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CLUSTERING RATES AND CHUNG TYPE LAWS OF THE ITERATED LOGARITHM FOR EMPIRICAL AND QUANTILE PROCESSES

D. VARRON

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Clustering rates and Chung type laws of the iterated logarithm for empirical and quantile processes

Davit Varron

Laboratoire de Statistiques et Modélisation

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Following the works of Berthet [1, 2], we first obtain exact clustering rates for the functional law of the iterated logarithm (FLIL) for empirical processes and for their increments (see [3, 4]). In a second time,we obtain functional Chung-type limit laws for the local empirical process in the spirit of Deheuvels [5].

Keywords: Empirical processes, Laws of the iterated Logarithm, Clustering rates, Chung-Mogulskii limit laws.

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1 Introduction

Define the uniform empirical process by $\alpha_n(t) := n^{1/2}(F_n(t) - t)$, where $F_n(t) := n^{-1}\sharp\{i :, U_i \leq t\},\ t \in [0,1]$, and $(U_i)_{n\geq 1}$ are i.i.d random variables

Laboratoire de Statistiques et Modélisation, ENSAI, 6 rue Blaise Pascal, 35172 Bruz Cedex, France

Tel. (+33) 2 99 05 32 76, Fax: (+33) 2 99 05 32 05, e-mail: *varron@ensai.fr*

uniformly distributed on $[0,1]$. Define the quantile process by

$$\beta_n(t) = n^{1/2}\Big(F_n^{-1}(t) - t\Big),\ t \in [0,1],$$

where $F_n^{-1}(t) := \inf\{u :\ F_n(u) \geq t\}$. These processes have been extensively investigated in the literature (see, e.g., [6] and [7] and the references therein). In a pioneering work, Finkelstein [3] has established the functional law of the iterated logarithm (FLIL) for α_n. Namely, the author showed that, almost surely,

$$\frac{\alpha_n}{b_n} \rightsquigarrow \mathcal{S}_2. \tag{1}$$

Here, we set $b_n = \sqrt{2\log_2 n}$ and $\log_2 u = \log(\log u \vee e)$. In a metric space $(\mathcal{E}, d)$ we set $u_n \rightsquigarrow \mathcal{H}$ whenever u_n is relatively compact with limit set $\mathcal{H}$ (see, e.g., [4]). In (1), we set $(\mathcal{E}, d) := (B[0,1], ||\cdot||)$ where $B[0,1]$ stands for the set of bounded functions on $[0,1]$ and $||\cdot||$ is the sup-norm over $[0,1]$. The set $\mathcal{S}_2$ in (1) is given by

$$\mathcal{S}_2 := \Big\{f(t) \in \mathcal{S}_1,\ f(1) = 0\Big\}, \tag{2}$$

where

$$\mathcal{S}_1 := \Big\{f(t) := \int_0^t f'(t)dt,\ \int_0^1 f'^2(t)dt \leq 1\Big\}. \tag{3}$$

Note that $\mathcal{S}_2$ (resp. $\mathcal{S}_1$) is the unit ball of the reproducing kernel Hilbert space of the Brownian bridge (resp. of the Wiener process on [0,1]). In the spirit of [3], Mason [4] has obtained the following FLIL for the local empirical process:

$$\frac{\alpha_n(a_n\cdot)}{\sqrt{a_n}b_n} \rightsquigarrow_{a.s.} \mathcal{S}_1. \tag{4}$$

Here, a_n is a sequence of constants satisfying $a_n \downarrow 0$, $na_n \uparrow \infty$ and $na_n/\log_2 n \to \infty$. Deheuvels and Mason [8] have established a related uniform functional limit law for the following sets of random trajectories.

$$\Theta_n := \Big\{\frac{\alpha_n(t+a_n\cdot)-\alpha_n(t)}{\sqrt{2a_n\log(1/a_n)}},\ t\in[0,1-a_n]\Big\}.$$

They showed that, with probability one, we have

$$\lim_{n\to\infty}\sup_{g_n\in\Theta_n}\inf_{f\in\mathcal{S}_1} || g_n - f ||= 0,$$

$$\lim_{n\to\infty}\sup_{f\in\mathcal{S}_1}\inf_{g_n\in\Theta_n} || g_n - f ||= 0, \tag{5}$$

where a_n is a sequence of constants fulfilling $a_n \downarrow 0$, $na_n \uparrow \infty$, $na_n/\log n \to \infty$, $\log(1/a_n)/\log_2 n \to \infty$. Berthet [1] refined (5) under stronger conditions imposed upon a_n. Making use of sharp upper bounds for Gaussian measures (refer to Talagrand [9]), he proved that for some suitable $\epsilon > \epsilon_0$ (where ϵ_0 is a universal constant), we have almost surely for all n large enough

$$\Theta_n \subset \mathcal{S}_1 + \epsilon\log(1/a_n)^{-2/3}\mathcal{B}_0. \tag{6}$$

Here $\mathcal{B}_0 := \{f \in B[0,1] :|| f ||\leq 1\}$. The first aim of the present article is to extend the just-mentioned result of Berthet [1] to some other random objects than that used for that given in 6 (see theorem 2.1 and 2.2 in the sequel). Results of this kind are usually called *clustering rates*, as they give us the follwing information: results (1) and (4) assert that sequences of random trajectories are asymptotically attracted by a given cluster set. Our results give the rates of these attraction phenomenon. Another related problem is finding rates of convergence of such random sequences to a specified function belonging to the cluster set. Such results are known under the name of functional *Chung-type limit laws*. We now focus on the local empirical process $\alpha_n(a_n \cdot)$, where $a_n \downarrow 0$ as $n \to \infty$. The works of Csáki [10], de Acosta [11], Grill [12], Gorn and Lifshits [13], and Berthet and Lifshits [14] on small ball probabilities for Wiener processes provide some crucial tools to establish such limit laws for $(\alpha_n(a_n \cdot))_{n \geq 1}$, as these are expected to asymptotically mimic Wiener processes (see Mason [4]). Along this line, Deheuvels [5] has given Chung-type limit laws for $(\alpha_n(a_n \cdot))_{n \geq 1}$, by showing that if a_n is a sequence of constants satisfying $na_n \uparrow \infty$, $a_n \downarrow 0$ and $na_n/(\log_2 n)^3 \to \infty$, we have almost surely, for each $f \in \mathcal{S}_1$ satisfying $\| f \|_H^2 := \int_0^1 f'^2(t)dt < 1$

$$\liminf_{n\to\infty} (\log_2 n) \Big|\Big| \frac{\alpha_n(a_n \cdot)}{\sqrt{a_n} b_n} - f \Big|\Big| = \frac{\pi}{4\sqrt{1 - \| f \|_H}}.$$

The proof of this theorem relies on strong approximation methods in combination with the results of de Acosta [11]. The latter provides useful exponential bounds for

$$\mathbb{P}\Big(\Big|\Big| \frac{W}{T} - f \Big|\Big| \leq \epsilon \Big),$$

with a small $\epsilon > 0$ and a large T. Here, W is a Wiener process on $[0,1]$ and f satisfies $\| f \|_H^2 < 1$. The study of related probabilities when $\| f \|_H = 1$ has required different arguments. In [12], rough estimates are given covering this case. In [13] and [14], some exact rates are given for functions with first derivatives having a variation either bounded or locally infinite. The sets of all functions of this type are called respectively $\mathcal{S}_1^{bv}$ and $\mathcal{S}_1^{liv}$. In the present paper, we shall make use of the latter results to extend the work of [5], to the case where f belongs to $\mathcal{S}_1^{bv} \cup \mathcal{S}_1^{liv}$. The remainder of our paper is organized as follows. Our main results are stated in §2 in theorems 2.1, 2.2 and 2.3. In §3, the proofs of these theorems 2.1, 2.2 and 2.3 are provided.

2 Main Results

Our first result gives a refinement of Finkelstein's FLIL [3].

THEOREM 2.1 *There exists a universal constant $\mathcal{L}_0$ >0 such that, for any choice of $\epsilon > \mathcal{L}_0$ we have almost surely, for all large n*

$$\frac{\alpha_n}{(2\log_2 n)^{1/2}} \in \mathcal{S}_2 + \epsilon(\log_2 n)^{-2/3}\mathcal{B}_0, \tag{7}$$

$$\frac{\beta_n}{(2\log_2 n)^{1/2}} \in \mathcal{S}_2 + \epsilon(\log_2 n)^{-2/3}\mathcal{B}_0. \tag{8}$$

Remark 1 The uniform Bahadur-Kiefer representation (see [15]) asserts that almost surely

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log_2 n)^{-1/4} \mid\mid \alpha_n + \beta_n \mid\mid = 2^{-1/4},$$

from where (8) is readily implied by (7).

Our second theorem concerns the FLIL for local increments of the empirical process.

THEOREM 2.2 *Let a_n be positive real numbers satisfying, as $n \to \infty$,*

$$na_n \uparrow \infty, \quad \frac{na_n}{(\log_2 n)^{7/3}} \to \infty, \quad a_n \downarrow 0. \tag{9}$$

Then there exists a universal constant $L > 0$ such that, for any choice of $\epsilon > L$ we have almost surely, for all large n,

$$\frac{\alpha_n(a_n \cdot)}{\sqrt{2a_n \log_2 n}} \in \mathcal{S}_1 + \epsilon (\log_2 n)^{-2/3} \mathcal{B}_0. \tag{10}$$

If moreover $na_n/(\log_2 n)^{11/3} \to \infty$ then we have almost surely, ultimately as $n \to \infty$,

$$\frac{\beta_n(a_n \cdot)}{\sqrt{2a_n \log_2 n}} \in \mathcal{S}_1 + \epsilon (\log_2 n)^{-2/3} \mathcal{B}_0. \tag{11}$$

Remark 2 We shall use the fact (see e.g. [16]) that under (9), we have almost surely

$$\limsup_{n\to\infty} (n/a_n)^{1/4} (\log_2 n)^{-1/4} (2\log_2 n + \log(na_n))^{-1/2} \parallel \alpha_n(a_n \cdot) + \beta_n(a_n \cdot) \parallel \le 2^{-1/4}.$$

from where (11) is implied by (10) after straightforward computations.

In order to state our last result, we need to give some definitionss. Recall that $f \in \mathcal{S}_1^{bv}$ whenever f' has a derivative with bounded variation and $\int_0^1 f'^2(t)dt = 1$. Resulults on small ball probabilities for a Wiener process when $f \in \mathcal{S}_1^{bv}$ have been established by Gorn and Lifshits [13]. For such a function f, we set $\nabla_f(L) := L^{2/3},\ L > 0$ and we denote by χ_f the constant which is the unique solution of equation (3.1) in [13] (we refer to the just mentioned paper for more details. The case where $f \in \mathcal{S}_1^{liv}$ (i.e. where the derivative of f' admits a version with locally infinite variation) has been treated by Berthet and Lifshits [14].

For such a function f, we set $\chi_f := 1$ and we denote by $\nabla_f(L)$ the unique solution of equation (2.1) in [2]. Note that $\nabla_f(L)$ is still a function of L, as equation (2.1) in [2] depends on the parameter L. Our third result is stated as follows.

THEOREM 2.3 *Let $f \in \mathcal{S}_1^{bv} \cup \mathcal{S}_1^{liv}$ be arbitrary and let a_n be a sequence of real numbers satisfying, as $n \to \infty$,*

$$na_n \uparrow \infty,\ a_n \downarrow 0,\ a_n \log_2 n \to 0, \tag{12}$$

$$\lim_{n\to\infty} \frac{na_n}{\log_2 n \nabla_f^2(\log_2 n)} = \infty. \tag{13}$$

Then we have almost surely

$$\liminf_{n\to\infty} \nabla_f(\log_2 n) \mid\mid \frac{\alpha_n(a_n\cdot)}{\sqrt{2a_n \log_2 n}} - f \mid\mid = \chi_f.$$

Remark 3 The conditions (12) and (13) imposed upon a_n turn out to be the best possible with respect to the methods used in the proof of theorem 2.3. The latter combines Poissonization techniques with strong approximation arguments. In recent papers Deheuvels and Lifshits [17] and Shmileva [18] have provided new tools to estimate probabilities of shifted small balls for a Poisson process without making use of strong approximation techniques. These results show up to be powerful enough to investigate Chung-Mogulskii limit laws for $\alpha_n(a_n.)$ without making use of strong approximation techniques, and thus relaxing condition (12). However, the results of Shmileva do not cover the case where $f \in \mathcal{S}_1^{liv}$ at this time.

3 Proofs

3.1 *Proof of Theorem 2.1*

Select an $\epsilon > 0$ and consider the sequence $\epsilon_n := \epsilon(\log_2 n)^{-2/3}$. The main tool to achieve our goal is the exponential inequality stated in the following fact, which follows directly from Talagrand [19]. Recall that $\mathcal{B}_0$ is the unit ball for $|| \cdot ||$.

FACT 3.1 Let B be a Brownian bridge. There exists three constants K, L_0 and u_0 >0 such that, for any $0 < u < u_0$ and $c > 0$, we have

$$P\big(B \notin c\mathcal{S}_2 + u\mathcal{B}_0\big) \leq K \exp\Big(\frac{L_0}{u^2} - \frac{cu}{2} - \frac{c^2}{2}\Big). \tag{14}$$

Let W be a Wiener process on $[0, 1]$. There exist two constants u_1 and L_1 such that, for any $0 < u < u_1$ and $c > 0$, we have

$$P\big(W \notin c\mathcal{S}_1 + u\mathcal{B}_0\big) \leq \exp\Big(\frac{L_1}{u^2} - \frac{cu}{2} - \frac{c^2}{2}\Big). \tag{15}$$

In the proof of theorem 2.1, we will make use of blocking techniques (see, e.g., [8] and [1]). For any real umber a, set $[a]$ as the unique integer q fulfilling $q \leq a < a + 1$, and set

$$n_k := \Big[\exp\Big(k \exp\Big(-(\log k)^{1/6}\Big)\Big)\Big],\ k \geq 1.$$

Set $N_k := \{n_k, \ldots, n_{k+1} - 1\}$ for $k \geq 5$. Given an integer $n \geq 1$, we set $k(n)$ as the unique integer k such that $n \in N_k$. We shall first study the following sequence of functions

$$g_n := (n_{k+1})^{-1/2} b_{n_{k+1}}^{-1} H_n,\ k = k(n),$$

with $H_n(t) := n(F_n(t) - t)$ and $b_n := (2\log_2 n)^{1/2}$. Let p_1 and q_1 be two conjugates numbers (such that $1/p_1 + 1/q_1 = 1$) with $p_1 > 1$. Set, for $k \geq 1$,

$$m_{p_1,k} := \min_{n \in N_k} \mathbb{P}\Big(\frac{1}{(n_{k+1})^{1/2} b_{n_{k+1}}} \mid\mid H_{n_{k+1}} - H_n \mid\mid \leq \frac{1}{p_1} \epsilon_{n_{k+1}} \Big).$$

A standard blocking argument based upon Ottaviani's lemma (see, e.g., [8]) yields

$$\begin{aligned} &\mathbb{P}\Big(\bigcup_{n \in N_k} \Big\{ \frac{1}{(n_{k+1})^{1/2} b_{n_{k+1}}} H_n \notin \mathcal{S}_2 + \epsilon_{n_{k+1}} \mathcal{B}_0 \Big\} \Big) \\ &\leq \frac{1}{m_{p_1,k}} \mathbb{P}\Big(\frac{1}{(n_{k+1})^{1/2} b_{n_{k+1}}} H_{n_{k+1}} \notin \mathcal{S}_2 + \frac{1}{q_1} \epsilon_{n_{k+1}} \mathcal{B}_0 \Big). \end{aligned}$$

Let k be integer and select $n \in N_k$. By the Dvoretsky-Kiefer-Wolfowitz inequality (see e.g. [20]) we have,

$$\begin{aligned} &\mathbb{P}\Big(\frac{1}{(n_{k+1})^{1/2} b_{n_{k+1}}} \mid\mid H_{n_{k+1}} - H_n \mid\mid \geq \frac{1}{p_1} \epsilon_{n_{k+1}} \Big) \\ &\leq \mathbb{P}\Big(\mid\mid \alpha_{n_{k+1}-n} \mid\mid \geq \frac{1}{p_1} \epsilon_{n_{k+1}} \big(\frac{1}{1 - \frac{n_k}{n_{k+1}}} \big)^{1/2} b_{n_{k+1}} \Big) \\ &\leq 2 \exp\Big(- \frac{\epsilon}{p_1}^{2} \Big((\log k)^{-1/3} \exp((\log k)^{1/6}) \big(1 - \frac{n_k}{n_{k+1}}\big) \Big) \Big), \end{aligned}$$

whence $m_{p_1,k} \geq 1/2$ for all large k, since $n_{k+1}/n_k \to 1$ as $k \to \infty$. Now let $p_2, q_2 > 1$ be two conjugate numbers $(1/p_2 + 1/q_2 = 1)$. For $k \geq 1$ we have,

$B_{n_{k+1}}$ denoting a Brownian bridge,

$$\mathbb{P}\Big(\frac{\alpha_{n_{k+1}}}{b_{n_{k+1}}} \notin \mathcal{S}_2 + \frac{1}{q_1}\epsilon_{n_{k+1}}\mathcal{B}_0\Big) \leq \mathbb{P}\Big(\| \alpha_{n_{k+1}} - B_{n_{k+1}} \| \geq \frac{1}{p_2 q_1}\epsilon_{n_{k+1}} b_{n_{k+1}}\Big)$$
$$+ \mathbb{P}\Big(B_{n_{k+1}} \notin b_{n_{k+1}}\mathcal{S}_2 + \frac{1}{q_2 q_1}\epsilon_{n_{k+1}} b_{n_{k+1}}\mathcal{B}_0\Big)$$
$$:= \mathbb{P}_k^{KMT} + \mathbb{P}_k^{Tal}.$$

Making use of the KMT approximation (see, e.g., [21])), we can choose a sequence $(B_{n_k})_{k\geq 1}$ satisfying, for some universal constants C_2, C_3 and for all k large enough,

$$\mathbb{P}_k^{KMT} \leq C_2 \exp\Big(- C_3 (n_{k+1})^{1/2} \frac{1}{2 p_2 q_1}\epsilon_{n_{k+1}} b_{n_{k+1}}\Big).$$

On the other hand, by applying fact (14) we have, for all large k,

$$\mathbb{P}_k^{Tal} \leq K \exp\Big[- \Big(\frac{\epsilon}{q_1 q_2} - \frac{L_0 (q_1 q_2)^2}{2\epsilon^2}\Big)(\log_2 n_{k+1})^{1/3} - \log_2 n_{k+1}\Big].$$

Routine analysis shows that both $\mathbb{P}_k^{KMT}$ and $\mathbb{P}_k^{Tal}$ are summable in k for any choice of $\epsilon > (L_0/2)^{1/3} =: \mathcal{L}_0$, provided that $q_1 > 1$ is chosen small enough. Now an application of (1) in combination with trivial properties of $(n_k)_{k\geq 1}$ shows that, almost surely, as $n \to \infty$,

$$\| g_n - b_n^{-1}\alpha_n \| = o((\log_2 n)^{-2/3}).$$

3.2 *Proof of Theorem 2.2*

Recall that $b_n := (2\log_2 n)^{1/2},\ n \geq 1$. Let $p_1, q_1 > 1$ be two conjugate numbers. Let $k \geq 5$ be an integer and, select $n \in N_k$. Set

$$m_{k,p_1} := \inf_{n \in N_k} \mathbb{P}\Big(\frac{1}{(n_{k+1}a_{n_{k+1}})^{1/2}b_{n_{k+1}}} \mid\mid H_n(a_{n_{k+1}}\cdot) - H_{n_{k+1}}(a_{n_{k+1}}\cdot) \mid\mid \leq \frac{1}{p_1}\epsilon_{n_{k+1}}\Big).$$

A standard blocking argument (see, e.g., [8], lemma 3.4) yields

$$\mathbb{P}\Big(\bigcup_{n \in N_k}\Big\{\frac{H_n(a_{n_{k+1}}\cdot)}{(n_{k+1}a_{n_{k+1}})^{1/2}b_{n_{k+1}}} \notin \mathcal{S}_1 + \epsilon_{n_{k+1}}\Big\}\Big)$$

$$\leq \frac{1}{m_{k,p_1}}\mathbb{P}\Big(\frac{H_{n_{k+1}}(a_{n_{k+1}}\cdot)}{(n_{k+1}a_{n_{k+1}})^{1/2}b_{n_{k+1}}} \notin \mathcal{S}_1 + \frac{1}{q_1}\epsilon_{n_{k+1}}\Big).$$

Now, for any integer $k \geq 5$ and $n \in N_k$, we have

$$\mathbb{P}\Big(\frac{1}{\sqrt{n_{k+1}-n}} \mid\mid H_{n_{k+1}}(a_{n_{k+1}}\cdot) - H_n(a_{n_{k+1}}\cdot) \mid\mid \geq \frac{1}{p_1}\epsilon_{n_{k+1}}b_{n_{k+1}}\big(\frac{n_{k+1}a_{n_{k+1}}}{n_{k+1}-n}\big)^{1/2}\Big)$$
$$\leq \mathbb{P}\Big(\sup_{t \leq a_{n_{k+1}}} \frac{\mid \alpha_{n_{k+1}-n}(t) \mid}{1-t} \geq \frac{1}{p_1}\epsilon_{n_{k+1}}b_{n_{k+1}}\big(\frac{n_{k+1}a_{n_{k+1}}}{n_{k+1}-n_k}\big)^{1/2}\Big).$$

It is well known (see, e.g., [6], proposition 1, p. 133) that for each n, the process $(1-t)^{-1}\alpha_n(t)$ is a martingale in t. The Doob-Kolmogorov inequality yields

$$1 - m_{k,p_1} \leq \frac{p_1^2(1-a_{n_{k+1}})(1-\frac{n_k}{n_{k+1}})(\log n_{k+1})^{1/3}}{2\epsilon^2}.$$

Hence for all k large enough we have $m_{k,p_1} \geq 1/2$. Now set for each integer $n \geq 1$

$$\widetilde{\Pi}_n(t) := n^{-1/2}\Big(\sum_{i=1}^{\eta_n} 1_{\{U_i \leq t\}} - t\Big),\ t \in [0,1],$$

where η_n is a Poisson variable with expectation equal to n and independent of $(U_i)_{i\geq 1}$. Let $\widetilde{\Pi}$ denote a standard centered Poisson process on $\mathbb{R}^+$ and let W be a Wiener process that we assume to be constructed on the same underlying probability space as $\widetilde{\Pi}$. Notice that $\widetilde{\Pi}_n(\cdot)$ and $n^{-1/2}\widetilde{\Pi}(n\cdot)$ are equal in distribution as processes on $[0,1]$. Now let $p_2, q_2 > 1$ be two conjugate numbers. By making use of Poissonization techniques (see, e.g., [8]) we see that for all sufficiently large k,

$$\begin{aligned}
&\mathbb{P}\Big(\frac{\alpha_{n_{k+1}}(a_{n_{k+1}}\cdot)}{a_{n_{k+1}}^{1/2}b_{n_{k+1}}} \notin \mathcal{S}_1 + \frac{1}{q_1}\epsilon_{n_{k+1}}\mathcal{B}_0\Big)\\
&\leq 2\mathbb{P}\Big(\frac{\widetilde{\Pi}_{n_{k+1}}(a_{n_{k+1}}\cdot)}{a_{n_{k+1}}^{1/2}b_{n_{k+1}}} \notin \mathcal{S}_1 + \frac{1}{q_1}\epsilon_{n_{k+1}}\mathcal{B}_0\Big)\\
&= 2\mathbb{P}\Big(\frac{\widetilde{\Pi}(n_{k+1}a_{n_{k+1}}\cdot)}{(2n_{k+1}a_{n_{k+1}}\log_2 n_{k+1})^{1/2}} \notin \mathcal{S}_1 + \frac{1}{q_1}\epsilon_{n_{k+1}}\mathcal{B}_0\Big)\\
&\leq 2\mathbb{P}\Big(\,||\, W(n_{k+1}a_{n_{k+1}}\cdot) - \widetilde{\Pi}(n_{k+1}a_{n_{k+1}}\cdot)\,|| \geq \frac{1}{q_1p_2}(n_{k+1}a_{n_{k+1}})^{1/2}b_{n_{k+1}}\epsilon_{n_{k+1}}\Big)\\
&\quad + 2\mathbb{P}\Big(\frac{W(n_{k+1}a_{n_{k+1}}\cdot)}{(n_{k+1}a_{n_{k+1}})^{1/2}b_{n_{k+1}}} \notin \mathcal{S}_1 + \frac{1}{q_1q_2}\epsilon_{n_{k+1}}\Big)\\
&:= \mathbb{P}_k^{KMT} + \mathbb{P}_k^{Tal}.
\end{aligned}$$

Now, making use of the strong approximation theorem of Komlós, Major ad Tusnady [22], we can assume that the process W involved in the former expression is satisfies, for some universal constants $C_1, C_2, C_3 > 0$, and for all $T > 0,\ z > 0$,

$$\mathbb{P}\big(\,||\,\widetilde{\Pi}(T\cdot) - W(\cdot)\,|| \geq z + C_1\log T\Big) \leq C_2\exp\big(-C_2 z\big). \tag{16}$$

Notice that, as $k \to \infty$,

$$\frac{(n_{k+1}a_{n_{k+1}})^{1/2} b_{n_{k+1}} \epsilon_{n_{k+1}}}{\log(n_{k+1}a_{n_{k+1}})} \to \infty.$$

Thus, we have, ultimately as $k \to \infty$,

$$\mathbb{P}_k^{KMT} \leq C_2 \exp\Big(- \frac{\epsilon C_3}{\sqrt{2} q_1 p_2} (n_{k+1}a_{n_{k+1}})^{1/2} (\log n_{k+1})^{-1/6} \Big). \qquad (17)$$

Recalling the assumption $na_n/(\log_2 n)^{7/3} \to \infty$ we see that $\mathbb{P}_k^{KMT}$ is summable in k. Now, recalling (15), we have for all large k,

$$\begin{aligned} \mathbb{P}_k^{Tal} &= \mathbb{P}\Big(W \notin b_{n_{k+1}} \mathcal{S}_1 + \frac{1}{q_1 q_2} \epsilon_{n_{k+1}} b_{n_{k+1}} \mathcal{B}_0 \Big) \\ &\leq \exp\Big(- \Big(\frac{\epsilon}{q_1 q_2} - \frac{L_1 (q_1 q_2)^2}{2\epsilon^2} \Big) (\log_2 n_{k+1})^{1/3} - \log_2 n_{k+1} \Big). \end{aligned}$$

The last inequality is due to Talagrand [19], with L_1 a universal constant. Now if $\epsilon > (L_1/2)^{1/3} =: \mathcal{L}_1$ and if $q_1, q_2 > 1$ are chose sufficiently small, then $\mathbb{P}_k^{Tal}$ is summable in k. By the Borel-Cantelli lemma, we see that for any $\epsilon > \mathcal{L}_1$ we have almost surely, for all large n,

$$g_n \in \mathcal{S}_1 + \epsilon_{n_{k+1}} \mathcal{B}_0,$$

where $g_n := (n_{k+1}a_{n_{k+1}})^{-1/2} b_{n_{k+1}}^{-1} H_n(a_{n_{k+1}} \cdot), \ \ n \in N_k$. To conclude the proof of theorem 2.2, it remains to control the distance between $a_n^{-1/2} b_n^{-1} \alpha_n(a_n \cdot)$ and g_n, which is the purpose of the following lemma.

LEMMA 3.2 *We have almost surely*

$$\limsup_{n\to\infty} (\log_2 n)^{2/3} \Big|\Big| \frac{\alpha_n(a_n \cdot)}{(2a_n \log_2 n)^{1/2}} - g_n \Big|\Big| = 0.$$

Proof

Set $\Gamma_n := 1 - (n/n_{k+1})^{1/2}(a_n/a_{n_{k+1}})^{1/2}(\log_2 n/\log_2 n_{k+1})^{1/2}$. The triangle inequality yields

$$\Big|\Big|\frac{\alpha_n(a_n\cdot)}{(2a_n\log_2 n)^{1/2}} - g_n\Big|\Big| \leq \Big|\Big|\frac{\alpha_n(a_n\cdot)}{(2a_n\log_2 n)^{1/2}}\Gamma_n\Big|\Big| + \Big|\Big|\frac{H_n(a_n\cdot) - H_n(a_{n_{k+1}}\cdot)}{(2n_{k+1}a_{n_{k+1}}\log_2 a_{n_{k+1}})^{1/2}}\Big|\Big| \tag{18}$$

$$:= A_n + B_n.$$

Clearly we have, as $k \to \infty$,

$$\max_{n\in N_k} \log_2 n_{k+1}\,\Gamma_n \leq \left(1 - \sqrt{\frac{n_k\log_2 n_k}{n_{k+1}\log_2 n_{k+1}}}\right)(\log_2 n_{k+1})^{2/3} \to 0.$$

Now, by applying (4) we have almost surely

$$\limsup_{n\to\infty}\Big|\Big|\frac{\alpha_n(a_n\cdot)}{(2a_n\log_2 n)^{1/2}}\Big|\Big| = 1. \tag{19}$$

Obviously (19) implies that, almost surely

$$\lim_{n\to\infty}\,(\log_2 n_{k+1})^{2/3}\max_{n\in N_k} A_n = 0.$$

We now focus on controlling B_n. Set $\rho_k := \frac{a_{n_k}}{a_{n_{k+1}}}$ and notice that

$$\mathbb{P}\Big(\max_{n\in N_k}(\log_2 n)^{2/3}\Big|\Big|\frac{H_n(a_n\cdot) - H_n(a_{n_{k+1}}\cdot)}{(2n_{k+1}a_{n_{k+1}}\log_2 a_{n_{k+1}})^{1/2}}\Big|\Big| \geq \epsilon\Big)$$
$$\leq \mathbb{P}\Big(\max_{n\in N_k}\sup_{1\leq\rho\leq\rho_k,\,0\leq t\leq 1}(\log_2 n_{k+1})^{2/3}\Big|\frac{\alpha_n(a_{n_{k+1}}\rho t) - \alpha_n(a_{n_{k+1}}t)}{(2a_{n_{k+1}}\log_2 n_{k+1})^{1/2}}\Big| \geq \epsilon\Big).$$

Now consider the Banach space $B\big([0,1]\times[0,2]\big)$ of all real bounded functions on $[0,1]\times[0,2]$, endowed with the usual sup norm $||\cdot||_{[0,1]\times[0,2]}$. We shall now make use of the powerful maximal inequality of Montgommery-Smith-Latala (see, e.g., [23]). This inequality implies that there exists a constant $\widetilde{c} > 0$ such that, for any finite sequence of i.i.d random variables $(X_i)_{i=1\ldots n}$ taking values in $B\big([0,1] \times [0,2]\big)$ and for all $\lambda > 0$ we have, assuming that the involved expressions are meaningful,

$$\mathbb{P}\Big(\max_{i=1,\ldots,n} \Big|\Big| \sum_{j=1}^{i} X_j \Big|\Big|_{[0,1]\times[0,2]} \geq \lambda \Big) \leq \widetilde{c}\, \mathbb{P}\Big(\Big|\Big| \sum_{i=1}^{n} X_i \Big|\Big|_{[0,1]\times[0,2]} \geq \frac{\lambda}{\widetilde{c}} \Big). \qquad (20)$$

For fixed $k \geq 1$, we apply (20) to the finite sequence $(X_i)_{i \in N_k}$, with $X_i(t,\rho) := 1_{[t,\rho t]}(U_i),\ t \in [0,1],\ \rho \in [1,\rho_k],\ rhot \leq 1$ and $X_i(t,\rho) = 0$ elsewhere. Hence

$$\begin{aligned}
&\mathbb{P}\Big(\max_{n \in N_k} \sup_{1 \leq \rho \leq \rho_k,\ 0 \leq t \leq 1} (\log_2 n_{k+1})^{2/3} \Big| \frac{\alpha_n(a_{n_{k+1}}\rho t) - \alpha_n(a_{n_{k+1}} t)}{(2a_{n_{k+1}} \log_2 n_{k+1})^{1/2}} \Big| \geq \epsilon \Big) \\
\leq& \widetilde{c}\, \mathbb{P}\Big(\sup_{1 \leq \rho \leq \rho_k,\ 0 \leq t \leq 1} (\log_2 n_{k+1})^{2/3} \Big| \frac{\alpha_{n_{k+1}}(a_{n_{k+1}}\rho t) - \alpha_{n_{k+1}}(a_{n_{k+1}} t)}{(2a_{n_{k+1}} \log_2 n_{k+1})^{1/2}} \Big| \geq \epsilon/\widetilde{c} \Big) \\
\leq& 2\widetilde{c}\, \mathbb{P}\Big(||\, \widetilde{\Pi}(n_{k+1}a_{n_k}\cdot) - W(n_{k+1}a_{n_k}\cdot)\, || \geq \frac{\epsilon}{8\widetilde{c}} \frac{(2n_{k+1}a_{n_{k+1}} \log_2 n_{k+1})^{1/2}}{(\log_2(n_{k+1}))^{2/3}} \Big) \\
&+ 2\widetilde{c}\, \mathbb{P}\Big(\Big|\Big| \frac{W(\rho_k \cdot) - W(\cdot)}{(2 \log_2 n_{k+1})^{1/2}} \Big|\Big| \geq \frac{\epsilon}{4\widetilde{c}(\log_2 n_{k+1})^{2/3}} \Big). \qquad (21)
\end{aligned}$$

In the last expression (which is the combination of usual poissonization techniques with the triangular inequality), $\widetilde{\Pi}$ and W denote respectively a centered Poisson process and a Wiener process based on the same underlying probability space. By the KMT construction (see, e.g., [22]), W can be constructed to satisfy (16) by making use of the same arguments as those onvoked to obtain (17), we conclude that the first term in (21) is widely sumable in k. To control the second term in (21), we shall make use of a well known inequality (see, e.g., [6], p. 536), with $a := \rho_k - 1,\ \lambda := (\rho_k - 1)^{-1/2}(\log_2 n_{k+1})^{-1/6}(\sqrt{2}\epsilon/4\widetilde{c})$

and $u = 1/2$, to get

$$\mathbb{P}\Big(\Big|\Big|\frac{W(\rho_k\cdot)-W(\cdot)}{\sqrt{2\log_2 n_{k+1}}}\Big|\Big| \geq \frac{\epsilon}{4\widetilde{c}(\log_2 n_{k+1})^{2/3}}\Big)$$
$$\leq \frac{1024\widetilde{c}}{\sqrt{2}\epsilon}(\rho_k-1)^{-1/2}((\log_2 n_{k+1})^{1/6}\exp\Big(-\frac{\epsilon^2}{64\widetilde{c}^2}(\rho_k-1)^{-1}(\log_2 n_{k+1})^{-1/3}\Big).$$

This expression is summable in k, and hence $\max_{n\in N_k} B_n \to 0$ almost surely as $k\to\infty$. □

3.3 *Proof of theorem 2.3*

Recall that χ_f, ∇_f, $\mathcal{S}_1^{bv}$ and $\mathcal{S}_1^{liv}$ are defined in §2. The main tool to achieve the proof of theorem 2.3 is the following inequality (see [2]), which sums up different results from Gorn and Lifshits [13], Berthet and Lifshits [14] and Grill [12] (see also de Acosta [11]).

INEQUALITY 3.3 For any $f \in \mathcal{S}_1^{BV}\cup\mathcal{S}_1^{LIV}$ and $\delta > 0$, there exist $\gamma^+ = \gamma^+(\delta, f) > 0$ and $\gamma- = \gamma^-(\delta, f) > 0$ such that for T sufficiently large:

$$\mathbb{P}\Big(\nabla_f\Big(\frac{T^2}{2}\Big)\Big|\Big|\frac{W}{T}-f\Big|\Big| \leq (1+\delta)\chi_f\Big) \geq \exp\Big(-\frac{T^2}{2}+\gamma^+\frac{\nabla_f^2(T^2/2)}{T^2}\Big),$$

$$\mathbb{P}\Big(\nabla_f\Big(\frac{T^2}{2}\Big)\Big|\Big|\frac{W}{T}-f\Big|\Big| \leq (1-\delta)\chi_f\Big) \leq \exp\Big(-\frac{T^2}{2}-\gamma^-\frac{\nabla_f^2(T^2/2)}{T^2}\Big).$$

Select $f \in \mathcal{S}_1^{BV}\cup\mathcal{S}_1^{LIV}$. We remind the two following properties of ∇_f (see [24]), namely $\lim_{L\to\infty} L^{-1}\nabla_f(L) < \infty$ and $\lim_{L\to\infty} L^{-2/3}\nabla_f(L) > 0$. We shall first show that almost surely

$$\liminf_{n\to\infty} \nabla_f(\log_2 n)\Big|\Big|\frac{\alpha_n(a_n\cdot)}{(2a_n\log_2 n)^{1/2}}-f\Big|\Big| \geq \chi_f.$$

Select $\epsilon > 0$. We start by applying Poissonization techniques in combination with the usual KMT approximation.

$$\begin{aligned}
&\mathbb{P}\Big(\nabla_f(\log_2 n)\Big|\Big|\frac{\alpha_n(a_n\cdot)}{(2a_n\log_2 n)^{1/2}}-f\Big|\Big|\leq\chi_f(1-2\epsilon)\Big)\\
\leq&2\mathbb{P}\Big(\nabla_f(\log_2 n)\Big|\Big|\frac{W(na_n\cdot)}{(2na_n\log_2 n)^{1/2}}-f\Big|\Big|\leq\chi_f(1-\epsilon)\Big)\\
&+2\mathbb{P}\Big(\Big|\Big|W(na_n\cdot)-\widetilde{\Pi}(na_n\cdot)\Big|\Big|\geq\frac{\epsilon(2na_n\log_2 n)^{1/2}}{\nabla_f(\log_2 n)}\Big).
\end{aligned}$$

These two terms are summable along the subsequence n_k, the first term beign controlled by the KMT approximation while the second is controlled by inequality 3.3. Now the control between n_k and n_{k+1} follows the same line as in lemma 3.2. We omit details for sake of brevity.

We now focus on showing that, almost surely,

$$\liminf_{n\rightarrow\infty}\nabla_f(\log_2 n)\Big|\Big|\frac{\alpha_n(a_n\cdot)}{(2a_n\log_2 n)^{1/2}}-f\Big|\Big|\leq\chi_f.$$

Set $n_k:=k^{2k}$, $v_k:=n_{k+1}-n_k$ and

$$h_k:=\frac{\sqrt{n_{k+1}}\alpha_{n_{k+1}}(a_{n_{k+1}}\cdot)-\sqrt{n_k}\alpha_{n_k}(a_{n_{k+1}}\cdot)}{\sqrt{2v_ka_{n_{k+1}}\log_2(v_k)}}.$$

Notice that the h_k are mutually independent processes, and that each h_k is distributed like $(2a_{n_{k+1}}\log_2 v_k)^{-1/2}\alpha_{v_k}(a_{n_{k+1}}\cdot)$. We now make use of the following "unpoissonization" lemma. Recall that $\widetilde{\Pi}(\cdot)$ denotes a centered standard Poisson process on $[0,\infty)$.

Lemma 3.4 *Under assumptions (12) and (13), there exist two sumable positive sequences $(c_k)_{k\geq1}$, $(b_k)_{k\geq1}$ and an integer $k_0\geq1$ such that, for any set $A\subset B([0,1])$ that is measurable for both empirical and Poisson processes and for* *all* $k \geq k_0$,

$$\mathbb{P}\Big(\widetilde{\Pi}(v_k a_{n_{k+1}} \cdot) \in A\Big) - c_k - b_k \leq 2\mathbb{P}\Big(v_k^{1/2} \alpha_{v_k}(a_{n_{k+1}} \cdot) \in A\Big).$$

Proof Given two sequences $(\theta_n)_{n\geq 1}$ and $(\nu_n)_{n\geq 1}$ we shall write $\theta_n \sim \nu_n$ whenever $\theta_n/\nu_n \to 1$ as $n \to \infty$. Set $u_k := (4\log_2(v_k)/n_{k+1}a_{n_{k+1}})^{1/2}$. By assumption (13) we have $u_k \to 0$ as$k \to \infty$. Now set $\Pi(t) := \widetilde{\Pi}(t) + t, \ \in [0,1]$ and $R_{1,k} := \Pi(v_k a_{n_{k+1}})$, $R_{2,k} := \Pi(v_k) - \Pi(v_k a_{n_{k+1}})$. For fixed k, $R_{1,k}$ and $R_{2,k}$ are independent random variables and are distributed as Poisson variables with respective expectations $v_k a_{n_{k+1}}$ and $v_k(1 - a_{n_{k+1}})$. Let $A \subset B([0,1])$ be an arbitrary set that we assume to be measurable for $\widetilde{\Pi}$ and α_n. Define the following events:

$$E_k := \Big\{\widetilde{\Pi}(v_k a_{n_{k+1}} \cdot) \in A\Big\}, \ k \geq 1.$$

We have, for any integer $k \geq 1$,

$$\begin{aligned}\mathbb{P}(E_k) \leq \mathbb{P}\Big(E_k \cap R_{1,k} \in [(1-u_k)v_k a_{n_{k+1}}, (1+u_k)v_k a_{n_{k+1}}]\Big) \\ + \mathbb{P}\Big(R_{1,k} < (1-u_k)v_k a_{n_{k+1}}\Big) + \mathbb{P}\Big(R_{1,k} > (1+u_k)v_k a_{n_{k+1}}\Big).\end{aligned}$$

Denote by c_k and b_k the two last terms of the RHS of the preceding inequality. We shall show that these two sequences have finite sums. Making use of Chernoff's inequality, we have:

$$c_k \leq \exp\Big(- v_k a_{n_{k+1}} \Big((1+u_k)\log(1+u_k) - u_k\Big)\Big).$$

Since $(1+u)\log(1+u)-u \sim \frac{u^2}{2}$ as $u \to 0$, it follows that for all large k,

$$\begin{aligned} c_k &\leq \exp\Big(-v_k a_{n_{k+1}} \frac{u_k^2}{2}\Big) \\ &= \exp\Big(-2\log_2 n_k\Big). \end{aligned}$$

We make use of a similar method to show that $(b_k)_{k\geq 1}$ is sumable. It remains to show that, for all $k \geq k_0$ (with k_0 independent of A), we have

$$\mathbb{P}\Big(E_k \cap R_{1,k} \in [(1-u_k)v_k a_{n_{k+1}}, (1+u_k)v_k a_{n_{k+1}}]\Big) \leq 2\mathbb{P}\Big(E_k \mid \Pi(v_k) = v_k\Big).$$

Now set

$$K_k := \inf\left\{\frac{\mathbb{P}(R_{2,k} = v_k - j)}{\mathbb{P}(\Pi(v_k) = v_k)} j \in \{(1-u_k)v_k a_{n_{k+1}}, (1+u_k)v_k a_{n_{k+1}}\}\right\}. \quad (22)$$

We have

$$\begin{aligned} &\mathbb{P}\Big(E_k \cap R_{1,k} \in [(1-u_k)v_k a_{n_{k+1}}, (1+u_k)v_k a_{n_{k+1}}]\Big) \\ &\leq \sum_{j=[(1-u_k)v_k a_{n_{k+1}}]}^{[(1+u_k)v_k a_{n_{k+1}}]+1} \mathbb{P}(E_k \cap R_{1,k} = j) \\ &\leq K_k^{-1} \sum_{j=[(1-u_k)v_k a_{n_{k+1}}]}^{[(1+u_k)v_k a_{n_{k+1}}]+1} \mathbb{P}(E_k \cap R_{1,k} = j) \frac{\mathbb{P}(R_{2,k} = v_k - j)}{\mathbb{P}(\Pi(v_k) = v_k)} \\ &= K_k^{-1} \sum_{j=[(1-u_k)v_k a_{n_{k+1}}]}^{[(1+u_k)v_k a_{n_{k+1}}]+1} \frac{\mathbb{P}(E_k \cap R_{1,k} = j \cap R_{2,k} = v_k - j)}{\mathbb{P}(\Pi(v_k) = v_k)} \\ &\leq K_k^{-1}\mathbb{P}(E_k \mid \Pi(v_k) = v_k). \end{aligned}$$

Hence, it suffices to show that $K_k \to 1$. Setting $l = v_k - j$ in (22) we have, as

$k \to \infty$,

$$K_k = \inf\left\{\frac{\mathbb{P}(R_{2,k} = l)}{\mathbb{P}(\Pi(v_k) = v_k)}, l \in [v_k(1-a_{n_{k+1}}) - v_k c_k u_k, v_k(1-a_{n_{k+1}}) + v_k c_k u_k]\right\}.$$

For clarity of notations, set $v'_k := v_k(1-a_{n_{k+1}})$, recalling that $R_{2,k}$ is a Poisson variable with parameter v'_k. Now, by Stirling's formula, we have $\mathbb{P}(\Pi(v_k) = v_k) \sim (2\pi v_k)^{-1/2}$ as $k \to \infty$. A routine study of the finite sequence

$$\Big(\mathbb{P}(R_{2,k}) = l,\ l = \big[v_k(1-a_{n_{k+1}}) - v_k c_k u_k\big], \ldots, \mathrm{Cov}_k(1-a_{n_{k+1}}) + v_k c_k u_k\big] + 1\Big)$$

shows that

$$\mathbb{P}(\Pi(v_k) = v_k)K_k = \min(\mathbb{P}_{1,k}, \mathbb{P}_{2,k}),$$

where

$$\mathbb{P}_{1,k} := \mathbb{P}(R_{2,k} = [v'_k - v_k a_{nk+1} u_k])$$

and

$$\mathbb{P}_{2,k} := \mathbb{P}(R_{2,k} = [v'_k + v_k a_{nk+1} u_k] + 1).$$

We set $u'_k = a_{n_{k+1}} u_k v_k / v'_k \sim u_k a_{n_{k+1}}$. Stirling's formula yields, ultimately as

$k \to \infty$,

$$\begin{aligned}
\mathbb{P}_{1,k} &= \frac{{v_k}'^{[{v_k}'-v_k a_{nk+1} u_k]}}{[{v_k}' - v_k a_{nk+1} u_k]!} \exp(-{v_k}') \\
&\sim \Big(\frac{{v_k}'}{{v_k}'(1-{u_k}')}\Big)^{{v_k}'(1-{u_k}')} \frac{\exp(-{v_k}')}{\exp(-{v_k}' + v_k u_k)} \sqrt{2\pi v_k'} \\
&\sim \sqrt{2\pi v_k}(1-{u_k}')^{-v_k(1-{u_k}')} \exp(-{v_k}'{u_k}') \\
&= \sqrt{2\pi v_k} \exp\Big(-v_k\big((1-{u_k}')\log(1-u_k) + {u_k}'\big)\Big).
\end{aligned}$$

Moreover, since $(1-\epsilon)\log(1-\epsilon) + \epsilon \sim \frac{\epsilon^2}{2}$ as $\epsilon \to 0$, we have, for all large k,

$$\exp\Big(-2a_{n_{k+1}} \log_2(n_{k+1})\Big) \leq \exp\Big(-v_k\big((1-{u_k}')\log(1-u_k) + {u_k}'\big)\Big).$$

By assumption (12) we have $a_n \log_2 n \to 0$, which ensures that $\mathbb{P}_{1,k} \sim \sqrt{2\pi v_k}$. The control of $\mathbb{P}_{2,k}$ is very similar. This achieves the proof of lemma 3.4. $\square$

We now apply the preceding lemma in conjunction with the KMT approximation. Let W be a Wiener process based on the same underlying probability space as $\widetilde{\Pi}$. For an arbitrary $\delta > 0$, we have

$$\begin{aligned}
&\mathbb{P}\Big(\nabla_f(\log_2(v_k)) \mid\mid h_k - f \mid\mid \leq (1+2\delta)\chi_f\Big) \\
&\geq \frac{1}{2}\mathbb{P}\Big(\nabla_f(\log_2(v_k))\Big|\Big|\frac{\widetilde{\Pi}(v_k a_{n_{k+1}}\cdot)}{(2v_k a_{n_{k+1}} \log_2 v_k)^{1/2}} - f\Big|\Big| \leq (1+2\delta)\chi_f\Big) - \frac{1}{2}c_k - \frac{1}{2}b_k \\
&\geq -\frac{1}{2}\mathbb{P}\Big(\Big|\Big|\widetilde{\Pi}(v_k a_{n_{k+1}}\cdot) - W(v_k a_{n_{k+1}}\cdot)\Big|\Big| \geq \frac{\delta\chi_f(2v_k a_{n_{k+1}} \log_2 v_k)^{1/2}}{\nabla_f(\log_2 v_k)}\Big) \\
&\quad + \frac{1}{2}\mathbb{P}\Big(\nabla_f(\log_2 v_k) \mid\mid \frac{W}{(2\log_2 v_k)^{1/2}} - f \mid\mid \leq \chi_f(1+\delta)\Big) - \frac{1}{2}c_k - \frac{1}{2}b_k \\
&=: -d_k + \frac{1}{2}\mathbb{P}_k - \frac{1}{2}c_k - \frac{1}{2}b_k.
\end{aligned}$$

Since $v_k \sim n_{k+1}$ it is easy to conclude that d_k is summable in k, by making use of the strong approximation (see [22]). Hence, making use of inequality 3.3, we have asymptotically

$$\mathbb{P}_k \geq \exp\Big(- \log_2 v_k \Big).$$

But $\log_2 v_k = \log(k+1) + \log_2 k + o\big(k^{-2}(\log k)^{-1}\big)$ and hence

$$\sum_{k\geq 1} \mathbb{P}\Big(\nabla_f(\log_2(v_k)) \mid\mid h_k - f \mid\mid \leq (1+2\delta)\chi_f\Big) = \infty.$$

Applying the second half of the Borel Cantelli lemma, we deduce that

$$\liminf_{k\to\infty} \nabla_f(\log_2 v_k) \mid\mid h_k - f \mid\mid \leq \chi_f.$$

To conclude the proof, it is enough to show that almost surely (recall that $\lim_{L\to\infty} L^{-1}\nabla_f(L) > 0$),

$$\lim_{k\to\infty} (\log_2 n_{k+1}) \Big|\Big| h_k - \frac{\alpha_{n_{k+1}}(a_{n_{k+1}}\cdot)}{(2n_{k+1}a_{n_{k+1}}\log_2(n_{k+1}))^{1/2}} \Big|\Big| = 0.$$

Routine algebra shows that

$$\begin{aligned}
&\log_2 n_{k+1} \Big|\Big| h_k - \frac{\alpha_{n_{k+1}}(a_{n_{k+1}}\cdot)}{(2a_{n_{k+1}}\log_2(n_{k+1}))^{1/2}} \Big|\Big| \\
\leq\ &\log_2 n_{k+1} \Big(\big(\frac{n_{k+1}\log_2 n_{k+1}}{v_k \log_2(v_k)}\big)^{1/2} - 1 \Big) \Big|\Big| \frac{\alpha_{n_{k+1}}(a_{n_{k+1}}\cdot)}{(2a_{n_{k+1}}\log_2 n_{k+1})^{1/2}} \Big|\Big| \\
&+ \log_2 n_{k+1} \Big|\Big| \frac{n_k^{1/2}\ \alpha_{n_k}(a_{n_{k+1}}\cdot)}{(2v_k a_{n_{k+1}}\log_2 n_{k+1})^{1/2}} \Big|\Big| \\
:=\ &A_k + B_k.
\end{aligned}$$

Applying theorem of Mason (4) we get $A_k \to 0$ almost surely as $k \to \infty$. We now apply Doob's inequality for positive submartingales

$$\begin{aligned}
&\mathbb{P}\Big(\log_2 n_{k+1}\Big|\Big|n_k\ \frac{\alpha_{n_k}(a_{n_{k+1}}\cdot)}{((2v_k a_{n_{k+1}}\log_2 n_{k+1}))^{1/2}}\Big|\Big| \geq \epsilon\Big)\\
&= \mathbb{P}\Big(\sup_{0\leq t\leq a_{n_{k+1}}}\Big|\frac{\alpha_n(t)}{1-t}\Big| \geq \frac{\epsilon(2v_k a_{n_{k+1}}\log_2 n_{k+1}))^{1/2}}{n_k^{1/2}\log_2 n_{k+1}}\Big)\\
&\leq \frac{1}{2\epsilon^2}(1-a_{n_{k+1}})\log_2 n_{k+1}\frac{n_k}{v_k}.
\end{aligned}$$

Since $n_k/v_k \sim 1/e^2k^2$ as $k \to \infty$, we conclude the proof of the lower bound in theorem 2.3 with the Borel-Cantelli lemma. □

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