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SOME UNIFORM IN BANDWITH FUNCTIONAL RESULTS FOR THE TAIL UNIFORM EMPIRICAL AND QUANTILE PROCESSES

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Some uniform in bandwidth functional results for the tail uniform empirical and quantile processes

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Abstract: For fixed $t \in [0,1)$ and $h > 0$, consider the local uniform empirical process

$$\mathcal{D}_{n,h,t}(s) := n^{-1/2}\Big[\sum_{i=1}^{n} 1_{[t,t+hs]}(U_i) - hs\Big],\ s \in [0,1],$$

where the U_i are independent and uniformly distributed on $[0,1]$. We investigate the functional limit behaviour of $\mathcal{D}_{n,h,t}$ uniformly in $\mathfrak{h}_n \leq h \leq h_n$ when $n\mathfrak{h}_n/\log\log n \to \infty$ and $h_n \to 0$.

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1. Introduction

Let $(U_i)_{i\geq 1}$ be an independent, identically distributed (i.i.d.) sequence of random variables that are uniformly distributed on $[0,1]$. Define the empirical distribution function based on $(U_1,\ldots,U_n)$ by $F_n(t) := n^{-1}\sharp\{1 \leq i \leq n,\ U_i \leq t\},\ t \in [0,1]$ and denote by $F_n^{\leftarrow}(t)$ the left-continuous inverse of F_n, namely $F_n^{\leftarrow}(t) := \inf\{s \geq 0,\ F_n(s) \geq t\}$. We also define the empirical (resp. quantile) process by $\alpha_n(t) := \sqrt{n}(F_n(t) - t), t \in [0,1]$ (resp. $\beta_n(t) := \sqrt{n}(F_n^{\leftarrow}(t) - t),\ t \in [0,1]$). The framework of this paper is the almost sure behaviour of the local empirical and quantile processes. Namely, given $t \in [0,1)$ we focus on studying the following processes, as $n \to \infty$ and $h \to 0$.

$$\mathcal{D}_{n,h,t}(s) := \alpha_n(t+hs) - \alpha_n(t),\ s \in [0,1], \tag{1.1}$$

$$\mathcal{D}'_{n,h,t}(s) := \beta_n(t+hs) - \beta_n(t),\ s \in [0,1]. \tag{1.2}$$

Mason (1988) was the first to establish a functional law of the iterated logarithm for the local empirical process (see also Einmahl and Mason (1997) for a generalization of this result to empirical processes indexed by functions). To cite this result, we need to introduce some further notations first. Write $\log_2(u) := \log(\log(u \vee 3))$. We say that a sequence $(h_n)_{n\geq 1}$ of strictly positive constants satisfies the local strong invariance conditions when, ultimately as $n \to \infty$,

$$h_n \downarrow 0,\ nh_n \uparrow \infty,\ nh_n/\log_2 n \to \infty. \tag{1.3}$$

Given a sequence $(x_n)_{n\geq 1}$ of elements of a metric space (E,d), we say that $x_n \rightsquigarrow K$ when K is non void and coincides with the set of all cluster points of $(x_n)_{n\geq 1}$. In our framework, (E,d) is the space $B([0,1])$ of all real bounded CADLAG trajectories on $[0,1]$, endowed with the usual sup norm, namely $|| g ||:= \sup\{| g(s) |,\ s \in [0,1]\}$. Consider the space $AC[0,1]$ of all absolutely continuous functions on $[0,1]$. For any $g \in AC[0,1]$, we define the usually called Hilbertian norm of g as

$$|| g ||_H^2 := \int_0^1 \dot{g}^2(x)dx, \tag{1.4}$$

where $\dot{g}$ is any version of the derivative of g with respect to the Lebesgue measure. The usually called Strassen ball can be defined as follows:

$$\mathcal{S} := \Big\{ g \in AC([0,1]),\ g(0) = 0, || g ||_H \leq 1 \Big\}. \tag{1.5}$$

As a corollary of a strong approximation result, Mason (1988) showed that, given a sequence $(h_n)_{n\geq 1}$ fulfilling (1.3) and given $t \in [0,1)$, we have, almost surely

$$\frac{\mathcal{D}_{n,h_n,t}}{(2h_n \log_2 n)^{1/2}} \rightsquigarrow \mathcal{S} \tag{1.6}$$

In the particular case where $t = 0$, Einmahl and Mason (1988) showed that $\mathcal{D}'_{n,h_n,t}$ also satisfies (1.6). They showed that result by making use of a local Bahadur Kiefer representation (see their Theorem 5). The almost sure limit behavior of $\mathcal{D}'_{n,h_n,t}$ when $t \in (0,1)$ has been investigated by Deheuvels (1997), who showed that the above mentioned process may obey functional limit laws that are different from (1.6). The aim of the present paper is the following: given two sequences $\mathfrak{h}_n < h_n$ fulfilling (1.3), does (1.6) still hold uniformly in $\mathfrak{h}_n \leq h \leq h_n$? Namely, do we have almost surely

$$\lim_{n\to\infty} \sup_{\mathfrak{h}_n \leq h \leq h_n} \inf_{g \in \mathcal{S}} \Big|\Big| \frac{\mathcal{D}_{n,h,t}}{(2h \log_2 n)^{1/2}} - g \Big|\Big| = 0, \tag{1.7}$$

$$\forall g \in \mathcal{S},\ \liminf_{n\to\infty} \sup_{\mathfrak{h}_n \leq h \leq h_n} \Big|\Big| \frac{\mathcal{D}_{n,h,t}}{(2h \log_2 n)^{1/2}} - g \Big|\Big| = 0\ ? \tag{1.8}$$

The remainder of this paper is organised as follows. In §2, we state our main results on $\mathcal{D}_{n,h,t}$. We then show how this results lead to a local Bahadur-Kiefer type representation that holds uniformly in h. The proofs of our main results follow in §3, 4 and 5.

2. Mains results

Our first result is a weaker form of assertion (1.7).

Theorem 1. *Let $(h_n)_{n\geq 1}$ and $(\mathfrak{h}_n)_{n\geq 1}$ be two sequences satisfying (1.3) as well as $\mathfrak{h}_n < \frac{1}{2} h_n$. Then, given $t \in [0,1)$, we have, almost surely:*

$$\lim_{n\to\infty} \sup_{\mathfrak{h}_n \leq h \leq h_n} \inf_{g \in \sqrt{2}\mathcal{S}} \Big|\Big| \frac{\mathcal{D}_{n,h,t}}{(2h \log_2 n)^{1/2}} - g \Big|\Big| = 0. \tag{2.1}$$

The proof of Theorem 1 is written in §3.

Remark: Condition $\mathfrak{h}_n < h_n/2$ is just technical, as this result is really interesting when $(\mathfrak{h}_n)_{n\geq 1}$ and $(h_n)_{n\geq 1}$ are sequences that tend to 0 at different rates (typically $n^{-\alpha_1}$ and $n^{-\alpha_2}$, $0 < \alpha_1 < \alpha_2 < 1$). Clearly, Theorem 1 seems unsatisfactory, as one would expect the limit set to be $\mathcal{S}$ instead of $\sqrt{2}\mathcal{S}$. As it will be pointed out in the proof of Theorem 1 (see §3.2), it is possible to prove (1.7) when

$$\forall\beta > 0,\ \lim_{n\to\infty} \log(h_n/\mathfrak{h}_n)/(\log n)^\beta = 0. \tag{2.2}$$

However, (2.2) is a very restrictive condition, imposing $(h_n)_{n\geq 1}$ and $(\mathfrak{h}_n)_{n\geq 1}$ to have rates of convergence to zero that are very close one to each other. In §3, we shall try to point out the main difficulty that imposes us to weaken (1.7) to (2.1). Showing that (1.7) is true or false without imposing (2.2) remains an open problem.

The second step of our investigation is to determine the validity of (1.8). This assertion turns out to be false as soon as $\mathfrak{h}_n/h_n \to 0$, which is a consequence of our next result. We first need to introduce some further notations. Given an integer $k \geq 2$, we endow the space $(B[0,1])^k$ with the product sup-norm, namely $|| g_1, \dots, g_k ||_k := \max\{|| g_1 ||, \dots, || g_k ||\}$, and we define

$$\mathcal{S}_k := \Big\{(g_1, \dots, g_k) \in (AC[0,1])^k, \sum_{j=1}^{k} || g_j ||_H^2 \leq 1\Big\}. \tag{2.3}$$

Now consider sequences $0 < h_{n,1} < \dots < h_{n,k} < 1$ satisfying, ultimately as $n \to \infty$,

$$h_{n,l}/h_{n,l+1} \downarrow 0,\ l = 1, \dots, k-1, \tag{2.4}$$

$$h_{n,k} \downarrow 0,\ nh_{n,1} \uparrow \infty. \tag{2.5}$$

Our second main result is the following functional limit law, which is proved in §4.

Theorem 2. *Under assumptions (2.4) and (2.5) we have almost surely*

$$\Big(\frac{\mathcal{D}_{n,h_{n,1},t}}{(2h_{n,1}\log_2 n)^{1/2}}, \dots, \frac{\mathcal{D}_{n,h_{n,k},t}}{(2h_{n,k}\log_2 n)^{1/2}}\Big) \rightsquigarrow \mathcal{S}_k. \tag{2.6}$$

Here $\rightsquigarrow$ *refers to the Banach space* $\big(B([0,1])^k, || \cdot ||_k\big)$.

Note that $\mathcal{S}_k$ is the unit ball of the reproducing kernel Hilbert space of $(W_1, \dots, W_k)$, where $W_1, \dots, W_k$ are independent Wiener processes on $[0,1]$.

Theorem 2 describes an asymptotic independence phenomenon which has been earlier investigated by Deheuvels (2000) and Deheuvels *et al.* (1999). The proof of Theorem 2 is provided in §4. Now, to see that (1.8) is false, choose g as the identity function so as $(g, g) \notin \mathcal{S}_2$, which entails that $\inf\{|| \ g - g_1, g - g_2 \ ||_2 \ , \ (g_1, g_2) \in \mathcal{S}_2\} > \epsilon_0$ for some $\epsilon_0 > 0$. By Theorem 2 we have, almost surely,

$$\begin{aligned}
&\liminf_{n\to\infty} \sup_{\mathfrak{h}_n \le h \le h_n} \Big|\Big| \frac{\mathcal{D}_{n,h,t}}{(2h \log_2 n)^{1/2}} - g \Big|\Big| \\
\ge &\liminf_{n\to\infty} \Big|\Big| \frac{\mathcal{D}_{n,\mathfrak{h}_n,t}}{(2\mathfrak{h}_n \log_2 n)^{1/2}} - g, \ \frac{\mathcal{D}_{n,h_n,t}}{(2h_n \log_2 n)^{1/2}} - g \Big|\Big|_2 \\
\ge &\epsilon_0,
\end{aligned}$$

which invalidates (1.8).

A local Bahadur-Kiefer representation

A consequence of Theorem 1 is the following local Bahadur-Kiefer representation, which is very largely inspired from Einmahl and Mason (1988, Theorem 5). For $0 < h < 1$ and $n \ge 1$ we set $a_n(h) := (h \log_2 n/n)^{1/2}, \ b_n(h) := \log(nh), \ d_n(h) := 2\log_2 n + b_n(h), \ r_n(h) := (a_n(h)d_n(h))^{1/2}$ and

$$R_n(h) := \Big|\Big| \mathcal{D}_{n,h,0} + \mathcal{D}'_{n,h,0} \Big|\Big|.$$

Theorem 3. *Under the conditions of Theorem 1, with $t = 0$, we have, almost surely*

$$\limsup_{n\to\infty} \sup_{\mathfrak{h}_n \le h \le h_n} r_n(h)^{-1} R_n(h) \le 2^{1/2}. \tag{2.7}$$

The proof of Theorem 3 is provided in §5.

Remark: In view of Theorem 5 of Einmahl and Mason (1988), Theorem 3 seems to be non optimal since a factor $2^{1/4}$ can be drop when $h_n = \mathfrak{h}_n$. This is a consequence of the fact that we were only able to prove (2.1) instead of (1.7).

3. Proof of Theorem 1

Our proof is divided into two subsections. In §3.1, we establish a large deviation result which holds uniformly in $\mathfrak{h}_n \le h \le h_n$. Then we make use of that (uniform) large deviation principle to prove Theorem 1 in §3.2.

3.1. A uniform large deviation principle

3.1.1. Definitions

Large deviation results are commonly used when proving functional laws of the iterated logarithm such as (1.6). As a uniformity in $\mathfrak{h}_n \leq h \leq h_n$ appears in Theorem 1, we shall make use of a large deviation principle that holds uniformly in h. This tool was first used by Mason (2004). From now on, $(\epsilon_{n,i})_{n\geq 1,\, i\leq p_n}$ will denote a triangular array of strictly positive numbers satisfying $\max_{1\leq i\leq p_n} \epsilon_{n,i} \rightarrow 0$ as $n \rightarrow \infty$. We call a rate function in a metric space (E,d) any positive real function J on E such that, for each $a \geq 0$, the set $\{g \in E,\ J(g) \leq a\}$ is a compact set of (E,d).

Definition 3.1. *Let (E,d) be a metric space and let $\mathcal{T}_0$ be a σ-algebra included in the Borel σ-algebra of (E,d). Let $(X_{n,i})_{n\geq 1,\, i\leq p_n}$ be a triangular array of random variables that are measurable for $(E,\mathcal{T}_0)$. We say that $(X_{n,i})_{n\geq 1,\, i\leq p_n}$ satisfies the uniform large deviation principle (ULDP) for $(\epsilon_{n,i})_{n\geq 1,\, i\leq p_n}$, a rate function J and $\mathcal{T}_0$ whenever*

1. *For each closed set $F \in \mathcal{T}_0$ we have*

$$\limsup_{n\rightarrow\infty} \max_{i\leq p_n} \epsilon_{n,i} \log\Big(\mathbb{P}\Big(X_{n,i} \in F\Big)\Big) \leq -J(F), \tag{3.1}$$

2. *For each open set $O \in \mathcal{T}_0$ we have*

$$\liminf_{n\rightarrow\infty} \min_{i\leq p_n} \epsilon_{n,i} \log\Big(\mathbb{P}\Big(X_{n,i} \in O\Big)\Big) \geq -J(O). \tag{3.2}$$

Remark: In this definition, we introduce a sub σ-algebra $\mathcal{T}_0$ because we will consider repeatedly (E,d) as the metric space $(B([0,1], ||\cdot||)$. As the $\mathcal{D}_{n,h,t}$ are not Borel measurable in that space, we shall consider $\mathcal{T}_0$ as the σ-algebra spawned by the open balls of $(B([0,1], ||\cdot||)$. We will sometimes take (E,d)

as a finite dimensional vector space, in which case $\mathcal{T}_0$ will denote the Borel σ-algebra. Another way to avoid measurability problems is to consider inner and outer probabilities (see, e.g.,Van der Vaart and Wellner (1996), Chapter 1). The next result is a consequence of the work of Arcones (2003).

Proposition 3.1. *Let $(X_{n,i})_{n\geq 1,\ i\leq p_n}$ be a triangular array of random variables taking values in $B([0,1])$ and measurable for $\mathcal{T}_0$. Let $(\epsilon_{n,i})_{n\geq 1,\ i\leq p_n}$ be a triangular array of strictly positive real numbers. Assume that the following conditions hold:*

1. *For each $p\geq 1$ and $(s_1,\ldots,s_p)\in(0,1)^p$ satisfying $s_i\neq s_j$ for each $i\neq j$, the triangular array $\big(X_{n,i}(s_1),\ldots,X_{n,i}(s_p)\big)_{n\geq 1,\ i\leq p_n}$ satisfies the ULDP in $\mathbb{R}^p$ for $(\epsilon_{n,i})_{n\geq 1,\ i\leq p_n}$ and a rate function $I_{s_1,\ldots,s_p}$.*
2. *For any $\tau>0$ we have*
$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\max_{i\leq p_n}\log\Big(\mathbb{P}\Big(\sup_{|s-s'|<\delta}\mid X_{n,i}(s')-X_{n,i}(s)\mid>\tau\Big)\Big)=-\infty.$$

Then $(X_{n,i})_{n\geq 1,\ i\leq p_n}$ satisfies the ULDP in $(B([0,1],||\cdot||)$ for $(\epsilon_{n,i})_{n\geq 1,\ i\leq p_n}$, $\mathcal{T}_0$ and the following rate function:

$$I(g):=\sup_{p\geq 1,\ (s_1,\ldots,s_p)\in(0,1)^p} I_{s_1,\ldots,s_p}\big(g(s_1),\ldots,g(s_p)\big),\ g\in B([0,1]).$$

Now consider the following rate function on $B([0,1])$ that is known to rule the large deviation properties of a Wiener process:

$$J(g):=\begin{cases} ||\ g\ ||_H^2, & \text{when } g\in AC[0,1];\\ \infty, & \text{when } g\notin AC[0,1].\end{cases}\tag{3.3}$$

Notice that $\mathcal{S}=\{g\in B([0,1]),\ g(0)=0,\ J(g)\leq 1\}$. The main tool that will be used to achieve our proof of Theorem 1 is the following ULDP.

Proposition 3.2. *Let $(h_n)_{n\geq 1}$ and $(\mathfrak{h}_n)_{n\geq 1}$ be two sequences satisfying conditions of Theorem 1 and let $(h_{n,i})_{n\geq 1,\ i\leq p_n}$ be a triangular array satisfying $\mathfrak{h}_n\leq h_{n,i}\leq h_n$ for each $n\geq 1,\ i\leq p_n$. Then the triangular array*

$$\Big((2h_{n,i}\log_2 n)^{-1/2}\mathcal{D}_{n,h_{n,i},t}\Big)_{n\geq 1,\ i\leq p_n}$$

satisfies the ULDP in $(B([0,1]), ||\cdot||)$ *for* $\mathcal{T}_0$*, the rate function* J *given in (3.3) and the (constant in* $i \leq p_n$*) triangular array* $(1/\log_2 n)_{n\geq 1,\, i\leq p_n}$.

Proof of Proposition 3.2: We shall make use of Proposition 3.1, and we hence have to show that conditions 1 and 2 of this proposition are satisfied. This verification will be a consequence of two separate lemmas. The next proposition, which shall be useful to prove our first lemma, follows directly from the arguments of Ellis (1984). Here $<\cdot,\cdot>$ denotes the usual scalar product on $\mathbb{R}^p$.

Proposition 3.3. *Let* $(X_{n,i})_{n\geq 1,\, i\leq p_n}$ *be a triangular array of random vectors taking values in* $\mathbb{R}^p$*, and let* $(\epsilon_{n,i})_{n\geq 1,\, i\leq p_n}$ *be a triangular array of strictly positive real numbers. Assume that there exists a positive real function* ℓ *(which may take infinite values) on* $\mathbb{R}^p$ *such that the following conditions are satisfied.*

1. ℓ *is convex and lower semi continuous on* $\mathbb{R}^p$.
2. *The definition set* $D(\ell) := \{\lambda \in \mathbb{R}^p,\ \ell(\lambda) < \infty\}$ *has an interior that contains the null vector.*
3. ℓ *is differentiable on the interior of* $D(\ell)$ *and, for each sequence* $(\lambda_n)_{n\geq 1}$ *converging to a boundary point of* $D(\ell)$ *we have* $||\ \nabla\ell(\lambda_n)\ ||_{\mathbb{R}^p} \to \infty$*. Here* $||\cdot||_{\mathbb{R}^p}$ *denotes the usual Euclidian norm.*
4. *For each* $\lambda \in D(\ell)$*, we have*
$$\lim_{n\to\infty} \max_{i\leq p_n} \Big|\epsilon_{n,i} \log\Big(\mathbb{E}\Big(\exp\big(\epsilon_{n,i}^{-1} < \lambda, X_{n,i} >\big)\Big)\Big) - \ell(\lambda)\Big| = 0.$$
5. *For each* $\lambda \notin D(\ell)$*, we have*
$$\lim_{n\to\infty} \min_{i\leq p_n} \epsilon_{n,i} \log\Big(\mathbb{E}\Big(\exp\big(\epsilon_{n,i}^{-1} < \lambda, X_{n,i} >\big)\Big)\Big) = \infty.$$

Then $(X_{n,i})_{n\geq 1,\, i\leq p_n}$ *satisfies the ULDP in* $\mathbb{R}^p$ *for* $(\epsilon_{n,i})_{n\geq 1,\, i\leq p_n}$ *with the following rate function:*

$$\widetilde{J}(s) := \sup_{\lambda\in\mathbb{R}^p} < \lambda, s > -\ell(\lambda),\ s \in \mathbb{R}^p.$$

We now state our first lemma.

Lemma 3.1. *Let $p \geq 1$ and $(s_1, \ldots, s_p) \in [0,1]^p$ be arbitrary, with $s_1 < s_2 < \ldots < s_p$. Under the assumptions of Proposition 3.2, the triangular array of $\mathbb{R}^p$-valued random vectors*

$$\Big((2h_{n,i}\log_2 n)^{-1/2}\big(\mathcal{D}_{n,h_{n,i},t}(s_1), \ldots, \mathcal{D}_{n,h_{n,i},t}(s_p)\big)\Big)_{n\geq 1,\ i\leq p_n}$$

satisfies the ULDP for $(\epsilon_{n,i})_{n\geq 1,\ i\leq p_n}$ with the following rate function (with $s_0 := 0$).

$$J_{s_1,\ldots,s_p}(x_1, \ldots, x_p) := \sum_{i=0}^{p}(s_{i+1} - s_i)\Big(\frac{x_{i+1} - x_i}{s_{i+1} - s_i}\Big)^2,\ (x_1, \ldots, x_p) \in \mathbb{R}^p.$$

Proof of Lemma 3.1.

We shall make use of Proposition 3.3. Fix $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p$ and and write the $\mathcal{D}_{n,h_{n,i},t}$ as sums of i.i.d. random variables, namely

$$(2h_{n,i}\log_2 n)^{-1/2}\sum_{j=1}^{p}\lambda_j \mathcal{D}_{n,h_{n,i},t}(s_j) = (2nh_{n,i}\log_2 n)^{-1/2}\sum_{k=1}^{n} Z^k_{n,h_{n,i},t}, \quad (3.4)$$

where

$$Z^k_{n,h_{n,i},t} := \sum_{j=1}^{p}\lambda_j\big(1_{[t,t+h_{n,i}s_j]}(U_k) - h_{n,i}s_j\big),\ k = 1, \ldots, n.$$

These n random variables are i.i.d with mean 0 and variance-covariance matrix given by $h_{n,i}\lambda'\Sigma_{n,i}\lambda$, with $\Sigma_{n,i}(l,l') := \min(s_l, s_{l'}) - h_{n,i}s_l s_{l'}$. Now define the matrix $\Sigma(l,l') := \min(s_l, s_{l'})$. Clearly, as $h_{n,i} \leq h_n \to 0$ we have $\Sigma_{n,i} \to \Sigma$ uniformly in i as $n \to \infty$. By standard computations we have, for each $n \geq 1$ and $i \leq p_n$:

$$(\log_2 n)^{-1}\log\Bigg(\mathbb{E}\Bigg(\exp\Big(\log_2 n(2h_{n,i}\log_2 n)^{-1/2}\sum_{j=1}^{p}\lambda_j\mathcal{D}_{n,h_{n,i},t}(s_j)\Big)\Bigg)\Bigg)$$
$$= \frac{n}{\log_2 n}\log\Big(\mathbb{E}\exp\big(r_{n,i}Z^1_{n,h_{n,i},t}\big)\Big), \quad (3.5)$$

where $r_{n,i} := (\log_2 n/2nh_{n,i})^{1/2}$. Recall that $\max_{i\leq p_n} r_{n,i} \to 0$ as $n \to \infty$, since $\mathfrak{h}_n$ satisfies (1.3), and notice that the $Z^k_{n,h_{n,i},t}$ are centered and almost surely bounded by $p\max_{j=1,\ldots,p} \mid \lambda_j \mid$. This ensures that the following Taylor expansion is valid, for each $n \geq 1,\ i \leq p_n$ (here ε denotes a real function satisfying $\varepsilon(u) \to 0$ as $u \to 0$):

$$\mathbb{E}\Big(\exp\big(r_{n,i} W^1_{n,h_{n,i},t}\big)\Big) = 1 + \frac{r_{n,i}^2 h_{n,i}}{2} \lambda' \Sigma_{n,i} \lambda (1+\varepsilon(r_{n,i})). \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$\lim_{n\to\infty} \max_{i\leq p_n} \left| \frac{\log\Big(\mathbb{E}\Big(\exp\Big(\frac{\log_2 n}{(2h_{n,i}\log_2 n)^{1/2}} \sum_{j=1}^p \lambda_j \mathcal{D}_{n,h_{n,i},t}(s_j)\Big)\Big)\Big)}{\log_2 n} - \frac{1}{4}\lambda'\Sigma\lambda \right| = 0.$$

As the function $\ell(\lambda) := \lambda'(\Sigma/4)\lambda$ obviously satisfies conditions of Proposition 3.3, the proof of Lemma 3.1 is concluded by noticing that

$$\sup_{t\in\mathbb{R}^p} < t, x > -\ell(t) = x'\Sigma^{-1}x = \sum_{i=0}^p (s_{i+1}-s_i)\Big(\frac{x_{i+1}-x_i}{s_{i+1}-s_i}\Big)^2 . \square$$

Our next lemma shows that condition 2 of Proposition 3.1 is fulfilled.

Lemma 3.2. *Under the assumptions of Proposition 3.2, we have, for each $\tau > 0$*

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \max_{i\leq p_n} \log\Big(\mathbb{P}\Big(\sup_{|s-s'|<\delta} \Big| \frac{\mathcal{D}_{n,h_{n,i},t}(s) - \mathcal{D}_{n,h_{n,i},t}(s')}{(2h_{n,i}\log_2 n)^{1/2}} \Big| \geq \tau \Big)\Big) = -\infty.$$

Proof of Lemma 3.2.

Fix $\tau > 0$ and introduce a parameter $\delta > 0$ that will be chosen small enough in the sequel. The proof of this lemma relies on an exponential inequality for the oscillations of the local empirical process, which is due to Einmahl and Mason (1988) (see their Inequality 1). For positive numbers a, b with $a + b \leq 1$, write

$$\omega_n(a,b) := \sup_{\substack{0\leq s\leq b,\\ 0\leq s'\leq a}} \mid \alpha_n(s+s') - \alpha_n(s) \mid . \tag{3.7}$$

Fact 1 (Einmahl, Mason, 1988). *Fix $0 < \varepsilon \leq 1/2$. There exists $K(\varepsilon) < \infty$ such that, for any $n \geq 1$, $\lambda > 0$, $a > 0$, $b > 0$ fulfilling $a+b \leq 1$ and $0 < a < 1/4$,*

$$\mathbb{P}\Big(\omega_n(a,b) \geq \lambda\Big) \leq K(\varepsilon) b a^{-1} \exp\Big(- \frac{(1-\varepsilon)\lambda^2}{2a} \Psi\Big(\frac{\lambda}{\sqrt{na}}\Big)\Big). \tag{3.8}$$

Here we write $\Psi(u) := 2u^{-2}((1+u)\log(1+u) - u)$.

Applying (3.8) to $b = h_{n,i},\ a = \delta h_{n,i},\ \varepsilon = 1/2$ and $\lambda = \tau(2h_{n,i}\log_2 n)^{1/2}$ we get, for all large n and $i \le p_n$ (so that $h_{n,i} \le h_n \le 1/4$)

$$\mathbb{P}\Big(\sup_{|s-s'|<\delta}\Big|\frac{\mathcal{D}_{n,h_{n,i},t}(s) - \mathcal{D}_{n,h_{n,i},t}(s')}{(2h_{n,i}\log_2 n)^{1/2}}\Big| \ge \tau\Big) \le \frac{K(\frac{1}{2})}{\delta}\exp\Big(-\frac{\tau^2\log_2 n}{2\delta}\Psi\Big(\frac{\tau\sqrt{2\log_2 n}}{\delta\sqrt{nh_{n,i}}}\Big)\Big)$$
$$\le \frac{K(\frac{1}{2})}{\delta}\exp\Big(-\frac{\tau^2\log_2 n}{4\delta}\Big). \quad (3.9)$$

The last inequality holds for all large n and $i \le p_n$ since $\Psi(u) \to 1$ as $u \to 0$, and since

$$\lim_{n\to\infty}\max_{i\le p_n}\frac{\log_2 n}{nh_{n,i}} = 0. \quad (3.10)$$

Now taking the logarithm in (3.9) concludes the proof of Lemma 3.2, then lemmas 3.1 and 3.2 in combination with Proposition 3.3 conclude the proof of Proposition 3.2. □

3.2. Proof of Theorem 1

We shall invoke usual blocking arguments along the following subsequence:

$$n_k := \Big[\exp\Big(k\exp\big(-(\log k)^{1/2}\big)\Big)\Big],\ k \ge 5. \quad (3.11)$$

Clearly, n_k satisfies, as $k \to \infty$,

$$\frac{n_k}{n_{k+1}} \to 1,\ \log_2(n_k) = \log k(1 + o(1)). \quad (3.12)$$

Now define the blocks $N_k := \{n_{k-1}, \ldots, n_k - 1\}$ for $k \ge 6$. Fix $\epsilon > 0$ and consider a parameter $\rho > 1$ that will be chosen small enough in the sequel. For any $k \ge 5$, consider the following discretisation of $[\mathfrak{h}_{n_k}, h_{n_{k-1}}]$

$$h_{n_k,R_k} := h_{n_{k-1}},\ \ h_{n_k,l} := \rho^l\mathfrak{h}_{n_k},\ l = 0, \ldots, R_k - 1, \quad (3.13)$$

where $R_k := [(\log(h_{n_{k-1}}/\mathfrak{h}_{n_k}))/\log(\rho)] + 1$, and $[u]$ denotes the only integer q fulfilling $q \le u < q+1$. Clearly, as $k \to \infty$, we have

$$R_k = O(\log n_k). \quad (3.14)$$

Our aim is to show that the following probabilities are summable in k so as the Borel-Cantelli lemma would complete the proof of Theorem 1.

$$\mathbb{P}_k := \mathbb{P}\Big(\max_{n\in N_k}\ \sup_{\mathfrak{h}_n\le h\le h_n}\ \inf_{g\in\sqrt{2}\mathcal{S}}\Big|\Big|\frac{\mathcal{D}_{n,h,t}}{(2h\log_2 n)^{1/2}}-g\Big|\Big|\ge 3\epsilon\Big). \tag{3.15}$$

Clearly we have

$$\begin{aligned}\mathbb{P}_k \le\ & \mathbb{P}\Big(\max_{0\le l\le R_k}\ \inf_{g\in\sqrt{2}\mathcal{S}}\Big|\Big|\frac{\mathcal{D}_{n_k,h_{n_k,l},t}}{(2h_{n_k,l}\log_2 n_k)^{1/2}}-g\Big|\Big|\ge\epsilon\Big)\\ &+\mathbb{P}\Big(\max_{n\in N_k}\ \max_{0\le l\le R_k-1}\ \sup_{h_{n_k,l}\le h\le\rho h_{n_k,l}}\Big|\Big|\frac{\mathcal{D}_{n,h,t}}{(2h\log_2 n)^{1/2}}-\frac{\mathcal{D}_{n_k,h_{n_k,l},t}}{(2h_{n_k,l}\log_2 n_k)^{1/2}}\Big|\Big|>2\epsilon\Big)\\ =:\ & \mathbb{P}_{1,k}+\mathbb{P}_{2,k}.\end{aligned}$$

To show that $\mathbb{P}_{1,k}$ is summable, we shall make use of Proposition 3.2. Consider the following subset of $B([0,1])$:

$$F := \Big\{f\in B([0,1]),\ \inf_{g\in\sqrt{2}\mathcal{S}}||\ f-g\ ||\ge\epsilon\Big\}.$$

Since the rate function J given in (3.3) is lower semi continuous on $(B([0,1],||\cdot||)$, there exists $\alpha_1>0$ satisfying $J(F)=2+2\alpha_1$. Hence, for all large k we have

$$\mathbb{P}_{1,k}\le(R_k+1)\exp\big(-(2+\alpha_1)\log_2 n_k\big). \tag{3.16}$$

Recalling (3.12) and (3.14), we conclude that $\mathbb{P}_{1,k}$ is summable in k. It remains to show the summability of $(\mathbb{P}_{2,k})_{k\ge1}$. First notice that

$$\begin{aligned}\mathbb{P}_{2,k}\le\ & \mathbb{P}\Big(\max_{l\le R_k-1}\max_{n\in N_k}\ \sup_{h_{n_k,l}\le h\le\rho h_{n_k,l}}\Big|\Big|\frac{\sqrt{n}\mathcal{D}_{n,h,t}-\sqrt{n}\mathcal{D}_{n,h_{n_k,l},t}}{(2n_k h_{n_k,l}\log_2 n_k)^{1/2}}\Big|\Big|>\epsilon\Big)\\ &+\mathbb{P}\Big(\max_{l\le R_k-1}\max_{n\in N_k}\ \sup_{h_{n_k,l}\le h\le\rho h_{n_k,l}}\mathcal{B}(n,h)\Big|\Big|\frac{\sqrt{n}\mathcal{D}_{n,h,t}}{(2n_k\rho h_{n_k,l}\log_2 n_k)^{1/2}}\Big|\Big|>\epsilon\Big)\\ =:\ & \mathbb{P}_{3,k}+\mathbb{P}_{4,k},\end{aligned} \tag{3.17}$$

where

$$\mathcal{B}(n,h):=\Big|\sqrt{\frac{n_k\rho h_{n_k,l}\log_2 n_k}{nh\log_2 n}}-1\Big|,\ n\in N_k,\ l\le R_k-1,\ h_{n_k,l}\le h\le\rho h_{n_k,l}. \tag{3.18}$$

We shall require a maximal inequality due to Montgomery-Smith (1993) (see also Latala (1993)).

Fact 2 (Montgomery-Smith, Latala, 1993). *There exists a constant $c>0$ such that, given a Banach space $(E,||\cdot||)$ and a finite sequence $(X_i)_{1\le i\le n}$ of i.i.d. random variables taking values in (E,d) we have, for each $\lambda>0$:*

$$\mathbb{P}\Big(\max_{1\le i\le n}\Big|\Big|\sum_{j=1}^{i}X_j\Big|\Big|\ge\lambda\Big)\le c\mathbb{P}\Big(\Big|\Big|\sum_{i=1}^{n}X_i\Big|\Big|\ge\frac{\lambda}{c}\Big). \tag{3.19}$$

Applying inequality (3.19), we get

$$\begin{aligned}\mathbb{P}_{3,k}\le&\sum_{l=0}^{R_k-1}\mathbb{P}\Big(\max_{n\in N_k}\sup_{h_{n_k,l}\le h\le\rho h_{n_k,l}}\Big|\Big|\frac{\sqrt{n}\mathcal{D}_{n,h,t}-\sqrt{n}\mathcal{D}_{n,h_{n_k,l},t}}{(2n_kh_{n_k,l}\log_2 n_k)^{1/2}}\Big|\Big|>\epsilon\Big)\\ \le& c\sum_{l=0}^{R_k-1}\mathbb{P}\Big(\sup_{h_{n_k,l}\le h\le\rho h_{n_k,l}}\Big|\Big|\frac{\sqrt{n_k}\mathcal{D}_{n_k,h,t}-\sqrt{n_k}\mathcal{D}_{n_k,h_{n_k,l},t}}{(2n_kh_{n_k,l}\log_2 n_k)^{1/2}}\Big|\Big|>\epsilon/c\Big). \quad (3.20)\end{aligned}$$

As $h_{n_k,l}\le h_{n_{k-1}}\to 0$, each term of (3.20) can be bounded by inequality (3.8), provided that $h_{n_{k-1}}<1/4$. In inequality (3.8), we repeatedly choose $b=h_{n_k,l},\ a=h_{n_k,l}(\rho-1),\ \varepsilon=1/2,\ \lambda=(2h_{n_k,l}\log_2 n_k)^{1/2}\epsilon/c$. Hence, for all large k we have

$$\begin{aligned}\mathbb{P}_{3,k}\le& c\sum_{l=0}^{R_k-1}\frac{K(\frac12)}{\rho-1}\exp\Big(-\frac{\epsilon^2\log_2 n_k}{2c^2(\rho-1)^2}\Psi\Big(\frac{\epsilon\sqrt{\log_2 n_k}}{c(\rho-1)\sqrt{n_kh_{n_k,l}}}\Big)\Big)\\ \le& c\sum_{l=0}^{R_k-1}\frac{K(\frac12)}{\rho-1}\exp\Big(-\frac{\epsilon^2\log_2 n_k}{4c^2(\rho-1)^2}\Big) \quad (3.21)\\ \le&\frac{cK(\frac12)}{\rho-1}R_kk^{-\epsilon/2c(\rho-1)^2}. \quad (3.22)\end{aligned}$$

Inequality (3.21) is true for all large k since $\Psi(u)\to 1$ as $u\to 0$, and since

$$\lim_{k\to\infty}\max_{l\le R_k-1}\frac{\log_2 n_k}{n_kh_{n_k,l}}=0. \tag{3.23}$$

Inequality (3.22) takes in account the fact that $\log_2 n_k=\log k(1+o(1))$ as $k\to\infty$. Hence for any choice of $1<\rho<1+\sqrt{\epsilon/2c}$ the general term (3.22) is summable in k and so are the $\mathbb{P}_{3,k}$ (recall (3.14)). Showing that $\sum\mathbb{P}_{4,k}<\infty$ will be done in a similar way. First notice that, as $n_k/n_{k-1} \to 1$ and $1 \leq \rho h_{n_k,l}/h \leq \rho$ we have

$$\lim_{k\to\infty} \max_{0\leq l\leq R_k-1} \max_{n\in N_k} \mathcal{B}(h,n) = \rho^{1/2} - 1 \leq 2(\rho-1). \tag{3.24}$$

Hence, for all large k we have

$$\begin{aligned}
\mathbb{P}_{4,k} &\leq \mathbb{P}\Big(\max_{0\leq l\leq R_k-1} \max_{n\in N_k} \Big|\Big|\frac{\sqrt{n}\mathcal{D}_{n,\rho h_{n_k,l},t}}{(2n_k\rho h_{n_k,l}\log_2 n_k)^{1/2}}\Big|\Big| > \frac{\epsilon}{2(\rho-1)}\Big)\\
&\leq c\sum_{l=0}^{R_k-1} \mathbb{P}\Big(\Big|\Big|\frac{\mathcal{D}_{n_k,\rho h_{n_k,l},t}}{(2\rho h_{n_k,l}\log_2 n_k)^{1/2}}\Big|\Big| > \frac{\epsilon}{2c(\rho-1)}\Big)\\
&\leq 2c\sum_{l=0}^{R_k-1} \exp\Big(-\frac{\epsilon^2(1-\rho h_{n_k,l})\log_2 n_k}{8c^2(\rho-1)^2}\Psi\Big(\frac{\epsilon(1-\rho h_{n_k,l})\sqrt{2\log_2 n_k}}{2c\sqrt{n_k\rho h_{n_k,l}}}\Big)\Big)
\end{aligned} \tag{3.25}$$

$$\leq 2cR_k \exp\Big(-\frac{\epsilon^2(1-\rho h_{n_k,l})\log_2 n_k}{16c^2(\rho-1)^2}\Big). \tag{3.26}$$

Here, (3.25) is a consequence of Inequality 2 in Shorack and Wellner (1986, p. 444), with $p = \rho h_{n_k,l},\ \lambda = \epsilon(1-\rho h_{n_k,l})(2\rho h_{n_k,l}\log_2 n_k)^{1/2}/4c(\rho-1)$. Recalling (3.23), we see that (3.26) holds for all large k, as $\Psi(u) \to 1$ when $u \to 0$. Now choosing $\rho > 1$ small enough leads to he summability of $(\mathbb{P}_{4,k})_{k\geq 1}$, which concludes the proof of Theorem 1.□

Remark: If we had replaced the limit set $\sqrt{2}\mathcal{S}$ by $\mathcal{S}$ in Theorem 1, then (3.16) would become

$$\mathbb{P}_{1,k} \leq (R_k+1)\exp\big(-(1+\alpha_1)\log_2 n_k\big).$$

Hence, we would be able to conclude that $\mathbb{P}_{1,k}$ is summable if the cardinality R_k+1 of the grids were smaller than $(\log n_k)^\beta$ for any $\beta > 0$. When constructing the $h_{n_k,l}$ as in (3.13), the just mentioned condition is violated as soon as $\mathfrak{h}_n$ and h_n have "really" different rates of convergence to zero (typically when $\mathfrak{h}_n = h^{-\beta_1} < n^{-\beta_2}$ with $0 < \beta_2 < \beta_1 < 1$). It seems however impossible to reduce the cardinality R_k+1 of our grids, since the oscillations between two consecutive $h_{n_k,l}$ become hardly controllable and hence the corresponding probabilities $\mathbb{P}_{2,k}$ might not be summable. One could expect some improvements of this proof, since the RHS of (3.16) is crudely obtained, but this turns out to be non trivial, as Proposition 3.2 would have to be improved to more accurate large deviation rates for the $\mathcal{D}_{n_k,h_{n_k,l},t}$, $0 \leq l \leq R_k$. Another possibility would be to "poissonize" the $\mathcal{D}_{n,h,t}$ and then make use of strong approximation of a centred Poisson process by a Wiener process W (see Komlòs *et al.*, 1977), which would reduce the problem to studying the summability of

$$\mathbb{P}^W_{1,k} := \mathbb{P}\Big(\exists \rho \in (\frac{\mathfrak{h}_{n_k}}{h_{n_{k-1}}}, 1),\ \rho^{-1/2} W(\rho \cdot) \notin (2\log_2 n_k)^{1/2}(\mathcal{S} + \epsilon \mathcal{B}_0)\Big), \tag{3.27}$$

and then try to make use of the isoperimetric properties of a Gaussian measures (here $\mathcal{B}_0$ denotes the unit ball of $B([0,1])$). This however fails to work by making brute use of the isoperimetric inequality, as long as $\mathfrak{h}_{n_k}/h_{n_{k-1}}$ is not negligible with respect to $\log_2 n_k$ as $k \to \infty$. We hope however, that (3.27) may be better controlled and we thus leave an open question to specialists in Gaussian measures.

4. Proof of Theorem 2

To avoid lengthy notations, we shall prove Theorem 2 only with $k = 2$ with no loss of generality. The key of our proof of Theorem 2 is the following lemma.

Lemma 4.1. *Under the assumptions of Theorem 2, for any $p \geq 1$, $0 < s_1^{(1)} < \ldots < s_p^{(1)} < 1$ and $0 < s_1^{(2)} < \ldots < s_p^{(2)} < 1$, the sequence of $\mathbb{R}^{2p}$-valued random vectors*

$$X_n := \Big(\frac{\mathcal{D}_{n,h_{n,1},t}(s_1^{(1)})}{(2h_{n,1}\log_2 n)^{1/2}}, \ldots, \frac{\mathcal{D}_{n,h_{n,1},t}(s_p^{(1)})}{(2h_{n,1}\log_2 n)^{1/2}}, \frac{\mathcal{D}_{n,h_{n,2},t}(s_1^{(2)})}{(2h_{n,2}\log_2 n)^{1/2}}, \ldots, \frac{\mathcal{D}_{n,h_{n,2},t}(s_p^{(2)})}{(2h_{n,2}\log_2 n)^{1/2}}\Big)$$

satisfies the large deviation principle for the sequence $(\log_2 n)^{-1}$ and the following rate function (writing $s_0^{(1)} = s_0^{(2)} = 0$).

$$\overline{J}_{s_1^{(1)},\ldots,s_p^{(1)},s_1^{(2)},\ldots,s_p^{(2)}}(x) := \sum_{i=1}^{p} (s_{i+1}^{(1)} - s_i^{(1)})\Big(\frac{x_{i+1}^{(1)} - x_i^{(1)}}{s_{i+1}^{(1)} - s_i^{(1)}}\Big)^2 + (s_{i+1}^{(2)} - s_i^{(2)})\Big(\frac{x_{i+1}^{(2)} - x_i^{(2)}}{s_{i+1}^{(2)} - s_i^{(2)}}\Big)^2,$$

$$x = x_1^{(1)}, \ldots, x_p^{(1)}, x_1^{(2)}, \ldots, x_p^{(2)} \in (0,1)^{2p}. \tag{4.1}$$

Proof of Lemma 4.1.

The proof follows the same lines as the proof of Lemma 3.1. Choose $\lambda := (\lambda_1^{(1)},\ldots,\lambda_p^{(1)},\lambda_1^{(2)},\ldots,\lambda_p^{(2)}) \in \mathbb{R}^{2p}$ arbitrarily and set (recall that U_1 is uniform on $[0,1]$).

$$X_{n,1} := \sum_{j=1}^{p} \lambda_j^{(1)} \big(1_{[t,t+h_{n,1}s_j^{(1)}]}(U_1) - h_{n,1}s_j^{(1)}\big),$$
$$X_{n,2} := \sum_{j=1}^{p} \lambda_j^{(2)} \big(1_{[t,t+h_{n,2}s_j^{(2)}]}(U_1) - h_{n,2}s_j^{(2)}\big).$$

By independence we have

$$\begin{aligned}&(\log_2 n)^{-1} \log\Big(\mathbb{E}\Big(\exp\big(\log_2 n \ <\lambda, X_n>\big)\Big)\Big)\\ =&\frac{n}{\log_2 n}\log\Big(\mathbb{E}\Big(\exp\big(r_{n,1}X_{n,1}+r_{n,2}X_{n,2}\big)\Big)\Big),\end{aligned}$$

with $r_{n,1} := \sqrt{\log_2 n/2nh_{n,1}}$ and $r_{n,2} := \sqrt{\log_2 n/2nh_{n,2}}$. As $X_{n,1}$ (resp $X_{n,2}$) is centered and almost surely bounded by $2p\max_{j=1,\ldots,2p} \mid \lambda_j \mid$, the following Taylor expansion is valid by the dominated convergence theorem (here $\lim_{|a|,|b|\to 0} \varepsilon(a,b)=0$):

$$\begin{aligned}&\log\Big(\mathbb{E}\Big(\exp\big(r_{n,1}X_{n,1}+r_{n,2}X_{n,2}\big)\Big)\Big)\\ =&\frac{1}{2}\Big(r_{n,1}^2\mathrm{Var}(X_{n,1})+r_{n,2}^2\mathrm{Var}(X_{n,2})+2r_{n,1}r_{n,2}\mathrm{Cov}(X_{n,1},X_{n,2})\Big)(1+\varepsilon(r_{n,1},r_{n,2})).\end{aligned}$$

Now, writing $\lambda_1 := (\lambda_1^{(1)},\ldots,\lambda_p^{(1)})$ and $\lambda_2 := (\lambda_1^{(2)},\ldots,\lambda_p^{(2)})$ we can write $\mathrm{Var}(X_{n,1}) = \lambda_1'\Sigma_n^{(1)}\lambda_1$ and $\mathrm{Var}(X_{n,2}) = \lambda_2'\Sigma_n^{(2)}\lambda_2$, where

$$\begin{aligned}\Sigma_n^{(1)}(i,j) :=& h_{n,1}\min(s_i^{(1)},s_j^{(1)}) - h_{n,1}^2 s_i^{(1)}s_j^{(1)}, \text{ and}\\ \Sigma_n^{(2)}(i,j) :=& h_{n,2}\min(s_i^{(2)},s_j^{(2)}) - h_{n,2}^2 s_i^{(2)}s_j^{(2)}.\end{aligned}$$

Hence, setting

$$\Sigma^{(1)}(i,j) := \min(s_i^{(1)},s_j^{(1)}) \text{ and } \Sigma^{(2)}(i,j) := \min(s_i^{(2)},s_j^{(2)}),$$

we obtain

$$\big(r_{n,1}^2\mathrm{Var}(X_{n,1})+r_{n,2}^2\mathrm{Var}(X_{n,2})\big) = \frac{\log_2 n}{2n}\big(\lambda_1'\Sigma^{(1)}\lambda_1+\lambda_2'\Sigma^{(2)}\lambda_2\big)(1+o(1)). \quad (4.2)$$

In a similar way, we can write $\mathrm{Cov}(X_{n,1}, X_{n,2}) = \lambda_1'\Sigma_n\lambda_2$, where $\Sigma_n(i,j) := \min(h_{n,1}s_i^{(1)}, h_{n,2}s_j^{(2)}) - h_{n,1}h_{n,2}s_i^{(1)}s_j^{(2)}$. Now recalling that $h_{n,1}/h_{n,2} \to 0$ we have $\Sigma_n(i,j) = h_{n,1}s_i^{(1)}(1 - s_j^{(2)}h_{n,2})$ for all large n, whence

$$\Big|r_{n,1}r_{n,2}\mathrm{Cov}(X_{n,1}, X_{n,2})\Big| = \frac{\log_2 n}{n}\sqrt{\frac{h_{n,1}}{h_{n,2}}}(1 + o(1)) = o\Big(\frac{\log_2 n}{n}\Big). \tag{4.3}$$

Combining (4.2) and (4.3) we get

$$\lim_{n\to\infty} (\log_2 n)^{-1} \log\Big(\mathbb{E}\Big(\exp\big(\log_2 n \ < \lambda, X_n >\big)\Big)\Big) = \frac{1}{4}(\lambda_1'\Sigma^{(1)}\lambda_1 + \lambda_2'\Sigma^{(2)}\lambda_2).$$

Then applying Proposition 3.3 leads to the claimed result.□

We shall now show that Lemma 4.1 is sufficient to infer a large deviation principle for the couples of processes $(2h_{n,1}\log_2 n)^{-1/2}\mathcal{D}_{n,h_{n,1},t}$ and $(2h_{n,2}\log_2 n)^{-1/2}\mathcal{D}_{n,h_{n,2},t}$. Consider the following processes on $[0,2]$ that are obtained by concatenation of $(2h_{n,1}\log_2 n)^{-1/2}\mathcal{D}_{n,h_{n,1},t}$ with $(2h_{n,2}\log_2 n)^{-1/2}\mathcal{D}_{n,h_{n,2},t}$:

$$\widetilde{\mathcal{D}_n}(s) := \begin{cases} \frac{\mathcal{D}_{n,h_{n,1},t}(s)}{(2h_{n,1}\log_2 n)^{1/2}}, & \text{when } 0 \le \mathrm{s} \le 1; \\ \frac{\mathcal{D}_{n,h_{n,2},t}(s-1)}{(2h_{n,2}\log_2 n)^{1/2}}, & \text{when } 1 < \mathrm{s} \le 2. \end{cases}$$

Combining Lemma 4.1 with Lemma 3.2 we conclude that conditions of Proposition 3.1 are fulfilled, and thus $\widetilde{\mathcal{D}}_n$ satisfies the large deviation principle for $\epsilon_n := (\log_2 n)^{-1}$ and for the following rate function:

$$\begin{aligned} &\overline{J}(g) \\ &:= \sup\Big\{\sum_{j=0}^{p}(s_{j+1}^{(1)} - s_j^{(1)})\Big(\frac{g(s_{j+1}^{(1)}) - g(s_j^{(1)})}{s_{j+1}^{(1)} - s_j^{(1)}}\Big)^2 + (s_{j+1}^{(2)} - s_j^{(2)})\Big(\frac{g(1+s_{j+1}^{(2)}) - g(1+s_j^{(2)})}{s_{j+1}^{(2)} - s_j^{(2)}}\Big)^2, \\ &\quad p \ge 1,\ 0 < s_1^{(1)} < \ldots < s_p^{(1)} < 1 < 1 + s_1^{(2)} < \ldots < 1 + s_p^{(2)} < 2\Big\} \\ &= \|\, g^{(1)} \,\|_H^2 + \|\, g^{(2)} \,\|_H^{(2)}, \end{aligned}$$

where $g^{(1)}(s) := g(s),\ g^{(2)}(s) := g(1+s),\ s \in [0,1]$. The remainder of the proof of Theorem 2 is a routine use of usual techniques in local empirical processes theory (refer, e.g., to Deheuvels and Mason (1990)). We omit details for sake of briefness. □

5. Proof of Theorem 3

We shall proceed in three steps. Recall that $a_n(h) := (h\log_2 n/n)^{1/2}$, $b_n(h) := \log(nh)$, $d_n(h) := 2\log_2 n + b_n(h)$, $r_n(h) := (a_n(h)d_n(h))^{1/2}$ and $R_n(h) := \Big|\Big|\mathcal{D}_{n,h,0} + \mathcal{D}'_{n,h,0}\Big|\Big|$.

Lemma 5.1. *Under the assumptions of Theorem 1, we have almost surely*

$$\limsup_{n\to\infty} \sup_{\mathfrak{h}_n \le h \le h_n} \frac{|| F_n^{\leftarrow}(h\cdot) ||}{h} = 1. \tag{5.1}$$

Proof of Lemma 5.1.

First notice that, almost surely, for each $\rho > 1,\ h > 0,\ n \ge 1$,

$$F_n^{\leftarrow}(h) \le \rho h \frac{\mathcal{D}_{n,\rho h,0}}{(2h\log_2 n)^{1/2}} + (\rho - 1)\Big(\frac{nh}{2\log_2 n}\Big)^{1/2} \ge 0.$$

Now, for fixed $\rho > 1$ we have $(\rho - 1)\inf\{nh/\log_2 n,\ \mathfrak{h}_n \le h \le h_n\} \to \infty$. Moreover, by a straightforward use of Theorem 1 and (1.6),

$$\liminf_{n\to\infty} \inf_{\mathfrak{h}_n \le h \le h_n} \frac{\mathcal{D}_{n,\rho h,0}}{(2h\log_2 n)^{1/2}} \ge -(2\rho)^{1/2} \text{ almost surely.} \tag{5.2}$$

This shows that (5.1) holds with $\le$ instead of $=$, while the converse inequality trivially holds by Kiefer (1972), Theorem 6.□

Lemma 5.2. *Under the assumptions of Theorem 1 we have almost surely*

$$\limsup_{n\to\infty} \sup_{\mathfrak{h}_n \le h \le h_n} \frac{|| \mathcal{D}'_{n,h,0} ||}{(2h\log_2 n)^{1/2}} \le 2^{1/2}.$$

Proof of Lemma 5.2.

From Inequality (2.23) in Einmahl and Mason (1988) we have, for each $n \ge 1$ and $h > 0$,

$$\frac{|| \mathcal{D}'_{n,h,0} ||}{(2h\log_2 n)^{1/2}} \le \frac{|| \mathcal{D}_{n,F_n^{\leftarrow}(h),0} ||}{(2h\log_2 n)^{1/2}} + \frac{1}{(2nh\log_2 n)^{1/2}}.$$

The second term can be drop since $n\mathfrak{h}_n \to \infty$. Fix $\rho > 0$. By Lemma 5.1 we have almost surely, for all large n and for all $\mathfrak{h}_n \le h \le h_n$,

$$\frac{||\, \mathcal{D}_{n,F_n^{\leftarrow}(h),0} \,||}{(2h \log_2 n)^{1/2}} \le \rho^{1/2} \frac{||\, \mathcal{D}_{n,\rho h,0} \,||}{(2\rho h \log_2 n)^{1/2}},$$

from where we readily obtain, by Theorem 1,

$$\limsup_{n\to\infty} \sup_{\mathfrak{h}_n \le h \le h_n} \frac{n^{-1/2} \,||\, \mathcal{D}'_{n,h,0} \,||}{(2h \log_2 n)^{1/2}} \le (2\rho)^{1/2} \text{ almost surely.}$$

As $\rho > 1$ was arbitrary, Lemma 5.2 is proved. □

The expression ω_n appearing in the next lemma has been defined in (3.7).

Lemma 5.3. *Under the assumptions of Theorem 1, and given $\eta > 0$, we have almost surely*

$$\limsup_{n\to\infty} \sup_{\mathfrak{h}_n \le h \le h_n} \frac{\omega_n(\eta a_n(h), h)}{r_n(h)} \le \eta^{1/2}. \tag{5.3}$$

Proof of Lemma 5.3.

This proof is largely inspired from the proof of Lemma 6 in Einmahl and Mason (1988). Fix $\epsilon > 0$ and consider the sequence (n_k) the sets N_k and the grids $h_{n_k,l}$, $0 \le l \le R_k$ as in §3.2. Also define, for each $k \ge 5$ and $l \le R_k$,

$$\begin{aligned} a_{k,l} :=& \eta(\rho h_{n_k,l} \log_2 n_k / n_{k-1})^{1/2} \text{ and} \\ r_{k,l} :=& (a_{k,l}(2 \log_2 n_k + \log(n_k h_{n_k,l})))^{1/2}. \end{aligned}$$

As $a_{k,l} \ge a_n(h)$ for each $n \in N_k$ and $h \in [h_{n_k,l}, \rho h_{n_k,l}]$, we have

$$\begin{aligned} &\mathbb{P}\Bigg(\bigcup_{n\in N_k} \bigcup_{\mathfrak{h}_n \le h \le h_n} \frac{\omega_n(\eta a_n(h), h)}{r_n(h)} \ge \eta^{1/2}(1+3\epsilon) \Bigg) \\ &\le \mathbb{P}\Bigg(\bigcup_{l=0}^{R_k-1} \bigcup_{n\in N_k} \bigcup_{h_{n_k,l} \le h \le \rho h_{n_k,l}} \frac{\omega_n(a_{k,l}, \rho h_{n_k,l})}{r_n(h)} > \eta^{1/2}(1+3\epsilon) \Bigg) \\ &\le \mathbb{P}\Bigg(\bigcup_{l=0}^{R_k-1} \bigcup_{n\in N_k} \frac{\omega_n(a_{k,l}, \rho h_{n_k,l})}{r_{k,l}} > \eta^{1/2}(1+2\epsilon) \Bigg) \qquad (5.4) \\ &=: \overline{\mathbb{P}}_k, \qquad (5.5) \end{aligned}$$

where (5.4) holds for any choice of $\rho > 1$ small enough, ultimately as $k \to \infty$, which is a consequence of the easily checked fact that

$$\lim_{\rho \to 1} \lim_{k \to \infty} \max_{n \in N_k} \max_{l \leq R_k - 1} \sup_{h \in [h_{n_k,l}, \rho h_{n_k,l}]} \Big| \frac{r_n(h)}{r_{k,l}} - 1 \Big| = 0. \tag{5.6}$$

By Bonferroni's inequality we can write

$$\begin{aligned} \overline{\mathbb{P}}_k &\leq \sum_{l=0}^{R_k-1} \mathbb{P}\Big(\bigcup_{n \in N_k} \frac{\omega_n(a_{k,l}, \rho h_{n_k,l})}{r_{k,l}} > \eta^{1/2}(1+2\epsilon) \Big) \\ &=: \sum_{l=0}^{R_k-1} \overline{\mathbb{P}}_{k,l}. \end{aligned} \tag{5.7}$$

Some straightforward verifications show that the blocking arguments of Inequality 2 in Einmahl and Mason (1988) can be used simultaneously to each $\overline{\mathbb{P}}_{k,l}$, for all large k and hence, by Fact 1,

$$\begin{aligned} \overline{\mathbb{P}}_{k,l} \leq & 2\mathbb{P}\Big(\omega_{n_k}(a_{k,l}, \rho h_{n_k,l}) \geq \eta^{1/2} r_{k,l}(1+\epsilon) \Big) \\ \leq & 2K\big(\frac{\epsilon}{2}\big) \frac{\rho h_{n_k,l}}{a_{k,l}} \exp\Big(- \frac{(1-\frac{\epsilon}{2})(1+\epsilon)^2}{2a_{k,l}} \eta r_{k,l}^2 \Psi\big(\Delta_{k,l}\big) \Big), \end{aligned}$$

where $\Delta_{k,l} := (1+\epsilon)\eta^{1/2} r_{k,l} n_k^{-1/2} a_{k,l}^{-1}$ converge to 0 uniformly in $l \leq R_k - 1$ when $k \to \infty$. Since Ψ (given in Fact 1) satisfies $\Psi(u) \to 1$ as $u \to 0$ we obtain, for all large k and for each $l \leq R_k - 1$,

$$\begin{aligned} \overline{\mathbb{P}}_{k,l} \leq & 2K\big(\frac{\epsilon}{2}\big) \sqrt{\frac{n_{k-1}\rho h_{n_k,l}}{\eta^2 \log_2 n_k}} \exp\Big(- \frac{(1-\frac{\epsilon}{2})^2(1+\epsilon)^2}{2} \big(2\log_2 n_k + \log(n_k \rho h_{n_k,l})\big) \Big) \\ \leq & 2K\big(\frac{\epsilon}{2}\big)\big(\frac{\eta^2}{\rho}\big)^{\epsilon/8} (n_{k-1}h_{n_k,l})^{-\epsilon/8} (\log_2 n_k)^{-1/2} (\log n_{k-1})^{-1-\epsilon/4}, \end{aligned}$$

for all large k and for each $0 \leq l \leq R_k - 1$, which entails by (5.7)

$$\begin{aligned} \overline{\mathbb{P}}_{1,k} \leq & 2K\big(\frac{\epsilon}{2}\big)\big(\frac{\eta^2}{\rho}\big)^{\epsilon/8} (\log_2 n_k)^{-1/2} (\log n_{k-1})^{-1-\epsilon/4} n_{k-1}^{-\epsilon/8} h_{n_k}^{-\epsilon/8} \sum_{l=0}^{R_k-1} \rho^{-l\epsilon/8} \\ \leq & 2K\big(\frac{\epsilon}{2}\big)\big(\frac{\eta^2}{\rho}\big)^{\epsilon/8} \frac{1}{1-\rho^{-\epsilon/8}} (\log_2 n_k)^{-1/2} (\log n_{k-1})^{-1-\epsilon/8} (n_{k-1}h_{n_k})^{-\epsilon/8}, \end{aligned}$$

from where $\overline{\mathbb{P}}_k$ is summable in k. □

The proof of Theorem 3 is concluded as follows. First, it is well known that, almost surely,

$$|| \alpha_n + \beta_n + (\alpha_n(F_n^{\leftarrow}) - \alpha_n) ||= n^{-1/2}, \tag{5.8}$$

whence, almost surely, for all $n \geq 1$ and $h > 0$,

$$R_n(h) \leq \sup_{0<s<h} || \alpha_n(s + n^{-1/2}\beta_n(s)) - \alpha_n(s) || + n^{-1/2}, \tag{5.9}$$

from where

$$r_n(h)^{-1}R_n(h) \leq r_n(h)^{-1}\omega_n(n^{-1/2} || \mathcal{D}_{n,h,0} ||, h)+(nh\log_2 n)^{-1/4}(2\log_2 n+\log(nh))^{-1/2},$$

which concludes the proof by combining lemmas 5.2 and 5.3 (with the choice of $\eta = 2$), as the second term of the RHS of 5.10 converges to 0 uniformly in $\mathfrak{h}_n \leq h \leq h_n$ as $n \to \infty$.□

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