0.000 | 26369 | 0.000 |
| 75 | 56250 | 58219 | 0.000 | 57590 | 0.000 | 58104 | 0.000 | 58173 | 0.000 | 58168 | 0.000 |
| 100 | 100000 | 103121 | 0.000 | 102243 | 0.000 | 102973 | 0.000 | 103053 | 0.000 | 103043 | 0.000 |

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